Tutorial: Continuous-Function on Closed Interval Basics, with Mean-Value and Taylor Theorem Upshots

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Abstract

This tutorial explores the relation of the local concept of a function's continuity to its global consequences on closed intervals, such as a continuous function's unavoidable boundedness on a closed interval, its attainment of its least upper and greatest lower bounds on that interval, and its unavoidable assumption on that closed interval of all of the values which lie between that minimum and maximum. In a nutshell, continuous functions map closed intervals into closed intervals. It is understandable that verifying this astonishing fact involves subtle and very intricate manipulation of the least-upper-bound/greatest-lower-bound postulate for the real numbers. In conjunction with the basic inequality properties of integrals, this continuous-function fact immediately implies the integral form of the mean-value theorem, which is parlayed into its differential form by the fundamental theorem of the calculus. Taylor expansion and its error estimation are additional fascinating developments intertwined with these lines of thinking.

The basic properties of a continuous function on a closed interval

The crucial basic properties of a continuous real function defined on a closed interval are that it *is bounded*, *attains* its least upper and greatest lower bounds, *and also takes on all values between that maximum and minimum*. To *establish these properties* involves *intricate manipulation* of the least-upper-bound postulate.

We begin with a real function $f(x)$ defined as a real number at every point of the positive-length closed interval $[a, b] = \{x|a \leq x \leq b,\ a < b\}$, *but we assume that* $f(x)$ *is unbounded above on* $[a, b]$. We then show that *this assumption implies the existence of a point* $c \in [a, b]$ *where* $f(x)$ *cannot be continuous*. To carry out that demonstration, we select an infinite set of *distinct* points $\{x_0, x_1, x_2, \ldots\}$ of $[a, b]$ such that $f(x_0) \geq 1$, $f(x_1) \geq 2f(x_0)$,..., $f(x_n)) \geq 2f(x_{n-1})$, $n = 1, 2, \ldots$. We next consider the following sequence of nested subsets $\{x_0, x_1, x_2, \ldots\} \supset \{x_1, x_2, x_3, \ldots\} \supset \{x_2, x_3, x_4, \ldots\} \supset \cdots$ of $[a, b]$, together with *the decreasing sequence* of the least upper bounds of each of these nested subsets, $r_0 \stackrel{\text{def}}{=} \sup\{x_0, x_1, x_2, \ldots\} \geq r_1 \stackrel{\text{def}}{=} \sup\{x_1, x_2, x_3, \ldots\} \geq r_2 \stackrel{\text{def}}{=} \sup\{x_2, x_3, x_4, \ldots\} \geq \cdots$, and also the greatest lower bound (in fact limit) $c = \inf\{r_0, r_1, r_2, \ldots\}$. Since $\{x_0, x_1, x_2, \ldots\} \subset [a, b]$, and $[a, b]$ is a *closed* interval, $r_0 \stackrel{\text{def}}{=} \sup\{x_0, x_1, x_2, \ldots\}$ is an element of $[a, b]$. Likewise r_1, r_2, ... are all elements of $[a, b]$, so the point $c = \inf\{r_0, r_1, r_2, \ldots\}$ is as well an element of the *closed* interval $[a, b]$. We note that *since* x_0, x_1, x_2, ... *are chosen so that no two of them can be equal,* $r_n > a$ *for all* $n = 0, 1, 2, \ldots$, where a is the left endpoint of $[a, b]$.

The *decreasing sequence* of least upper bounds $r_0 = \sup\{x_0, x_1, x_2, \ldots\} \geq r_1 = \sup\{x_1, x_2, x_3, \ldots\} \geq \ldots$ can exhibit *two distinct behaviors vis-à-vis its greatest lower bound* c: (1) starting from a $k \geq 0$, $r_k = c$, $r_{k+1} = c, \ldots,$ *or* (2) for *all* of the $l = 0, 1, \ldots,$ $r_l > c$. In *both cases* $f(x)$ *is unbounded above in every positive-length open interval* $(c - \delta,\ c + \delta) \stackrel{\text{def}}{=} \{x|c - \delta < x < c + \delta,\ \delta > 0\}$ *around* c*, so* $f(x)$ *isn't continuous at* $x = c$. We now show for cases (1) and (2) successively that *given an arbitrary* $\delta > 0$ *and an arbitrary nonnegative integer* m, there exists an $x \in (c - \delta,\ c + \delta)$ for which $f(x) \geq 2^m$.

