PROPERTIES OF IDEAL BITOPOLOGICAL $\alpha$-OPEN SETS

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Dedicated to Professor Valeriu Popa on the Occasion of His 80th Birthday

ABSTRACT. The aim of this paper is to introduced and characterized the concepts of $\alpha$-open sets and their related notions in ideal bitopological spaces.

1. Introduction

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [10] and Vaidyanathasamy [11]. An ideal $I$ on a topological space $(X,\tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a bitopological space $(X,\tau_1,\tau_2)$ with an ideal $I$ on $X$ and if $P(X)$ is the set of all subsets of $X$, a set operator $(.;)_i^*: P(X) \to P(X)$, called the local function [11] of $A$ with respect to $\tau_i$ and $I$, is defined as follows: for $A \subset X$, $(.;)_i^*(\tau_i, I) = \{x \in X | U \cap A \notin I \text{ for every } U \in \tau_i(x)\}$, where $\tau_i(x) = \{U \in \tau_i | x \in U\}$. For every ideal topological space $(X,\tau,I)$, there exists topology $\tau^*(I)$, finer than $\tau$, generated by the base $\beta(I,\tau) = \{U \setminus I | U \in \tau \text{ and } I \in I\}$, but in general $\beta(I,\tau)$ is not always a topology [7]. Observe additionally that $\tau_i-\text{Cl}^*(A) = A \cup A_i^*(\tau_i, I)$ defines a Kuratowski closure operator for $\tau^*(I)$, when there is no chance of confusion, $A_i^*(I)$ is denoted by $A_i^*$ and $\tau_i-\text{Int}^*(A)$ denotes the interior of $A$ in $\tau_i^*(I)$. The aim of this paper is to introduced and characterized the concepts of $\alpha$-open sets and their related notions in ideal bitopological spaces.

2. Preliminaries

Let $A$ be a subset of a bitopological space $(X,\tau_1,\tau_2)$. We denote the closure of $A$ and the interior of $A$ with respect to $\tau_i$ by $\tau_i-\text{Cl}(A)$ and $\tau_i-\text{Int}(A)$, respectively.

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Definition 2.1. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-$\alpha$-open $[9]$ if $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}(A)))$, where $i, j = 1, 2$ and $i \neq j$.

Definition 2.2. A subset $S$ of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\alpha$-$\mathcal{I}$-open $[8]$ if $S \subset \text{Int}(\text{Cl}^*(\text{Int}(S)))$. The family of all $\alpha$-$\mathcal{I}$-open sets of $(X, \tau, \mathcal{I})$ is denoted by $\alpha\mathcal{I}O(X, \tau)$.

Definition 2.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(i, j)$-$\alpha$-continuous $[9]$ if the inverse image of every $\sigma_j$-open set in $(Y, \sigma_1, \sigma_2)$ is $(i, j)$-$\alpha$-open in $(X, \tau_1, \tau_2, \mathcal{I})$, where $i \neq j, i, j = 1, 2$.

Definition 2.4. A subset $A$ of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be

- (i) $(i, j)$-$R$-$\mathcal{I}$-open $[1]$ if $A = \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))$.
- (ii) $(i, j)$-semi-$\mathcal{I}$-open $[3]$ if $A \subset \tau_j\text{-Cl}^*(\tau_i\text{-Int}(A))$.
- (iii) $(i, j)$-pre-$\mathcal{I}$-open $[2]$ if $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))$.
- (iv) $(i, j)$-$b$-$\mathcal{I}$-open $[4]$ if $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)) \cup \tau_j\text{-Cl}^*(\tau_i\text{-Int}(A))$.
- (v) $(i, j)$-$\beta$-$\mathcal{I}$-open $[5]$ if $A \subset \tau_j\text{-Cl}(\tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)))$.
- (vi) $(i, j)$-$\delta$-$\mathcal{I}$-open $[1]$ if $\tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)) \subset \tau_j\text{-Cl}^*(\tau_i\text{-Int}(A))$.

The complement of an $(i, j)$-pre-$\mathcal{I}$-open (resp. $(i, j)$-$\beta$-$\mathcal{I}$-open) set is called an $(i, j)$-pre-$\mathcal{I}$-closed (resp. $(i, j)$-$\beta$-$\mathcal{I}$-closed) set.

Lemma 2.5. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space. Then

- (i) A subset $A$ is $(i, j)$-pre-$\mathcal{I}$-closed if and only if $\tau_i\text{-Cl}(\tau_j\text{-Int}^*(A)) \subset A$ $[2]$;
- (i) A subset $A$ is $(i, j)$-$\beta$-$\mathcal{I}$-closed if and only if $\tau_j\text{-Int}(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(A))) \subset A$ $[5]$.

Definition 2.6. A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be

- (i) pairwise pre-$\mathcal{I}$-continuous $[2]$ if the inverse image of every $\sigma_i$-open set of $Y$ is $(i, j)$-pre-$\mathcal{I}$-open in $X$, where $i \neq j, i, j = 1, 2$.
- (i) pairwise semi-$\mathcal{I}$-continuous $[3]$ if the inverse image of every $\sigma_i$-open set of $Y$ is $(i, j)$-semi-$\mathcal{I}$-open in $X$, where $i \neq j, i, j = 1, 2$.
- (i) pairwise $b$-$\mathcal{I}$-continuous $[4]$ if the inverse image of every $\sigma_i$-open set of $Y$ is $(i, j)$-$b$-$\mathcal{I}$-open in $X$, where $i \neq j, i, j = 1, 2$.
- (i) pairwise $\beta$-$\mathcal{I}$-continuous $[5]$ if the inverse image of every $\sigma_i$-open set of $Y$ is $(i, j)$-$\beta$-$\mathcal{I}$-open in $X$, where $i \neq j, i, j = 1, 2$.
- (i) pairwise strongly $\beta$-$\mathcal{I}$-continuous $[5]$ if the inverse image of every $\sigma_i$-open set of $Y$ is strongly $(i, j)$-$\beta$-$\mathcal{I}$-open in $X$, where $i \neq j, i, j = 1, 2$. 

Definition 3.1. A subset \( A \) of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\) is said to be \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open if and only if \( A \subset \tau_i\text{-}\text{Int}(\tau_j\text{-}\text{Cl}^i(\tau_i\text{-}\text{Int}(A))) \), where \( i, j = 1, 2 \) and \( i \neq j \).

The family of all \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open sets of \((X, \tau_1, \tau_2, \mathcal{I})\) is denoted by \( \alpha \mathcal{IO}(X, \tau_1, \tau_2, \mathcal{I}) \) or \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)O\((X)\). Also, The family of all \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open sets of \((X, \tau_1, \tau_2, \mathcal{I})\) containing \( x \) is denoted by \( \alpha \mathcal{IO}(X, x) \).

Remark 3.2. Let \( \mathcal{I} \) and \( \mathcal{J} \) be two ideals on \((X, \tau_1, \tau_2)\). If \( \mathcal{I} \subset \mathcal{J} \), then \( \alpha \mathcal{IO}(X, \tau_1, \tau_2) \subset \alpha \mathcal{IO}(X, \tau_1, \tau_2) \).

Proposition 3.3. (i) Every \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open set is \((i, j)\)-semi-\(\mathcal{I}\)-open.

(ii) Every \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open set is \((i, j)\)-\(\alpha\)-open.

(iii) Every \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open set is \((i, j)\)-pre-\(\mathcal{I}\)-open.

(iv) Every \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open set is \((i, j)\)-\(\alpha\)-open.

(v) Every \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open set is \((i, j)\)-\(\alpha\)-open.

Proof. The proof follows from the definitions.

The following example show that the converses of Proposition 3.3 is not true in general.

Example 3.4. Let \( X = \{a, b, c\} \), \( \tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \), \( \tau_2 = \{\emptyset, \{a\}, X\} \) and \( \mathcal{I} = \{\emptyset, \{a\}\} \). Then the set \( \{a, c\} \) is \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open but not \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open. Also, the set \( \{b, c\} \) is \((i, j)\)-semi-\(\mathcal{I}\)-open but not \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open and the set \( \{a, c\} \) is \((i, j)\)-pre-\(\mathcal{I}\)-open and \((i, j)\)-\(\alpha\)-open but not \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open.

Proposition 3.5. For an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\) and \( A \subset X \) we have:

(i) If \( \mathcal{I} = \{\emptyset\} \), then \( A \) is \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open if and only if \( A \) is \((i, j)\)-\(\alpha\)-open.

(ii) If \( \mathcal{I} = \mathcal{P}(X) \), then \( A \) is \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open if and only if \( A \) is \( \tau_i \)-open.

Proof. The proof follows from the fact that

(i) If \( \mathcal{I} = \{\emptyset\} \), then \( A^\ast = \text{Cl}(A) \).

(ii) If \( \mathcal{I} = \mathcal{P}(X) \), then \( A^\ast = \emptyset \) for every subset \( A \) of \( X \).

Proposition 3.6. Let \( A \) be a subset of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\). If \( B \) is an \((i, j)\)-semi-\(\mathcal{I}\)-open set of \( X \) such that \( B \subset A \subset \tau_i\text{-}\text{Int}(\tau_j\text{-}\text{Cl}^i(\tau_i\text{-}\text{Int}(B))) \), then \( A \) is an \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open set of \( X \).

Proof. Since \( B \) is an \((i, j)\)-semi-\(\mathcal{I}\)-open set of \( X \), we have \( B \subset \tau_j\text{-}\text{Cl}^i(\tau_i\text{-}\text{Int}(B)) \). Thus, \( A \subset \tau_i\text{-}\text{Int}(\tau_j\text{-}\text{Cl}^i(\tau_i\text{-}\text{Int}(B))) \subset \tau_i\text{-}\text{Int}(\tau_j\text{-}\text{Cl}^i(\tau_i\text{-}\text{Int}(B))) \subset \tau_i\text{-}\text{Int}(\tau_j\text{-}\text{Cl}^i(\tau_i\text{-}\text{Int}(A))) \), and so \( A \) is an \((i, j)\)-\(\alpha\)-\(\mathcal{I}\)-open set of \( X \).
Proposition 3.7. Let \((X, \tau_1, \tau_2, \mathcal{I})\) be an ideal bitopological space. Then a subset of \(X\) is \((i, j)-\alpha-\mathcal{I}\)-open if and only if it is both \(\delta-\mathcal{I}\)-open and \(\pre-\mathcal{I}\)-open.

