**Theorem 1** If $V$ is finite-dimensional vector space over a field $F$ and $T$ is a homomorphism of $V$ onto $V$, prove that $T$ must be one-to-one, and so an isomorphism.

**Proof.**

Let $n = \dim V$ and $v_1, \ldots, v_n$ be a basis of $V$. Since $T$ is onto, $v_i = w_i T$ for some $w_i \in V$ and $i = 1, \ldots, n$. To show the linear independence of the $w_i$, consider $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ with $\alpha_1, \ldots, \alpha_n$ in $F$. It follows that

$$\begin{align*}
\alpha_1 v_1 + \cdots + \alpha_n v_n &= \alpha_1 (w_1 T) + \cdots + \alpha_n (w_n T) \\
&= (\alpha_1 w_1) T + \cdots + (\alpha_n w_n) T \\
&= (\alpha_1 w_1 + \cdots + \alpha_n w_n) T \\
&= 0 T \\
&= 0
\end{align*}$$

and hence by the linear independence of $v_1, \ldots, v_n$ forces $\alpha_i = 0$ for $i = 1, \ldots, n$. Since $V$ is of dimension $n$, any set of $n$ linearly independent vectors in $V$ forms a basis of $V$. Therefore $w_1, \ldots, w_n$ is a basis of $V$. Now suppose $vT = 0$ for some $v \in V$. Thus $v = \lambda_1 w_1 + \cdots + \lambda_n w_n$ with $\lambda_1, \ldots, \lambda_n$ in $F$. Moreover

$$\begin{align*}
\lambda_1 v_1 + \cdots + \lambda_n v_n &= \lambda_1 (w_1 T) + \cdots + \lambda_n (w_n T) \\
&= (\lambda_1 w_1) T + \cdots + (\lambda_n w_n) T \\
&= (\lambda_1 w_1 + \cdots + \lambda_n w_n) T \\
&= v T \\
&= 0
\end{align*}$$

and hence by the linear independence of $v_1, \ldots, v_n$ forces $\lambda_i = 0$ for $i = 1, \ldots, n$. So $v = 0$. Since its kernel is $(0)$, $T$ is an isomorphism.

**References**