In case (1), let $n = \max(k, m)$. Then $c = r_n$, where $r_n = \sup S_n$, and $S_n \stackrel{\text{def}}{=} \{x_n, x_{n+1}, x_{n+2}, \ldots\}$. For all $x \in S_n$, $f(x) \geq 2^n \geq 2^m$. Because $c = \sup S_n$, there *exists* an $x \in S_n$ such that $x \leq c$ and $c - x < \delta$, *so that* $x \in (c - \delta,\ c + \delta)$ *and also* $f(x) \geq 2^m$.

In case (2), there *exists* a nonnegative integer k such that for all nonnegative integers l which satisfy $l \geq k$, it is still true that $r_l > c$, but it is *also* true that $r_l < c + (\delta/2)$ because c *is the greatest lower bound (in fact the limit) of the decreasing sequence* $r_0 \geq r_1 \geq r_2 \cdots$ *of least upper bounds*. We let $n = \max(k, m)$, and note that for all $x \in S_n \stackrel{\text{def}}{=} \{x_n, x_{n+1}, x_{n+2} \ldots\}$, $f(x) \geq 2^n \geq 2^m$, and since $r_n = \sup S_n$, there *exists* an $x \in S_n$ such that $x \leq r_n$ and $(r_n - x) \leq (\delta/2)$. Since $r_n < c + (\delta/2)$ and $x \leq r_n$, $x \leq c + (\delta/2)$. Also, since $r_n > c$ and $r_n \leq x + (\delta/2)$, $c \leq x + (\delta/2) \Rightarrow x \geq c - (\delta/2)$, which together with $x \leq c + (\delta/2)$ implies that $x \in (c - \delta,\ c + \delta)$. So *in case* (2) *as well as case* (1) there exists an $x \in (c - \delta,\ c + \delta)$ for which $f(x) \geq 2^m$.

Thus if a function $f(x)$ that is defined as a real number everywhere on the positive-length closed interval $[a, b]$ *is unbounded above on that interval, there exists a point* $c \in [a, b]$ *where* $f(x)$ *cannot be continuous*; this lemma is obviously readily extended to the case that $f(x)$ is *unbounded below* on $[a, b]$.

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To show that a *continuous* real function $f(x)$ on a closed interval $[a,b]$ *attains* its least upper bound $\sup_{x'\in[a,b]} f(x')$ on that interval *again involves selecting a set of distinct points* $\{x_0, x_1, x_2, \ldots\} \subset [a,b]$, but *in this case the distance by which* $f(x_n)$ *is below* $\sup_{x'\in[a,b]} f(x')$ *is required to be half or less of the distance by which* $f(x_{n-1})$ *is below* $\sup_{x'\in[a,b]} f(x')$*,* $n = 1, 2, \ldots$. Of course *it is initially supposed that* $f(x)$ *is strictly below* $\sup_{x'\in[a,b]} f(x')$ *at every* $x \in [a,b]$*, since if that wasn't the case, no argument would be needed*. Then exactly as in the foregoing discussion one considers the point $c = \inf\{r_0, r_1, r_2, \ldots\}$, where $r_0 \stackrel{\text{def}}{=} \sup\{x_0, x_1, x_2, \ldots\} \geq r_1 \stackrel{\text{def}}{=} \sup\{x_1, x_2, x_3, \ldots\} \geq r_2 \stackrel{\text{def}}{=} \sup\{x_2, x_3, x_4, \ldots\} \geq \ldots$. Since $f(x)$ *is assumed to be continuous at every point in* $[a,b]$*, including at the point* c, study of the situation which obtains in every positive-length open interval around c *shows that it is impossible for* $f(c)$ *to have any value other than* $\sup_{x'\in[a,b]} f(x')$, which *quashes* the initial supposition *that* $f(x)$ *is strictly below* $\sup_{x'\in[a,b]} f(x')$ *at every* $x \in [a,b]$. The reader is invited *to fill in the details of the situation which obtains in every positive-length open interval around* c, with the foregoing discussion of the situation which obtained in positive-length open intervals around the point which in that discussion was denoted c *serving as a guide*. Showing that a *continuous* real function $f(x)$ on a closed interval $[a,b]$ *attains its greatest lower bound* $\inf_{x'\in[a,b]} f(x')$ would obviously *be very closely analogous indeed* to what was shown here regarding attainment by the continuous real function $f(x)$ of its least upper bound $\sup_{x'\in[a,b]} f(x')$ on $[a,b]$.