Proof. Let \(A\) be an \((i, j)-\alpha-\mathcal{I}\)-open set. Since every \((i, j)-\alpha-\mathcal{I}\)-open set is \((i, j)-\text{semi}-\mathcal{I}\)-open, by Proposition 3.3 \(A\) is an \((i, j)-\delta-\mathcal{I}\)-open. Now we prove that \(A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*_{i}(A))\). Since \(A\) is an \((i, j)-\alpha-\mathcal{I}\)-open, we have \(A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A))) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))\). Hence \(A\) is \((i, j)-\text{pre}-\mathcal{I}\)-open. Conversely, let \(A\) be an \((i, j)-\delta-\mathcal{I}\)-open and \((i, j)-\pre-\mathcal{I}\)-open set. Then we have \(\tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)) \subset \tau_j\text{-Cl}^*(\tau_i\text{-Int}(A))\) and hence \(\tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))\). Since \(A\) is \((i, j)-\pre-\mathcal{I}\)-open, we have \(A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))\). Therefore, we obtain that \(A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))\); hence \(A\) is \((i, j)-\alpha-\mathcal{I}\)-open. \(\square\)

Lemma 3.8. A subset \(A\) is \((i, j)-\alpha-\mathcal{I}\)-open if and only if \((i, j)-\text{semi}-\mathcal{I}\)-open and \((i, j)-\pre-\mathcal{I}\)-open.

Proof. Let \(A\) be \((i, j)-\text{semi}-\mathcal{I}\)-open and \((i, j)-\pre-\mathcal{I}\)-open. Then, \(A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)) \subset \tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))\). This shows that \(A\) is \((i, j)-\alpha-\mathcal{I}\)-open. The converse is obvious. \(\square\)

Corollary 3.9. The following properties are equivalent for subsets of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\):

(i) Every \((i, j)-\pre-\mathcal{I}\)-open set is \((i, j)-\text{semi}-\mathcal{I}\)-open.
(ii) A subset \(A\) of \(X\) is \((i, j)-\alpha-\mathcal{I}\)-open if and only if it is \((i, j)-\pre-\mathcal{I}\)-open.

Corollary 3.10. The following properties are equivalent for subsets of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\):

(i) Every \((i, j)-\text{semi}-\mathcal{I}\)-open set is \((i, j)-\pre-\mathcal{I}\)-open.
(ii) A subset \(A\) of \(X\) is \((i, j)-\alpha-\mathcal{I}\)-open if and only if it is \((i, j)-\text{semi}-\mathcal{I}\)-open.

Proposition 3.11. Let \(A\) be a subset of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\). If \(A\) is \((i, j)-\pre-\mathcal{I}\)-closed and \((i, j)-\alpha-\mathcal{I}\)-open, then it is \(\tau_i\)-open.

Proof. Suppose \(A\) is \((i, j)-\pre-\mathcal{I}\)-closed and \((i, j)-\alpha-\mathcal{I}\)-open. Then by Lemma 2.5 \(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(A)) \subset A\) and \(A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))\). Now \(\tau_i\text{-Cl}(\tau_i\text{-Int}(A)) \subset \tau_i\text{-Cl}(\tau_i\text{-Int}^*(A)) \subset A\) and so \(A \subset \tau_i\text{-Int}(\tau_i\text{-Cl}^*(\tau_i\text{-Int}(A))) \subset A \subset \tau_i\text{-Int}(A)\). Therefore, \(A\) is \(\tau_i\)-open. \(\square\)

Lemma 3.12. [1] If \(A\) is any subset of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\), then \(\tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))\) is \((i, j)-R-\mathcal{I}\)-open.

Proposition 3.13. Let \(A\) be a subset of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\). If \(A\) is \((i, j)-\alpha-\mathcal{I}\)-open and \((i, j)-\beta-\mathcal{I}\)-closed, then it is \((i, j)-R-\mathcal{I}\)-open.
Proof. Let $A$ be $(i, j)$-$\alpha$-$\mathcal{I}$-open and $(i, j)$-$\beta$-$\mathcal{I}$-closed. We have by Lemma 2.5, $A \subset \tau_j$-$\text{Int}(\tau_i$-$\text{Cl}^*(\tau_j$-$\text{Int}(A)))$ and $\tau_j$-$\text{Int}(\tau_i$-$\text{Cl}^*(\tau_j$-$\text{Int}(A))) \subset \tau_j$-$\text{Int}(\tau_i$-$\text{Cl}^*(\tau_j$-$\text{Int}(A))) \subset A$; hence $A = \tau_j$-$\text{Int}(\tau_i$-$\text{Cl}^*(\tau_j$-$\text{Int}(A)))$. Thus, by Lemma 3.12, $A$ is $(i, j)$-$\beta$-$\mathcal{I}$-open. \hfill $\Box$

An ideal bitopological space is said to satisfy the condition $(\mathcal{A})$ if $U \cap \tau_j$-$\text{Cl}^*(A) \subset \tau_j$-$\text{Cl}^*(U \cap A)$ for every $U \in \tau_i$.

**Theorem 3.14.** Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space that satisfies the condition $(\mathcal{A})$. Then we have the following

(i) If $V \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$ and $A \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$, then $V \cap A \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$.

(ii) If $V \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$ and $A \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$, then $V \cap A \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$.

Proof. (i). Let $V \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$ and $A \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$. Then $V \cap A \subset \tau_i$-$\text{Int}(\tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(V))) \cap \tau_j$-$\text{Int}(\tau_i$-$\text{Cl}^*(\tau_j$-$\text{Int}(A))) \subset \tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(V \cap \tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(A)))) \subset \tau_j$-$\text{Cl}^*(\tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(V \cap \tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(A))))). This shows that $V \cap A \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$. 

(ii). Let $V \in (i, j)$-$\beta$-$\mathcal{I}O(X)$ and $A \in (i, j)$-$\alpha$-$\mathcal{I}O(X)$. Then $V \cap A \subset \tau_i$-$\text{Int}(\tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(A))) \cap \tau_j$-$\text{Int}(\tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(V))) = \tau_j$-$\text{Int}(\tau_j$-$\text{Cl}^*(\tau_j$-$\text{Int}(V))) \subset \tau_j$-$\text{Cl}^*(\tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(V))) \subset \tau_i$-$\text{Int}(\tau_j$-$\text{Cl}^*(\tau_i$-$\text{Int}(V))). This shows that $V \cap A \in (i, j)$-$\beta$-$\mathcal{I}O(X)$.
Theorem 3.18. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space satisfies the condition \((A)\). If \(A \in (i, j)-\alpha IO(X)\) and \(A \subset B \in (i, j)-\alpha IO(B)\), then \(A \in (i, j)-\alpha IO(O).\)

Proof. By definition, \(A \subset \tau_i-\text{Int}(\tau_j-\text{Cl}^*(\tau_i-\text{Int}(A \cap B))) \cap B = \tau_i-\text{Int}(\tau_j-\text{Cl}^*(\tau_i-\text{Int}(A \cap B))) \cap B \subset \tau_i-\text{Int}(\tau_j-\text{Cl}^*(\tau_i-\text{Int}(A \cap B))) \subset \tau_i-\text{Int}(\tau_j-\text{Cl}^*(\tau_i-\text{Int}(A \cap B))) \subset \tau_i-\text{Int}(\tau_j-\text{Cl}^*(\tau_i-\text{Int}(A \cap B))) \subset \tau_i-\text{Int}(\tau_j-\text{Cl}^*(\tau_i-\text{Int}(A \cap B))) \subset \tau_i-\text{Int}(\tau_j-\text{Cl}^*(\tau_i-\text{Int}(A \cap B))).\) Therefore, we obtain \(A \in (i, j)-\alpha IO(O).\)

Definition 3.19. In an ideal bitopological space \((X, \tau_1, \tau_2, I)\), \(A \subset X\) is said to be \((i, j)-\alpha I\)-closed if \(X \setminus A\) is \((i, j)-\alpha I\)-open in \(X\), \(i, j = 1, 2\) and \(i \neq j\).

Theorem 3.20. If \(A\) is an \((i, j)-\alpha I\)-closed set in an ideal bitopological space \((X, \tau_1, \tau_2, I)\) if and only if \(\tau_i-\text{Cl}(\tau_j-\text{Int}(\tau_i-\text{Cl}(A))) \subset A\).

Proof. The proof follows from the definitions.

Theorem 3.21. If \(A\) is an \((i, j)-\alpha I\)-closed set in an ideal bitopological space \((X, \tau_1, \tau_2, I)\), then \(\tau_i-\text{Cl}(\tau_j-\text{Int}(\tau_i-\text{Cl}(A))) \subset A\).

Proof. Since \(A \in (i, j)-\alpha IO(X)\), \(X \setminus A \in (i, j)-\alpha IO(X).\) Hence, \(X \setminus A \subset \tau_i-\text{Int}(\tau_j-\text{Cl}^*(\tau_i-\text{Int}(X \setminus A))) \subset \tau_i-\text{Int}(\tau_j-\text{Cl}(\tau_i-\text{Int}(X \setminus A))) = X \setminus \tau_i-\text{Cl}(\tau_j-\text{Int}((\tau_i-\text{Cl}(A))))).\) Therefore, we obtain \(\tau_i-\text{Cl}(\tau_j-\text{Int}((\tau_i-\text{Cl}(A))) \subset A\).

Proposition 3.22. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. If a subset of \(X\) is \((i, j)-\beta I\)-closed and \((i, j)-\delta I\)-open, then it is \((i, j)-\alpha I\)-closed.

Proof. The proof follows from the definitions.

Theorem 3.23. Arbitrary intersection of \((i, j)-\alpha I\)-closed sets is always \((i, j)-\alpha I\)-closed.

Proof. Follows from Theorems 3.17 and 3.21.

Definition 3.24. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space, \(S\) a subset of \(X\) and \(x\) be a point of \(X\). Then

(i) \(x\) is called an \((i, j)-\alpha I\)-interior point of \(S\) if there exists \(V \in (i, j)-\alpha IO(X, \tau_1, \tau_2)\) such that \(x \in V \subset S\).

(ii) the set of all \((i, j)-\alpha I\)-interior points of \(S\) is called \((i, j)-\alpha I\)-interior of \(S\) and is denoted by \((i, j)-\alpha I\) Int(S).

Theorem 3.25. Let \(A\) and \(B\) be subsets of \((X, \tau_1, \tau_2, I)\). Then the following properties hold:

(i) \((i, j)-\alpha I\) Int(A) = \(\{T : T \subset A\) and \(A \in (i, j)-\alpha IO(X)\}\).

(ii) \((i, j)-\alpha I\) Int(A) is the largest \((i, j)-\alpha I\)-open subset of \(X\) contained in \(A\).

(iii) \(A\) is \((i, j)-\alpha I\)-open if and only if \(A = (i, j)-\alpha I\) Int(A).
(iv) \((i,j)-\alpha I\ Int((i,j)-\alpha I\ Int(A)) = (i,j)-\alpha I\ Int(A)\).
(v) If \(A \subset B\), then \((i,j)-\alpha I\ Int(A) \subset (i,j)-\alpha I\ Int(B)\).
(vi) \((i,j)-\alpha I\ Int(A \cap B) = (i,j)-\alpha I\ Int(A) \cap (i,j)-\alpha I\ Int(B)\).
(vii) \((i,j)-\alpha I\ Int(A \cup B) \subset (i,j)-\alpha I\ Int(A) \cup (i,j)-\alpha I\ Int(B)\).

Proof. (i). Let \(x \in \bigcup\{T : T \subset A \text{ and } A \in (i,j)-\alpha I\ 0(X)\}\). Then, there exists \(T \in (i,j)-\alpha I\ 0(X,x)\) such that \(x \in T \subset A\) and hence \(x \in (i,j)-\alpha I\ Int(A)\). This shows that \(\bigcup\{T : T \subset A \text{ and } A \in (i,j)-\alpha I\ 0(X)\}\) \(\subset (i,j)-\alpha I\ Int(A)\). For the reverse inclusion, let \(x \in (i,j)-\alpha I\ Int(A)\). Then there exists \(T \in (i,j)-\alpha I\ 0(X,x)\) such that \(x \in T \subset A\). we obtain \(x \in \bigcup\{T : T \subset A \text{ and } A \in (i,j)-\alpha I\ 0(X)\}\). This shows that \((i,j)-\alpha I\ Int(A) \subset \bigcup\{T : T \subset A \text{ and } A \in (i,j)-\alpha I\ 0(X)\}\). Therefore, we obtain \((i,j)-\alpha I\ Int(A) = \bigcup\{T : T \subset A \text{ and } A \in (i,j)-\alpha I\ 0(X)\}\).

The proof of (ii) – (v) are obvious.
(vi). By (v), we have \((i,j)-\alpha I\ Int(A) \subset (i,j)-\alpha I\ Int(A \cup B)\) and \((i,j)-\alpha I\ Int(B) \subset (i,j)-\alpha I\ Int(A \cup B)\). Then we obtain \((i,j)-\alpha I\ Int(A) \cup (i,j)-\alpha I\ Int(B) \subset (i,j)-\alpha I\ Int(A \cup B)\) Since \((i,j)-\alpha I\ Int(A) \subset A\) and \((i,j)-\alpha I\ Int(B) \subset B\), we obtain \((i,j)-\alpha I\ Int(A \cup B) \subset (i,j)-\alpha I\ Int(A) \cup (i,j)-\alpha I\ Int(B)\) It follows that \((i,j)-\alpha I\ Int(A \cap B) = (i,j)-\alpha I\ Int(A) \cap (i,j)-\alpha I\ Int(B)\).
(vii). Since \(A \cap B \subset A\) and \(A \cap B \subset B\), by (v), we have \((i,j)-\alpha I\ Int(A \cap B) \subset (i,j)-\alpha I\ Int(A)\) and \((i,j)-\alpha I\ Int(A \cap B) \subset (i,j)-\alpha I\ Int(B)\). Therefore, \((i,j)-\alpha I\ Int(A) \cup (i,j)-\alpha I\ Int(B) \subset (i,j)-\alpha I\ Int(A \cap B)\).

Theorem 3.26. If \((X, \tau_1, \tau_2, I)\) is an ideal bitopological space satisfying the condition \((A)\), then \((i,j)-\alpha I\ Int(A) = A \cap \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A)))\) holds for every subset \(A\) of \(X\).

Proof. Since \(A \cap \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A))) \subset \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A))) = \tau_1-Int(\tau_i-Int(\tau_j-Cl^*(\tau_i-Int(A)))) = \tau_1-Int(\tau_i-Int(\tau_j-Cl^*(\tau_i-Int(A))) \cap (\tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A)))) = \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A) \cap \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A)))) = \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A) \cap \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A))))) = A \cap \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A)))\) is an \((i,j)-\alpha I\)-open set contained in \(A\) and so \(A \subset \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A))) \subset (i,j)-\alpha I\ Int(A)\). Since \((i,j)-\alpha I\ Int(A) \subset (i,j)-\alpha I\ Int(A) \subset \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A))) \subset \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A)))\) and so \((i,j)-\alpha I\ Int(A) \subset A \cap \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A)))\). Hence \((i,j)-\alpha I\ Int(A) = A \cap \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A)))\).

Definition 3.27. The union of all \((i,j)-\alpha I\)-open sets of \((X, \tau_1, \tau_2, I)\) containing \(A\) is called the \((i,j)-\alpha I\)-interior of \(A\) and is denote by \((i,j)-\alpha I\ Int(A)\).

Lemma 3.28. If \((X, \tau_1, \tau_2, I)\) is an ideal bitopological space satisfying the condition \((A)\), then \((i,j)-\alpha I\ Int(A) = A \cap \tau_1-Int(\tau_j-Cl^*(\tau_i-Int(A)))\) holds for every subset \(A\) of \(X\).

Theorem 3.29. If \((X, \tau_1, \tau_2, I)\) is an ideal bitopological space satisfying the condition \((A)\), then \((i,j)-\alpha I\ Int(A) = (i,j)-\alpha I\ Int(A)\) holds for every \((i,j)-\delta I\)-open subset \(A\) of \(X\).
Proof. Since every $(i, j)$-$\alpha I$-open set is $(i, j)$-$\delta$-$\alpha I$-open, $(i, j)$-$\alpha I$ $\operatorname{Int}(A) \subset (i, j)$-$p\alpha I$ $\operatorname{Int}(A)$. By Theorem 3.26, $\alpha I$ $\operatorname{Int}(A) = A \cap \tau_1$-$\operatorname{Int}(\tau_1$-$\operatorname{Cl}(\tau_1$-$\operatorname{Int}(A)))$. Since $A$ is $(i, j)$-$\delta$-$\alpha I$-open, $(i, j)$-$\alpha I$ $\operatorname{Int}(A) \supset A \cap \tau_1$-$\operatorname{Int}(\tau_1$-$\operatorname{Cl}(\tau_1$-$\operatorname{Int}(A))) = (i, j)$-$p\alpha I$ $\operatorname{Int}(A)$. Therefore, $(i, j)$-$\alpha I$ $\operatorname{Int}(A) = (i, j)$-$p\alpha I$ $\operatorname{Int}(A)$. □

**Definition 3.30.** Let $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space, $S$ a subset of $X$ and $x$ be a point of $X$. Then

(i) $x$ is called an $(i, j)$-$\alpha I$-cluster point of $S$ if $V \cap S \neq \emptyset$ for every $V \in (i, j)$-$\alpha I$ $\operatorname{Cl}(X, x)$.

(ii) the set of all $(i, j)$-$\alpha I$-cluster points of $S$ is called $(i, j)$-$\alpha I$-closure of $S$ and is denoted by $(i, j)$-$\alpha I$ $\operatorname{Cl}(S)$.

**Theorem 3.31.** Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2, I)$. Then the following properties hold:

(i) $(i, j)$-$\alpha I$ $\operatorname{Cl}(A) = \cap\{F : A \subset F$ and $F \in (i, j)$-$\alpha I$ $\operatorname{Cl}(X)\}$.

(ii) $(i, j)$-$\alpha I$ $\operatorname{Cl}(A)$ is the smallest $(i, j)$-$\alpha I$-closed subset of $X$ containing $A$.

(iii) $A$ is $(i, j)$-$\alpha I$-closed if and only if $A = (i, j)$-$\alpha I$ $\operatorname{Cl}(A)$.

(iv) $(i, j)$-$\alpha I$ $\operatorname{Cl}((i, j)$-$\alpha I$ $\operatorname{Cl}(A) = (i, j)$-$\alpha I$ $\operatorname{Cl}(A)$.

(v) If $A \subset B$, then $(i, j)$-$\alpha I$ $\operatorname{Cl}(A) \subset (i, j)$-$\alpha I$ $\operatorname{Cl}(B)$.

(vi) $(i, j)$-$\alpha I$ $\operatorname{Cl}(A \cup B) = (i, j)$-$\alpha I$ $\operatorname{Cl}(A) \cup (i, j)$-$\alpha I$ $\operatorname{Cl}(B)$.

(vii) $(i, j)$-$\alpha I$ $\operatorname{Cl}(A \cap B) \subset (i, j)$-$\alpha I$ $\operatorname{Cl}(A) \cap (i, j)$-$\alpha I$ $\operatorname{Cl}(B)$.