If for a *continuous* real function $f(x)$ on $[a,b]$, $f(a) < f(b)$, then for any real number r such that $f(a) < r < f(b)$, there exists a point $c \in [a,b]$ such that $f(c) = r$. To show this, we consider the set $S^{\text{re}}_{f<r}$ *of right endpoints* y *of the closed sub-intervals* $[a,y]$ *of* $[a,b]$ *that satisfy* $x \in [a,y] \Rightarrow f(x) < r$. Every positive-length open interval around $\sup S^{\text{re}}_{f<r}$ has points x' at which $f(x') < r$, since for any $x' < \sup S^{\text{re}}_{f<r}$ of course $f(x') < r$. Every positive-length open interval around $\sup S^{\text{re}}_{f<r}$ *also* has points x' at which $f(x') \geq r$, since points x such that $f(x) \geq r$ *definitely do occur in* $[a,b]$—e.g., $f(b) > r$—and the *nature* of the point $\sup S^{\text{re}}_{f<r}$ *ensures* that for every $\delta > 0$ there exists an x' such that $\sup S^{\text{re}}_{f<r} \leq x' \leq \delta + \sup S^{\text{re}}_{f<r}$ and $f(x') \geq r$. Since $f(x)$ thus assumes *both* values which are less than r *and* values which are greater than or equal to r in every positive-length open interval around $\sup S^{\text{re}}_{f<r}$, then *since* $f(x)$ *is continuous at* $x = \sup S^{\text{re}}_{f<r}$, $f(\sup S^{\text{re}}_{f<r}) = r$. This demonstration is readily extended to $f(a) > f(b)$; in that case, for any real number r such that $f(a) > r > f(b)$, there is a point $c \in [a,b]$ such that $f(c) = r$.

These lemmas concerning "filling in" the function values between the $[a,b]$ closed-interval endpoint values $f(a)$ and $f(b)$ of a continuous function $f(x)$ are readily extended to "filling in" the function values *between the minimum and maximum of* $f(x)$ *on the closed interval* $[a,b]$ because there is a point $c \in [a,b]$ where $f(x)$ attains its minimum value $\min_{x'\in[a,b]} f(x')$ and a point $d \in [a,b]$ where $f(x)$ attains its maximum value $\max_{x''\in[a,b]} f(x'')$, so to demonstrate the "filling in" between the minimum and maximum values of $f(x)$ on the closed interval $[a,b]$ *entails simply shifting focus to the particular closed sub-interval* $[c,d]$ *of* $[a,b]$. The *upshot* is that a continuous function $f(x)$ on a closed interval $[a,b]$ *assumes every value between its minimum and maximum value on* $[a,b]$—it in fact maps the closed interval $[a,b]$ onto the closed interval $[\min_{x'\in[a,b]} f(x'),\ \max_{x''\in[a,b]} f(x'')]$, so *continuous functions preserve closed intervals*. These last five words *distill all of the properties established in the foregoing paragraphs into a single compact theorem*.

The mean-value theorems from continuous-function and calculus basics

A continuous function $f(x)$ defined for $x \in [a,b]$, where $b > a$, of course has on $[a,b]$ a well-defined finite minimum $\min_{x'\in[a,b]} f(x')$ and also a well-defined finite maximum $\max_{x''\in[a,b]} f(x'')$. This fact *in conjunction with the various usual definitions of the integral*—for example, the Riemann-integral definition—implies that,

$$\int_a^b f(x)dx \geq \left(\min_{x'\in[a,b]} f(x')\right)(b-a) \ \text{ and } \ \int_a^b f(x)dx \leq \left(\max_{x''\in[a,b]} f(x'')\right)(b-a). \tag{1a}$$