Proof. (i). Suppose that $x \notin (i, j)$-$\alpha I$ $\operatorname{Cl}(A)$. Then there exists $F \in (i, j)$-$\alpha I$ $\operatorname{O}(X) \setminus S \neq \emptyset$. Since $X \setminus V$ is $(i, j)$-$\alpha I$-closed set containing $A$ and $x \notin X \setminus V$, we obtain $x \notin \cap\{F : A \subset F$ and $F \in (i, j)$-$\alpha I$ $\operatorname{Cl}(X)\}$. Then there exists $F \in (i, j)$-$\alpha I$ $\operatorname{Cl}(X)$ such that $A \subset F$ and $x \notin F$. Since $X \setminus V$ is $(i, j)$-$\alpha I$-closed set containing $x$, we obtain $(X \setminus V)$-$\operatorname{Cl} = \emptyset$. This shows that $x \notin (i, j)$-$\alpha I$ $\operatorname{Cl}(A)$. Therefore, we obtain $(i, j)$-$\alpha I$ $\operatorname{Cl}(A) = \cap\{F : A \subset F$ and $F \in (i, j)$-$\alpha I$ $\operatorname{Cl}(X)\}$.

The other proofs are obvious. □

**Theorem 3.32.** Let $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space and $A \subset X$. A point $x \in (i, j)$-$\alpha I$ $\operatorname{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (i, j)$-$\alpha I$ $\operatorname{O}(X, x)$.

Proof. Suppose that $x \in (i, j)$-$\alpha I$ $\operatorname{Cl}(A)$. We shall show that $U \cap A \neq \emptyset$ for every $U \in (i, j)$-$\alpha I$ $\operatorname{O}(X, x)$. Suppose that there exists $U \in (i, j)$-$\alpha I$ $\operatorname{O}(X, x)$ such that $U \cap A = \emptyset$. Then $A \subset X \setminus U$ and $X \setminus U$ is $(i, j)$-$\alpha I$-closed. Since $A \subset X \setminus U$, $(i, j)$-$\alpha I$ $\operatorname{Cl}(A) \subset (i, j)$-$\alpha I$ $\operatorname{Cl}(X \setminus U)$. Since $x \in (i, j)$-$\alpha I$ $\operatorname{Cl}(A)$, we have $x \in (i, j)$-$\alpha I$ $\operatorname{Cl}(X \setminus U)$. Since $X \setminus U$ is $(i, j)$-$\alpha I$-closed, we have $x \in X \setminus U$; hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in (i, j)$-$\alpha I$ $\operatorname{O}(X, x)$. We shall show that
Suppose \( G \). Theorem 3.36. A subset of an ideal bitopological space

Proof. of its points.

If \( \text{Theorem 3.34.} \) the condition \( \alpha \) is said to be an \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open.

\( A \) holds for every subset \( X \). Therefore, we obtain \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open. Conversely, let \( x \in (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open set \( x \in (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open.

Theorem 3.33. Let \((X,\tau_1,\tau_2,\mathcal{I})\) be an ideal bitopological space and \( A \subset X \). Then the following properties hold:

(i) \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A) = \( X \setminus (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A);

(ii) \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Int(A).

Proof. Let \( x \in (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A). Since \( x \notin (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A), there exists \( V \in (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A,\( X,x \)) such that \( V \cap A \neq \emptyset \); hence we obtain \( x \in (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Int(X \setminus A). This shows that \( X \setminus (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A) \( \subset \) \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Int(X \setminus A). Let \( x \in (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Int(X \setminus A). Since \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Int(X \setminus A) \( \cap \) A = \( \emptyset \), we obtain \( x \notin (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A); hence \( x \in X \setminus (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A). Therefore, we obtain \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Int(X \setminus A) = \( X \setminus (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A).

\( \square \)

Theorem 3.34. If \((X,\tau_1,\tau_2,\mathcal{I})\) is an ideal bitopological space satisfies the condition \((A)\), then \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-Cl(A) = \( A \cup \tau_1\)-\( \text{Cl}(\tau_j\text{-Int}^*(\tau_1\text{-Cl}(A))) \) holds for every subset \( A \subset X \).

Proof. The proof follows from the definitions. \( \square \)

Definition 3.35. A subset \( B_x \) of an ideal bitopological space \((X,\tau_1,\tau_2,\mathcal{I})\) is said to be an \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-neighbourhood of a point \( x \in X \) if there exists an \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open \( U \) such that \( x \in U \subset B_x \).

Theorem 3.36. A subset of an ideal bitopological space \((X,\tau_1,\tau_2,\mathcal{I})\) is \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open if and only if it is an \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-neighbourhood of each of its points.

Proof. Let \( G \) be an \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open set \( x \in X \). Then by definition, it is clear that \( G \) is an \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-neighbourhood of each of its points, since for every \( x \in G, x \in G \cap G \) and \( G \) is \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open. Conversely, suppose \( G \) is an \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-neighbourhood of each of its points. Then for each \( x \in G \), there exists \( S_x \in (i,j)\)-\( \alpha \)-\( \mathcal{I} \)-\( \mathcal{O} \)(X) such that \( S_x \subset G \). Then \( G = \bigcup \{S_x : x \in G\} \). Since each \( S_x \) is \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open, \( G \) is \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open in \((X,\tau_1,\tau_2,\mathcal{I})\). \( \square \)

Proposition 3.37. The product of two \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open sets is \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open.

Proof. The proof follows from Lemma 3.3 of [12]. \( \square \)

4. Pairwise \( \alpha \)-\( \mathcal{I} \)-Continuous Functions

Definition 4.1. A function \( f : (X,\tau_1,\tau_2,\mathcal{I}) \rightarrow (Y,\sigma_1,\sigma_2) \) is said to be \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-continuous if the inverse image of every \( \sigma_i \)-open set of \( Y \) is \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-open in \( X \), where \( i \neq j \), \( i,j=1,2 \).

Proposition 4.2. (i) Every \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-continuous function is \((i,j)\)-\( \alpha \)-\( \mathcal{I} \)-semi-\( \mathcal{I} \)-continuous but not conversely.
(ii) Every \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-continuous function is \((i,j)\)-\(\alpha\)-continuous but not conversely.

(iii) Every \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-continuous function is \((i,j)\)-\(\text{pre}\)-\(\mathcal{I}\)-continuous but not conversely.

(iv) Every \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-continuous function is \((i,j)\)-\(b\)-\(\mathcal{I}\)-continuous but not conversely.

(v) Every \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-continuous function is \((i,j)\)-\(\beta\)-\(\mathcal{I}\)-continuous but not conversely.

**Proof.** The proof follows from Proposition 3.3 and Example 3.4. \(\square\)

**Theorem 4.3.** A function \(f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)\) is \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-continuous if and only if it is \((i,j)\)-semi-\(\mathcal{I}\)-continuous and \((i,j)\)-\(\text{pre}\)-\(\mathcal{I}\)-continuous.

**Proof.** This is an immediate consequence of Lemma 3.8. \(\square\)

**Theorem 4.4.** For a function \(f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)\), the following statements are equivalent:

(i) \(f\) is pairwise \(\alpha\)-\(\mathcal{I}\)-continuous;

(ii) For each point \(x\) in \(X\) and each \(\sigma_i\)-open set \(F\) in \(Y\) such that \(f(x) \in F\), there is a \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-open set \(A\) in \(X\) such that \(x \in A\), \(f(A) \subset F\);

(iii) The inverse image of each \(\sigma_i\)-closed set in \(Y\) is \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-closed in \(X\);

(iv) For each subset \(A\) of \(X\), \(f(((i,j)\)-\(\alpha\)-\(\mathcal{I}\)\(\text{Cl}(A)\)) \subset \sigma_i\)-\(\text{Cl}(f(A))\);

(v) For each subset \(B\) of \(Y\), \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)\(\text{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\)-\(\text{Cl}(B))\);

(vi) For each subset \(C\) of \(Y\), \(f^{-1}(\sigma_i\)-\(\text{Int}(C)\)) \(\subset (i,j)\)-\(\alpha\)-\(\mathcal{I}\)\(\text{Int}(f^{-1}(C))\).

(vii) \(\tau_i\)-\(\text{Cl}(\tau_j\)-\(\text{Int}^+(\tau_i\)-\(\text{Cl}(f^{-1}(B)))) \subset f^{-1}(\tau_i\)-\(\text{Cl}(B))\) for each subset \(B\) of \(Y\).

(viii) \(f((\tau_i\)-\(\text{Cl}(\tau_j\)-\(\text{Int}^+(\tau_i\)-\(\text{Cl}(A)))))) \subset \tau_i\)-\(\text{Cl}(f(A))\) for each subset \(A\) of \(X\).

**Proof.**

(i) \(\Rightarrow\) (ii): Let \(x \in X\) and \(F\) be a \(\sigma_j\)-open set of \(Y\) containing \(f(x)\). By (i), \(f^{-1}(F)\) is \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-open in \(X\). Let \(A = f^{-1}(F)\). Then \(x \in A\) and \(f(A) \subset F\).

(ii) \(\Rightarrow\) (i): Let \(F\) be \(\sigma_j\)-open in \(Y\) and let \(x \in f^{-1}(F)\). Then \(f(x) \in F\).

By (ii), there is an \((i,j)\)-\(\mathcal{I}\)-open set \(U_x\) in \(X\) such that \(x \in U_x\) and \(f(U_x) \subset F\). Then \(x \in U_x \subset f^{-1}(F)\). Hence \(f^{-1}(F)\) is \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-open in \(X\).

(i) \(\iff\) (iii): This follows due to the fact that for any subset \(B\) of \(Y\), \(f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)\).

(iii) \(\Rightarrow\) (iv): Let \(A\) be a subset of \(X\). Since \(A \subset f^{-1}(f(A))\) we have \(A \subset f^{-1}(\sigma_j\)-\(\text{Cl}(f(A)))\). Now, \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)\(\text{Cl}(f(A))\) is \(\sigma_j\)-closed in \(Y\) and hence \(f^{-1}(\sigma_j\)-\(\text{Cl}(f(A))) \subset f^{-1}(\sigma_j\)-\(\text{Cl}(f(A)))\), for \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)\(\text{Cl}(A)\) is the smallest \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-closed set containing \(A\). Then \(f((i,j)\)-\(\alpha\)-\(\mathcal{I}\)\(\text{Cl}(A)) \subset \sigma_j\)-\(\text{Cl}(f(A))\).
Then $X = \text{Int}(\text{Int}(\tau \text{-} \text{Cl}(f^{-1}(F)))) \subset (i,j)\text{-}\sigma_{\tau}\text{-Cl}(f(f^{-1}(F))) = (i,j)\text{-}\sigma_{\tau}\text{-Cl}(F) = F$. Therefore, $(i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-Cl}(f^{-1}(F)) \subset f^{-1}(F)$. Consequently, $f^{-1}(F)$ is $(i,j)\text{-}\alpha\text{-}\mathcal{I}$-closed in $X$.

$(iv) \Rightarrow (v)$: Let $B$ be any subset of $Y$. Now, $f((i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-Cl}(f^{-1}(B))) \subset (i,j)\text{-}\sigma_{\tau}\text{-Cl}(f(f^{-1}(B))) \subset \sigma_{\tau}\text{-Cl}(B)$. Consequently, $(i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_{\tau}\text{-Cl}(B))$.

$(v) \Rightarrow (iv)$: Let $B = f(A)$ where $A$ is a subset of $X$. Then $(i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-Cl}(A) \subset (i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_{:\tau}\text{-Cl}(B)) = f^{-1}(\sigma_{:\tau}\text{-Cl}(f(A)))$. This shows that $f((i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-Cl}(A)) \subset \sigma_{\tau}\text{-Cl}(f(A))$.

$(i) \Rightarrow (vi)$: Let $B$ be a $\sigma_{\tau}$-open set in $Y$. Clearly, $f^{-1}(\sigma_{\tau}\text{-} \text{Int}(B))$ is $(i,j)$-$\alpha\text{-}\mathcal{I}$-open and we have $f^{-1}(\sigma_{\tau}\text{-} \text{Int}(B)) \subset (i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-Int}(f^{-1}\sigma_{\tau}\text{-} \text{Int}(B)) \subset (i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is $(i,j)\text{-}\alpha\text{-}\mathcal{I}$-open in $X$.

$(iii) \Rightarrow (viii)$: Let $B$ be any subset os $Y$. Since $\tau_{\tau}\text{-Cl}(B)$ is $\tau_{\tau}$-closed in $Y$, by $(iii)$, $f^{-1}(\tau_{\tau}\text{-Cl}(B))$ is $\alpha\text{-}\mathcal{I}$-closed and $X\setminus f^{-1}(\tau_{\tau}\text{-Cl}(B))$ is $\alpha\text{-}\mathcal{I}$-open.

Then $X\setminus f^{-1}(\tau_{\tau}\text{-Cl}(B)) \subset \tau_{\tau}\text{-}\text{Int}(\tau_{\tau}\text{-Cl}((\tau_{\tau}\text{-Cl}(f^{-1}(\tau_{\tau}\text{-Cl}(B))))) = X\setminus \tau_{\tau}\text{-Cl}(\tau_{\tau}\text{-}\text{Int}^{\ast}((\tau_{\tau}\text{-Cl}(f^{-1}(\tau_{\tau}\text{-Cl}(B)))))$. Hence we obtain $\tau_{\tau}\text{-Cl}(\tau_{\tau}\text{-Cl}^\ast((\tau_{\tau}\text{-Cl}(f^{-1}(B))))) \subset f^{-1}(\tau_{\tau}\text{-Cl}(B))$.

$(vii) \Rightarrow (viii)$: Let $A$ be any subset of $X$. By $(iv)$, we have $\text{Cl}(\tau_{\tau}\text{-Cl}^\ast((\tau_{\tau}\text{-Cl}(A))) \subset \tau_{\tau}\text{-Cl}(\tau_{\tau}\text{-Cl}^\ast((\tau_{\tau}\text{-Cl}(f^{-1}(f(A))))) \subset f^{-1}(\tau_{\tau}\text{-Cl}(f(A)))$ and hence $f(\tau_{\tau}\text{-Cl}(\tau_{\tau}\text{-Cl}^\ast((\tau_{\tau}\text{-Cl}(A)))) \subset \tau_{\tau}\text{-Cl}(f(A))$.

$(viii) \Rightarrow (i)$: Let $V$ be any open set of $Y$. Then by $(v)$, $f(\tau_{\tau}\text{-Cl}(\tau_{\tau}\text{-Cl}^\ast((\tau_{\tau}\text{-Cl}(f^{-1}(f(V))))))) \subset \tau_{\tau}\text{-Cl}(f(f^{-1}(f(V)))) \subset \tau_{\tau}\text{-Cl}(f(V)) = \tau_{\tau}\text{-Cl}(f(V)) = f^{-1}(f(V)) \subset X\setminus f^{-1}(V)$. Consequently, we obtain $f^{-1}(V) \subset \tau_{\tau}\text{-Int}(\tau_{\tau}\text{-Cl}^\ast((\tau_{\tau}\text{-Cl}(f^{-1}(f(V)))))))$. This shows that $f^{-1}(V)$ is $\alpha\text{-}\mathcal{I}$-open. Thus, $f$ is $\alpha\text{-}\mathcal{I}$-continuous. □

**Corollary 4.5.** Let $f : (X, \tau_{\tau}, \tau_{\tau}, \mathcal{I}) \rightarrow (Y, \sigma_{\tau_{\tau}}, \sigma_{\tau_{\tau}}, \mathcal{I})$ be an $(i,j)$-$\alpha\text{-}\mathcal{I}$-continuous function, then

$(i)$ $f(\tau_{\tau}\text{-Cl}^\ast(U)) \subset \tau_{\tau}\text{-Cl}(f(U))$ for every $(i,j)$-$\text{-}\mathcal{I}$-pre-$\mathcal{I}$-open set $U$ of $X$.

$(ii)$ $\tau_{\tau}\text{-Cl}^\ast(f^{-1}(V)) \subset f^{-1}(\tau_{\tau}\text{-Cl}(V))$ for every $(i,j)$-$\text{-}\mathcal{I}$-pre-$\mathcal{I}$-open set $V$ of $Y$.

**Proof.** $(1)$. Let $U$ be any $(i,j)$-$\text{-}\mathcal{I}$-pre-$\mathcal{I}$-open set of $X$, then $U \subset \tau_{\tau}\text{-Int}(\tau_{\tau}\text{-Cl}^\ast(U))$. Therefore, by Theorem 4.4, we have $f(\tau_{\tau}\text{-Cl}^\ast(U)) \subset f(\tau_{\tau}\text{-Cl}(U)) \subset f(\tau_{\tau}\text{-Cl}(\tau_{\tau}\text{-Cl}^\ast(U)))) \subset f(\tau_{\tau}\text{-Cl}(\tau_{\tau}\text{-Cl}^\ast(U)))) \subset \tau_{\tau}\text{-Cl}(f(U))$.  


(2). Let $V$ be any $(i, j)$-pre-$\mathcal{L}$-open set of $Y$. By Theorem 4.4, $\tau_j - \text{Cl}^*(f^{-1}(V)) \subset \tau_j - \text{Cl}(f^{-1}(\tau_j - \text{Int}(\tau_j - \text{Cl}^*(V)))) \subset \tau_j - \text{Cl}(\tau_j - \text{Int}(\tau_j - \text{Cl}^*(V))) \subset \tau_j - \text{Cl}(\tau_j - \text{Cl}(f^{-1}(\tau_j - \text{Int}(\tau_j - \text{Cl}^*(V))))) \subset f^{-1}(\tau_j - \text{Cl}(\tau_j - \text{Cl}(f^{-1}(\tau_j - \text{Int}(\tau_j - \text{Cl}^*(V))))) \subset f^{-1}(\tau_j - \text{Cl}(f^{-1}(V))).$

\begin{flushright}$\square$
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\textbf{Theorem 4.6.} Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)$ be a pairwise $\alpha$-$\mathcal{L}$-continuous function. Then for each subset $V$ of $Y$, $f^{-1}(\sigma_1 - \text{Int}(V)) \subset \tau_j - \text{Cl}^*(f^{-1}(V)).$

\textit{Proof.} Let $V$ be any subset of $Y$. Then $\sigma_1 - \text{Int}(V)$ is $\sigma_1$-open in $Y$ and so $f^{-1}(\sigma_1 - \text{Int}(V))$ is $(i, j)$-$\alpha$-$\mathcal{L}$-open in $X$. Hence $f^{-1}(\sigma_1 - \text{Int}(V)) \subset \tau_i - \text{Int}(\tau_j - \text{Cl}^*(\tau_j - \text{Int}(f^{-1}(\sigma_1 - \text{Int}(V))))) \subset \tau_j - \text{Cl}^*(f^{-1}(V)).$

\begin{flushright}$\square$
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\textbf{Theorem 4.7.} Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)$ be a bijective. Then $f$ is pairwise $\alpha$-$\mathcal{L}$-continuous if and only if $\sigma_1 - \text{Int}(f(U)) \subset f((i, j)-\alpha \mathcal{I} \text{Int}(U))$ for each subset $U$ of $X$.

\textit{Proof.} Let $U$ be any subset of $X$. Then by Theorem 4.4, $f^{-1}(\sigma_1 - \text{Int}(f(U))) \subset (i, j)-\alpha \mathcal{I} \text{Int}(f^{-1}(f(U)))$. Since $f$ is bijection, $\sigma_1 - \text{Int}(f(U)) = f(f^{-1}(\sigma_1 - \text{Int}(f(U)))) \subset f((i, j)-\alpha \mathcal{I} \text{Int}(U))$. Conversely, let $V$ be any subset of $Y$. Then $\sigma_1 - \text{Int}(f(f^{-1}(V))) \subset f((i, j)-\alpha \mathcal{I} \text{Int}(f^{-1}(V)))$. Since $f$ is bijection, $\sigma_1 - \text{Int}(V) = \sigma_1 - \text{Int}(f(f^{-1}(V))) \subset f((i, j)-\alpha \mathcal{I} \text{Int}(f^{-1}(V)))$; hence $f^{-1}(\sigma_1 - \text{Int}(V)) \subset (i, j)-\alpha \mathcal{I} \text{Int}(f^{-1}(V))$. Therefore, by Theorem 4.4, $f$ is pairwise $\alpha$-$\mathcal{L}$-continuous.