Therefore,

$$\min_{x'\in[a,b]} f(x') \ \leq \ (b-a)^{-1} \int_a^b f(x)dx \ \leq \ \max_{x''\in[a,b]} f(x''), \tag{1b}$$

and since for $x \in [a,b]$, $f(x)$ takes on all of the values which lie between $\min_{x'\in[a,b]} f(x')$ and $\max_{x''\in[a,b]} f(x'')$, there exists a $c \in [a,b]$ such that,

$$f(c) = (b-a)^{-1} \int_a^b f(x)dx, \text{ where } c \in [a,b], \tag{1c}$$

which after being multiplied through by $(b-a)$ yields that,

$$\int_a^b f(x)dx = f(c)(b-a), \text{ where } c \in [a,b]. \tag{1d}$$

This is *the integral form of the mean-value theorem, which assumes that* $f(x)$ *is continuous for* $x \in [a,b]$. If we now consider *a different function* $F(x)$ *whose derivative* $F'(x)$ *is continuous for* $x \in [a,b]$, we are permitted *to substitute* $F'(x)$ *for* $f(x)$ *in* Eq. (1d), which produces,

$$\int_a^b F'(x)dx = F'(c)(b-a), \text{ where } c \in [a,b]. \tag{1e}$$

We now apply the fundamental theorem of the calculus to the left side of Eq. (1e) to obtain,

$$(F(b) - F(a)) = F'(c)(b-a), \text{ where } c \in [a,b]. \tag{1f}$$

This is *the differential form of the mean-value theorem, which assumes that $F'(x)$ is continuous for $x \in [a,b]$*.

Taylor expansion and estimation of the error entailed

The coefficient c_k, $k = 0, 1, 2, \ldots, n$, of the kth power of x in the polynomial $P_n(x)$,

$$P_n(x) \stackrel{\text{def}}{=} \textstyle\sum_{k=0}^{n} c_k\, x^k, \tag{2a}$$

is rather simply related to the kth-order derivative with respect to x of $P_n(x)$, evaluated at $x = 0$,

$$\begin{gathered} d^l(P_n(x))/dx^l = \theta(n - l + \tfrac{1}{2}) \textstyle\sum_{k=l}^{n} c_k\, k(k-1)\cdots(k-l+1)x^{k-l} \;\Rightarrow \\ \big(d^l(P_n(x))/dx^l\big)_{x=0} = \theta(n-l+\tfrac{1}{2})\, c_l\, l! \;\Rightarrow\; c_k = \big(d^k(P_n(x))/dx^k\big)_{x=0}/k!\,,\ k = 0, 1, 2 \ldots . \end{gathered} \tag{2b}$$

The *upshot* of the Eq. (2b) determination of the coefficient c_k of the kth power of x in the polynomial $P_n(x)$ in terms of the kth-order derivative of $P_n(x)$, evaluated at $x = 0$, *is a representation of $P_n(x)$ in terms of its derivatives to all orders, evaluated at* $x = 0$, namely,

$$P_n(x) = P_n(x=0) + \textstyle\sum_{k=1}^{\infty} \big(d^k(P_n(x))/dx^k\big)_{x=0}\, x^k/k!. \tag{2c}$$

The Eq. (2c) and (2a) representations of $P_n(x)$ are in terms of powers of x, which is advantageous when x has a value close to zero. Since that isn't always the case, we now *extend* the Eq. (2a) and (2c) representations of $P_n(x)$ from powers of x to powers of $(x - x_0)$, where x_0 is an arbitrary constant,

$$\begin{gathered} P_n(x) = \textstyle\sum_{k=0}^{n} b_k\,(x-x_0)^k \;\Rightarrow\; \big(d^l(P_n(x))/dx^l\big)_{x=x_0} = \theta(n-l+\tfrac{1}{2})\, b_l\, l! \;\Rightarrow \\ \big(P_n(x) - P_n(x_0)\big) = \textstyle\sum_{k=1}^{\infty} \big(d^k(P_n(x))/dx^k\big)_{x=x_0}\,(x-x_0)^k/k!. \end{gathered} \tag{2d}$$

These Taylor representations of any polynomial $P_n(x)$ are exact in principle since the kth-order derivatives $d^k(P_n(x))/dx^k$ that occur in Eq. (2d) vanish identically for all values of k such that $k > n$. However, notwithstanding that we have *developed* the Eq. (2d) Taylor representations *by using polynomials* $P_n(x)$, for which these Taylor representations *are exact in principle*, we obviously *would like to apply the* Eq. (2d) *Taylor representations to any function $F(x)$ which possesses the requisite kth-order derivatives at $x = x_0$*. For such *general-function application* of the Eq. (2d) Taylor representations, we obviously *also* would wish to have a systematic means *of estimating the errors which are made*. To *begin thinking* about error estimation for Eq. (2d) applied to a general $F(x)$, *we first drastically downgrade* Eq. (2d) *by truncating its entire right side*, thus producing *the most rudimentary conceivable extension of Taylor expansion to $F(x)$, i.e.*,

$$\big(F(x) - F(x_0)\big) = 0. \tag{2e}$$

For this most rudimentary conceivable Taylor expansion of $F(x)$, the Eq. (1f) differential form of the mean-value theorem *supplies the desired error estimate*. Taking $b = x$ and $a = x_0$ in Eq. (1f) yields,

$$\big(F(x) - F(x_0)\big) = F'(c)(x - x_0), \text{ where } c \in [x_0, x], \tag{2f}$$