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\textbf{Theorem 4.8.} Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function. If $g : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (X \times Y, \sigma_1 \times \sigma_2)$ defined by $g(x) = (x, f(x))$ is a pairwise $\alpha$-$\mathcal{L}$-continuous function, then $f$ is pairwise $\alpha$-$\mathcal{L}$-continuous.

\textit{Proof.} Let $V$ be a $\sigma_1$-open set of $Y$. Then $f^{-1}(V) = X \cap f^{-1}(V) = g^{-1}(X \times V)$. Since $g$ is a pairwise $\alpha$-$\mathcal{L}$-continuous function and $X \times V$ is a $\tau_1 \times \sigma_1$-open set of $X \times Y$, $f^{-1}(V)$ is a $(i, j)$-$\alpha$-$\mathcal{L}$-open set of $X$. Hence $f$ is pairwise $\alpha$-$\mathcal{L}$-continuous.

\begin{flushright}$\square$
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\textbf{Definition 4.9.} A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2, \mathcal{I})$ is said to be:

(i) pairwise $\alpha$-$\mathcal{L}$-open (resp. pairwise semi-$\mathcal{L}$-open [3], pairwise pre-$\mathcal{L}$-open [6]) if $f(U)$ is a $(i, j)$-$\alpha$-$\mathcal{L}$-open (resp. $(i, j)$-semi-$\mathcal{L}$-open, $(i, j)$-pre-$\mathcal{L}$-open) set of $Y$ for every $\tau_1$-open set $U$ of $X$.

(ii) pairwise $\alpha$-$\mathcal{L}$-closed (resp. pairwise semi-$\mathcal{L}$-closed [3], pairwise pre-$\mathcal{L}$-closed [6]) if $f(U)$ is a $(i, j)$-$\alpha$-$\mathcal{L}$-closed set of $Y$ for every $\tau_1$-closed set $U$ of $X$.

\textbf{Theorem 4.10.} A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2, \mathcal{I})$ is $(i, j)$-$\alpha$-$\mathcal{L}$-open if and only if it is $(i, j)$-semi-$\mathcal{L}$-open and $(i, j)$-pre-$\mathcal{L}$-open.

\textit{Proof.} This is an immediate consequence of Lemma 3.8. 

\begin{flushright}$\square$
\end{flushright}
Theorem 4.11. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$, the following statements are equivalent:

(i) $f$ is pairwise $\alpha\mathcal{I}$-open;
(ii) $f(\tau_1\text{-Int}(U)) \subset (i,j)\alpha\mathcal{I}\text{Int}(f(U))$ for each subset $U$ of $X$;
(iii) $\tau_1\text{-Int}(f^{-1}(V)) \subset f^{-1}((i,j)\alpha\mathcal{I}\text{Int}(V))$ for each subset $V$ of $Y$.

Proof. (i) $\Rightarrow$ (ii): Let $U$ be any subset of $X$. Then $\tau_1\text{-Int}(U)$ is a $\tau_1$-open set of $X$. Then $f(\tau_1\text{-Int}(U))$ is a $(i,j)\alpha\mathcal{I}$-open set of $Y$. Since $f(\tau_1\text{-Int}(U)) \subset f(U)$, $f(\tau_1\text{-Int}(U)) = (i,j)\alpha\mathcal{I}\text{Int}(f(\tau_1\text{-Int}(U))) \subset (i,j)\alpha\mathcal{I}\text{Int}(f(U))$.

(ii) $\Rightarrow$ (iii): Let $V$ be any subset of $Y$. Then $f^{-1}(V)$ is a subset of $X$. Hence $f(\tau_1\text{-Int}(f^{-1}(V))) \subset (i,j)\alpha\mathcal{I}\text{Int}(f(f^{-1}(V))) \subset (i,j)\alpha\mathcal{I}\text{Int}(V)$. Then $\tau_1\text{-Int}(f^{-1}(V)) \subset f^{-1}(f(\tau_1\text{-Int}(f^{-1}(V)))) \subset f^{-1}((i,j)\alpha\mathcal{I}\text{Int}(V))$.

(iii) $\Rightarrow$ (i): Let $U$ be any $\tau_1$-open set of $X$. Then $\tau_1\text{-Int}(U) = U$ and $f(U)$ is a subset of $Y$. Now, $V = \tau_1\text{-Int}(V) \subset \tau_1\text{-Int}(f^{-1}(f(V))) \subset f^{-1}((i,j)\alpha\mathcal{I}\text{Int}(f(V)))$. Then $f(V) \subset f(f^{-1}((i,j)\alpha\mathcal{I}\text{Int}(f(V)))) \subset (i,j)\alpha\mathcal{I}\text{Int}(f(V))$ and $(i,j)\alpha\mathcal{I}\text{Int}(f(V)) \subset (i,j)\alpha\mathcal{I}\text{Int}(f(V))$. Hence $f(V)$ is a $(i,j)\alpha\mathcal{I}$-open set of $Y$; hence $f$ is pairwise $\alpha\mathcal{I}$-open. □

Theorem 4.12. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a function. Then $f$ is a pairwise $\alpha\mathcal{I}$-closed function if and only if for each subset $V$ of $X$, $(i,j)\alpha\mathcal{I}\text{Cl}(f(V)) \subset f(\tau_1\text{-Cl}(V))$.

Proof. Let $f$ be a pairwise $\alpha\mathcal{I}$-closed function and $V$ any subset of $X$. Then $f(V) \subset f(\tau_1\text{-Cl}(V))$ and $f(\tau_1\text{-Cl}(V))$ is a $(i,j)\alpha\mathcal{I}$-closed set of $Y$. We have $(i,j)\alpha\mathcal{I}\text{Cl}(f(V)) \subset (i,j)\alpha\mathcal{I}\text{Cl}(f(\tau_1\text{-Cl}(V))) = f(\tau_1\text{-Cl}(V))$. Conversely, let $V$ be a $\tau_1$-open set of $X$. Then $f(V) \subset (i,j)\alpha\mathcal{I}\text{Cl}(f(V)) \subset f(\tau_1\text{-Cl}(V)) = f(V)$; hence $f(V)$ is a $(i,j)\alpha\mathcal{I}$-closed subset of $Y$. Therefore, $f$ is a pairwise $\alpha\mathcal{I}$-closed function. □

Theorem 4.13. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a function. Then $f$ is a pairwise $\alpha\mathcal{I}$-closed function if and only if for each subset $V$ of $Y$, $f^{-1}((i,j)\alpha\mathcal{I}\text{Cl}(f^{-1}(V))) \subset \tau_1\text{-Cl}(f^{-1}(V))$.

Proof. Let $V$ be any subset of $Y$. Then by Theorem 4.12, $(i,j)\alpha\mathcal{I}\text{Cl}(V) \subset f(\tau_1\text{-Cl}(f^{-1}(V)))$. Since $f$ is bijection, $f^{-1}((i,j)\alpha\mathcal{I}\text{Cl}(V)) = f^{-1}(f^{-1}((i,j)\alpha\mathcal{I}\text{Cl}(f^{-1}(V)))) \subset f^{-1}(f(\tau_1\text{-Cl}(f^{-1}(V)))) = \tau_1\text{-Cl}(f^{-1}(V))$. Conversely, let $U$ be any subset of $X$. Since $f$ is bijection, $((i,j)\alpha\mathcal{I}\text{Cl}(f(U)) = f(f^{-1}((i,j)\alpha\mathcal{I}\text{Cl}(f(U)))) \subset f(\tau_1\text{-Cl}(f^{-1}(f(U)))) = f(\tau_1\text{-Cl}(U))$. Therefore, by Theorem 4.12, $f$ is a pairwise $\alpha\mathcal{I}$-closed function. □

Theorem 4.14. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a pairwise $\alpha\mathcal{I}$-open function. If $V$ is a subset of $Y$ and $U$ is a $\tau_1$-closed subset of $X$ containing $f^{-1}(V)$, then there exists a $(i,j)\alpha\mathcal{I}$-closed set $F$ of $Y$ containing $V$ such that $f^{-1}(F) \subset U$. 
Proof. Let $V$ be any subset of $Y$ and $U$ a $\tau_i$-closed subset of $X$ containing $f^{-1}(V)$, and let $F = Y \setminus (f(X \setminus V))$. Then $f(X \setminus V) \subseteq f(f^{-1}(X \setminus V)) \subseteq X \setminus V$ and $X \setminus V$ is a $\tau_i$-open set of $X$. Since $f$ is pairwise $\alpha$-$\mathcal{I}$-open, $f(X \setminus U)$ is a $(i,j)$-$\alpha$-$\mathcal{I}$-open set of $Y$. Hence $F$ is an $(i,j)$-$\alpha$-$\mathcal{I}$-closed set of $Y$ and $f^{-1}(F) = f^{-1}(Y \setminus (f(X \setminus U))) \subseteq U$.

**Theorem 4.15.** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a pairwise $\alpha$-$\mathcal{I}$-closed function. If $V$ is a subset of $Y$ and $U$ is a open subset of $X$ containing $f^{-1}(V)$, then there exists $(i,j)$-$\alpha$-$\mathcal{I}$-open set $F$ of $Y$ containing $V$ such that $f^{-1}(F) \subseteq U$.

Proof. The proof is similar to the Theorem 4.14.

**Theorem 4.16.** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a pairwise $\alpha$-$\mathcal{I}$-open function. Then for each subset $V$ of $Y$, $f^{-1}(\tau_i\text{-Cl}(\tau_j\text{-Int}^*)(\tau_i\text{-Cl}(V))) \subseteq \tau_i\text{-Cl}(f^{-1}(V))$.

Proof. Let $V$ be any subset of $Y$. Then $\tau_i\text{-Cl}(f^{-1}(V))$ is a $\tau_i$-closed set of $X$. Then by Theorem 4.14, there exists an $(i,j)$-$\alpha$-$\mathcal{I}$-closed set $F$ of $Y$ containing $V$ such that $f^{-1}(F) \subseteq \tau_i\text{-Cl}(f^{-1}(V))$. Since $Y \setminus F$ is $(i,j)$-$\alpha$-$\mathcal{I}$-open, $f^{-1}(Y \setminus F) \subseteq f^{-1}(\tau_j\text{-Int}(\tau_i\text{-Cl}(Y \setminus F)))$ and $X \setminus f^{-1}(F) \subseteq X \setminus f^{-1}(\tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}(Y \setminus F)))) = X \setminus f^{-1}(\tau_i\text{-Cl}(\tau_i\text{-Cl}(f(F))))$. Thus we obtain that $f^{-1}(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(V)))) \subseteq f^{-1}(F) \subseteq \tau_i\text{-Cl}(f^{-1}(V))$. Therefore, we have $f^{-1}(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(V)))) \subseteq \tau_i\text{-Cl}(f^{-1}(V))$.

**Definition 4.17.** A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ is said to be:

(i) pairwise $\alpha$-($\mathcal{I}, \mathcal{J}$)-open if $f(U)$ is a $(i,j)$-$\alpha$-$\mathcal{J}$-open set of $Y$ for every $(i,j)$-$\alpha$-$\mathcal{I}$-open set $U$ of $X$.
(ii) pairwise $\alpha$-($\mathcal{I}, \mathcal{J}$)-closed if $f(U)$ is a $(i,j)$-$\alpha$-$\mathcal{J}$-closed set of $Y$ for every $(i,j)$-$\alpha$-$\mathcal{I}$-closed set $U$ of $X$.

It is clear that every pairwise $\alpha$-($\mathcal{I}, \mathcal{J}$)-open (resp. pairwise $\alpha$-($\mathcal{I}, \mathcal{J}$)-closed) function is pairwise $\alpha$-$\mathcal{J}$-open (resp. pairwise $\alpha$-$\mathcal{J}$-closed) function. But the converse is not true in general.

**Example 4.18.** Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\alpha$-$\mathcal{I}$-open but not pairwise $\alpha$-($\mathcal{I}, \mathcal{I}$)-open.

**Theorem 4.19.** For a function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$, the following statements are equivalent:

(i) $f$ is pairwise $\alpha$-($\mathcal{I}, \mathcal{J}$)-open;
(ii) $f((i,j)\alpha\mathcal{I}\text{Int}(U)) \subseteq (i,j)\alpha\mathcal{J}\text{Int}(f(U))$ for each subset $U$ of $X$;
(iii) $(i,j)\alpha\mathcal{I}\text{Int}(f^{-1}(V)) \subseteq f^{-1}((i,j)\alpha\mathcal{J}\text{Int}(V))$ for each subset $V$ of $Y$.
Proof. (i) $\Rightarrow$ (ii): Let $U$ be any subset of $X$. Then $(i, j)-\alpha I \text{Int}(U)$ is a $(i, j)-\alpha I$-open set of $X$. Then $f((i, j)-\alpha I \text{Int}(U))$ is a $(i, j)-\alpha I$-open set of $Y$. Since $f((i, j)-\alpha I \text{Int}(U)) \subset f(U)$, $f((i, j)-\alpha I \text{Int}(U)) = (i, j)-\alpha I \text{Int}(f((i, j)-\alpha I \text{Int}(U))) \subset (i, j)-\alpha J \text{Int}(f(U))$.

(ii) $\Rightarrow$ (iii): Let $V$ be any subset of $Y$. Then $f^{-1}(V)$ is a subset of $X$. Hence $f((i, j)-\alpha I \text{Int}(f^{-1}(V))) \subset (i, j)-\alpha J \text{Int}(f(f^{-1}(V))) \subset (i, j)-\alpha I \text{Int}(f^{-1}(V)))$. Then $(i, j)-\alpha I \text{Int}(f^{-1}(V)) \subset f^{-1}(f((i, j)-\alpha I \text{Int}(f^{-1}(V)))) \subset f^{-1}((i, j)-\alpha I \text{Int}(V))$.

(iii) $\Rightarrow$ (i): Let $U$ be any $(i, j)-\alpha I$-open set of $X$. Then $(i, j)-\alpha I \text{Int}(U) = U$ and $f(U)$ is a subset of $Y$. Now, $U = (i, j)-\alpha I \text{Int}(U) \subset (i, j)-\alpha I \text{Int}(f^{-1}(f(U))) \subset f^{-1}((i, j)-\alpha J \text{Int}(f(U)))$. Then $f(U) \subset f(f^{-1}((i, j)-\alpha J \text{Int}(f(U)))) \subset (i, j)-\alpha J \text{Int}(f(U))$ and $(i, j)-\alpha J \text{Int}(f(U)) \subset f(U)$. Hence $f(U)$ is a $(i, j)-\alpha J$-closed set of $Y$; hence $f$ is a pairwise $\alpha-(I, J)$-open.

Theorem 4.20. Let $f : (X, \tau_{1}, \tau_{2}, I) \to (Y, \sigma_{1}, \sigma_{2}, J)$ be a function. Then $f$ is a pairwise $\alpha-(I, J)$-closed function if and only if for each subset $U$ of $X$, $(i, j)-\alpha J \text{Cl}(f(U)) \subset f((i, j)-\alpha I \text{Cl}(U))$.

Proof. Let $f$ be a pairwise $\alpha-(I, J)$-closed function and $U$ any subset of $X$. Then $f(U) \subset f((i, j)-\alpha I \text{Cl}(U))$ and $f((i, j)-\alpha I \text{Cl}(U))$ is a $(i, j)-\alpha J$-closed set of $Y$. We have $(i, j)-\alpha J \text{Cl}(f(U)) \subset (i, j)-\alpha J \text{Cl}(f((i, j)-\alpha I \text{Cl}(U))) = f((i, j)-\alpha I \text{Cl}(U))$. Conversely, let $U$ be a $(i, j)-\alpha I$-open set of $X$. Then $f(U) \subset (i, j)-\alpha J \text{Cl}(f(U)) \subset f((i, j)-\alpha I \text{Cl}(U)) = f(U)$; hence $f(U)$ is a pairwise $\alpha-(I, J)$-closed function.

Theorem 4.21. Let $f : (X, \tau_{1}, \tau_{2}, I) \to (Y, \sigma_{1}, \sigma_{2}, J)$ be a function. Then $f$ is a pairwise $\alpha-(I, J)$-closed function if and only if for each subset $V$ of $Y$, $f^{-1}((i, j)-\alpha J \text{Cl}(f(V))) \subset (i, j)-\alpha I \text{Cl}(f^{-1}(V))$.

Proof. Let $V$ be any subset of $Y$. Then by Theorem 4.20, $(i, j)-\alpha J \text{Cl}(f(f^{-1}(V))) \subset f((i, j)-\alpha I \text{Cl}(f^{-1}(V)))$. Since $f$ is bijection, $f^{-1}((i, j)-\alpha J \text{Cl}(V)) \subset (i, j)-\alpha I \text{Cl}(f^{-1}(V))$. Conversely, let $U$ be any subset of $X$. Then $f^{-1}((i, j)-\alpha J \text{Cl}(f(U))) \subset (i, j)-\alpha I \text{Cl}(f^{-1}(f(U)))$. Hence $(i, j)-\alpha J \text{Cl}(f(U)) \subset (i, j)-\alpha I \text{Cl}(f^{-1}(f(U)))$. Therefore, by Theorem 4.20 $f$ is a pairwise $\alpha-(I, J)$-closed function.

Theorem 4.22. Let $f : (X, \tau_{1}, \tau_{2}, I) \to (Y, \sigma_{1}, \sigma_{2}, J)$ be a pairwise $\alpha-(I, J)$-open function. If $V$ is a subset of $Y$ and $U$ is a $(i, j)-\alpha I$-closed subset of $X$ containing $f^{-1}(V)$, then there exists $(i, j)-\alpha I$-closed set $F$ of $Y$ containing $V$ such that $f^{-1}(F) \subset V$.

Proof. The proof is similar to the Theorem 4.14.

Theorem 4.23. Let $f : (X, \tau_{1}, \tau_{2}, I) \to (Y, \sigma_{1}, \sigma_{2}, J)$ be a pairwise $\alpha-(I, J)$-closed function. If $V$ is a subset of $Y$ and $U$ is a $(i, j)-\alpha I$-open subset of $X$ containing $f^{-1}(V)$, then there exists $(i, j)-\alpha J$-open set $F$ of $Y$ containing $V$ such that $f^{-1}(F) \subset V$. 
Proof. The proof is similar to the Theorem 4.14.

**Theorem 4.24.** For a bijective function \( f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2, \mathcal{J}) \), the following statements are equivalent:

(i) \( f \) is pairwise \( \alpha-(\mathcal{I}, \mathcal{J}) \)-closed;

(ii) \( f \) is pairwise \( \alpha-(\mathcal{I}, \mathcal{J}) \)-open.

**Proof.** The proof is clear.

## 5. PAIRWISE \( \alpha-\mathcal{I} \)-IRRRESOLUTE FUNCTIONS

**Definition 5.1.** A function \( f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2, \mathcal{J}) \) is said to be \((i, j)\)-\( \alpha-\mathcal{I} \)-irresolute if the inverse image of every \((i, j)\)-\( \alpha-\mathcal{J} \)-open set of \( Y \) is \((i, j)\)-\( \alpha-\mathcal{I} \)-open in \( X \), where \( i \neq j, i, j = 1, 2 \).

**Proposition 5.2.** Every pairwise \( \alpha-\mathcal{I} \)-irresolute function is pairwise \( \alpha-\mathcal{I} \)-continuous but not conversely.

**Proof.** Straightforward.

**Theorem 5.3.** Let \( f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2, \mathcal{J}) \) be a function, then

1. \( f \) is pairwise \( \alpha-\mathcal{I} \)-irresolute;
2. the inverse image of each \((i, j)\)-\( \alpha-\mathcal{J} \)-closed subset of \( Y \) is \((i, j)\)-\( \alpha-\mathcal{I} \)-closed in \( X \);
3. for each \( x \in X \) and each \( V \in S\mathcal{J}O(Y) \) containing \( f(x) \), there exists \( U \in \alpha\mathcal{I}O(X) \) containing \( x \) such that \( f(U) \subset V \).

**Proof.** The proof is obvious from that fact that the arbitrary union of \((i, j)\)-\( \alpha-\mathcal{I} \)-open subsets is \((i, j)\)-\( \alpha-\mathcal{I} \)-open.

**Theorem 5.4.** Let \( f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2, \mathcal{J}) \) be a function, then

(i) \( f \) is pairwise \( \alpha-\mathcal{I} \)-irresolute;
(ii) \((i, j)\)-\( \alpha\mathcal{I} Cl(f^{-1}(V)) \subset f^{-1}((i, j)\)-\( \alpha\mathcal{J} Cl(V)) \) for each subset \( V \) of \( Y \);
(iii) \( f((i, j)\)-\( \alpha\mathcal{I} Cl(U) \subset (i, j)\)-\( \alpha\mathcal{J} Cl(f(U)) \) for each subset \( U \) of \( X \).

**Proof.** (i) \( \Rightarrow \) (ii): Let \( V \) be any subset of \( Y \). Then \( V \subset (i, j)-\alpha\mathcal{J} Cl(V) \) and \( f^{-1}(V) \subset f^{-1}((i, j)-\alpha\mathcal{J} Cl(V)) \). Since \( f \) is pairwise \( \alpha-\mathcal{I} \)-irresolute, \( f^{-1}((i, j)-\alpha\mathcal{J} Cl(V)) \) is a \((i, j)\)-\( \alpha-\mathcal{I} \)-closed subset of \( X \). Hence \((i, j)-\alpha\mathcal{I} Cl(f^{-1}(V)) \subset (i, j)-\alpha\mathcal{I} Cl(f^{-1}((i, j)-\alpha\mathcal{J} Cl(V))) = f^{-1}((i, j)-\alpha\mathcal{J} Cl(V)) \).

(ii) \( \Rightarrow \) (iii): Let \( U \) be any subset of \( X \). Then \( f(U) \subset (i, j)-\alpha\mathcal{J} Cl(f(U)) \) and \((i, j)-\alpha\mathcal{J} Cl(U) \subset (i, j)-\alpha\mathcal{I} Cl(f^{-1}(f(U))) \subset f^{-1}((i, j)-\alpha\mathcal{J} Cl(f(U))) \). This implies that \( f((i, j)-\alpha\mathcal{I} Cl(U)) \subset f(f^{-1}((i, j)-\alpha\mathcal{J} Cl(f(U)))) \subset (i, j)-\alpha\mathcal{J} Cl(f(U)) \).

(iii) \( \Rightarrow \) (i): Let \( V \) be a \((i, j)\)-\( \alpha-\mathcal{J} \)-closed subset of \( Y \). Then \( f((i, j)-\alpha\mathcal{I} Cl(f^{-1}(V)) \subset (i, j)-\alpha\mathcal{I} Cl(f^{-1}(f(V))) \subset (i, j)-\alpha\mathcal{I} Cl(V) = V \). This implies that \((i, j)-\alpha\mathcal{I} Cl(f^{-1}(V)) \subset f^{-1}(f((i, j)-\alpha\mathcal{I} Cl(f^{-1}(V)))) \subset (i, j)-\alpha\mathcal{I} Cl(V) \).
\( f^{-1}(V) \). Therefore, \( f^{-1}(V) \) is a \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-closed subset of \(X\) and consequently \(f\) is a pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute function. \(\square\)

**Theorem 5.5.** A function \(f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})\) is a pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute if and only if \(f^{-1}((i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Int}(V)) \subset (i,j)\)-\(\alpha\)\(\mathcal{I}\) \(\text{Int}(f^{-1}(V))\) for each subset \(V\) of \(Y\).

**Proof.** Let \(V\) be any subset of \(Y\). Then \((i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Int}(V) \subset V\). Since \(f\) is pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute, \(f^{-1}((i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Int}(V))\) is a \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-open subset of \(X\). Hence \(f^{-1}((i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Int}(V)) = (i,j)\)-\(\alpha\)\(\mathcal{I}\) \(\text{Int}(f^{-1}((i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Int}(V)) \subset (i,j)\)-\(\alpha\)\(\mathcal{I}\) \(\text{Int}(f^{-1}(V))\). Conversely, let \(V\) be a \((i,j)\)-\(\alpha\)-\(\mathcal{J}\)-open subset of \(Y\). Then \(f^{-1}(V) = f^{-1}((i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Int}(V)) \subset (i,j)\)-\(\alpha\)\(\mathcal{I}\) \(\text{Int}(f^{-1}(V))\). Therefore, \(f^{-1}(V)\) is a \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-open subset of \(X\) and consequently \(f\) is a pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute function. \(\square\)

**Corollary 5.6.** Let \(f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})\) be a function. Then \(f\) is pairwise \(\alpha\)-\(\mathcal{I}\)-closed and pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute if and only if \(f((i,j)\)-\(\alpha\)\(\mathcal{I}\) \(\text{Cl}(V)) = (i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Cl}(f(V))\) for every subset \(V\) of \(X\).

**Definition 5.7.** An ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\) is called pairwise \(\alpha\)-\(\mathcal{I}\)-Hausdorff if for each two distinct points \(x \neq y\), there exist disjoint \((i,j)\)-\(\alpha\)-\(\mathcal{I}\)-open sets \(U\) and \(V\) containing \(x\) and \(y\), respectively.

**Theorem 5.8.** Let \(f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})\) be a pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute function. If \(Y\) is pairwise \(\alpha\)-\(\mathcal{J}\)-Hausdorff, then \(X\) is pairwise \(\alpha\)-\(\mathcal{I}\)-Hausdorff.

**Proof.** The proof is clear. \(\square\)

**Corollary 5.9.** Let \(f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})\) be a function. Then \(f\) is pairwise \(\alpha\)-\(\mathcal{I}\)-open and pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute if and only if \(f^{-1}((i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Cl}(V)) = (i,j)\)-\(\alpha\)\(\mathcal{I}\) \(\text{Cl}(f^{-1}(V))\) for every subset \(V\) of \(Y\).

**Definition 5.10.** A function \(f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})\) is said to be pairwise \(\alpha\)-\(\mathcal{I}\)-homeomorphism if \(f\) and \(f^{-1}\) are pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute.

**Theorem 5.11.** Let \(f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})\) be a bijection. Then the following statements are equivalent:

(i) \(f\) is pairwise \(\alpha\)-\(\mathcal{I}\)-homeomorphism;

(ii) \(f^{-1}\) is pairwise \(\alpha\)-\(\mathcal{I}\)-homeomorphism;

(iii) \(f\) and \(f^{-1}\) are pairwise \(\alpha\)-\(\mathcal{J}\)-open (pairwise \(\alpha\)-\(\mathcal{J}\)-closed);

(1) \(f\) is pairwise \(\alpha\)-\(\mathcal{I}\)-irresolute and pairwise \(\alpha\)-\(\mathcal{J}\)-open (pairwise \(\alpha\)-\(\mathcal{J}\)-closed);

(2) \(f((i,j)\)-\(\alpha\)\(\mathcal{I}\) \(\text{Cl}(V)) = (i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Cl}(f(V))\) for each subset \(V\) of \(X\);

(3) \(f((i,j)\)-\(\alpha\)\(\mathcal{I}\) \(\text{Int}(V)) = (i,j)\)-\(\alpha\)\(\mathcal{J}\) \(\text{Int}(f(V))\) for each subset \(V\) of \(X\);
\[ f^{-1}((i, j) - \alpha J \text{Int}(V)) = (i, j) - \alpha I \text{Int}(f^{-1}(V)) \text{ for each subset } V \text{ of } Y; \]
\[ (i, j) - \alpha I \text{Cl}(f^{-1}(V)) = f^{-1}((i, j) - \alpha J \text{Cl}(V)) \text{ for each subset } V \text{ of } Y; \]

**Proof.** (1) \(\Rightarrow\) (2): It follows immediately from the definition of a pairwise \(\alpha I\)-homeomorphism.

(2) \(\Rightarrow\) (3) \(\Rightarrow\) (4): It follows from Theorem 4.24.

(4) \(\Rightarrow\) (5): It follows from Theorem 4.21 and Corollary 5.6.

(5) \(\Rightarrow\) (6): Let \(U\) be a subset of \(X\). Then by Theorem 3.33, \(f((i, j) - \alpha I \text{Int}(U)) = X \setminus f((i, j) - \alpha I \text{Cl}(X \setminus U)) = X \setminus ((i, j) - \alpha I \text{Cl}(f(X \setminus U)) = (i, j) - \alpha I \text{Int}(f(U)). \]

(6) \(\Rightarrow\) (7): Let \(V\) be a subset of \(Y\). Then \(f(((i, j) - \alpha I \text{Int}(f^{-1}(V)))) = (i, j) - \alpha I \text{Int}(f^{-1}(f(V))) = (i, j) - \alpha I \text{Int}(f(V)). \) Hence \(f^{-1}(f(((i, j) - \alpha I \text{Int}(f^{-1}(V))))) = f^{-1}((i, j) - \alpha I \text{Int}(V)). \) Therefore, \(f^{-1}((i, j) - \alpha J \text{Int}(V)) = (i, j) - \alpha I \text{Int}(f^{-1}(V)). \)

(7) \(\Rightarrow\) (8): Let \(V\) be a subset of \(Y\). Then by Theorem 3.33, \((i, j) - \alpha I \text{Cl}(f^{-1}(V)) = X \setminus (f^{-1}(((i, j) - \alpha J \text{Int}(Y \setminus V)) = X \setminus ((i, j) - \alpha I \text{Int}(f^{-1}((X \setminus V)))) = f^{-1}((i, j) - \alpha I \text{Cl}(V)). \]

(8) \(\Rightarrow\) (1): It follows from Theorem 4.21 and Corollary 5.9. \(\square\)

**References**

PROPERTIES OF IDEAL BITOPOLOGICAL $\alpha$-OPEN SETS

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