Of course this application of Eq. (1f) *assumes that the derivative $F'(x')$ of $F(x')$ is continuous for all values of x' in the closed interval* $[x_0, x]$. The *derivation* of the Eq. (1f) differential form of the mean-value theorem from Eq. (1e) *involved the fundamental theorem of the calculus*, and it turns out that the fundamental theorem *itself* can be made to generate error renditions *for less and less severely truncated versions of the* Eq. (2d) *Taylor expansion*. To *begin* to show this, we *slightly modify* that theorem's *presentation* as follows,

$$\textstyle\int_{x_0}^{x} F'(x')dx' = F(x) - F(x_0) \;\Rightarrow\; \big(F(x) - F(x_0)\big) = \int_{x_0}^{x} F^{(1)}(x')dx'. \tag{2g}$$

From Eq. (2g) we see *in particular* that for the Eq. (2e) *most rudimentary conceivable Taylor expansion* $(F(x) - F(x_0)) = 0$, the Eq. (2g) fundamental theorem of the calculus *yields the error as $\int_{x_0}^{x} F^{(1)}(x')dx'$, to which application of the integral form of the mean-value theorem yields precisely the* Eq. (2f) *error estimate*.

But *vastly more interesting* is that *successive integrations by parts* of the integral term $\int_{x_0}^{x} F^{(1)}(x')dx'$ of the Eq. (2g) fundamental theorem *generates successive refinements* of the Eq. (2e) most rudimentary conceivable Taylor expansion $(F(x) - F(x_0)) = 0$ *in complete correspondence with successive terms of the* Eq. (2d) *Taylor sum* (which of course is exact for polynomials). In *addition* to generating the successive terms of the Eq. (2d) Taylor sum for $F(x)$, the successive integrations by parts of the fundamental theorem of the calculus *also feature a compact integral rendition of the error entailed by approximating* $F(x)$ *by the Taylor-sum terms which they have generated*. We now display the results *of the first three integrations by parts* of the integral $\int_{x_0}^{x} F^{(1)}(x')dx'$ of the fundamental theorem of the calculus,

$$\big(F(x) - F(x_0)\big) = \int_{x_0}^{x} F^{(1)}(x')dx' = F^{(1)}(x_0)(x - x_0) + \int_{x_0}^{x}(x - x')F^{(2)}(x')dx' =$$

$$F^{(1)}(x_0)(x - x_0) + F^{(2)}(x_0)(x - x_0)^2/2 + (1/2)\int_{x_0}^{x}(x - x')^2 F^{(3)}(x')dx' = \tag{2h}$$

$$F^{(1)}(x_0)(x - x_0) + F^{(2)}(x_0)(x - x_0)^2/2 + F^{(3)}(x_0)(x - x_0)^3/6 + (1/6)\int_{x_0}^{x}(x - x')^3 F^{(4)}(x')dx'.$$

It now seems apparent that nth integration by parts of the Eq. (2g) fundamental calculus theorem yields,

$$\big(F(x) - F(x_0)\big) = \theta(n - \tfrac{1}{2})\sum_{k=1}^{n} F^{(k)}(x_0)(x - x_0)^k/k! + (1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')dx', \tag{2i}$$

which is readily *inductively verified* via integration by parts of its last term. Eq. (2i) presents only the first n terms of the Eq. (2d) *infinite* Taylor sum, *but its last term is an integral rendition of the error entailed by using those first n terms of the* Eq. (2d) *infinite Taylor sum*. The Eq. (2d) *infinite* Taylor sum is exact for polynomials and entire functions, *but it can also easily diverge*, e.g., for analytic functions which hem in its radius of convergence with complex-plane points of non-analyticity. A *sufficient* condition *for the overall validity of* Eq. (2i) *fully conjoined with its error integral* is continuity of $F^{(n+1)}(x')$ at all $x' \in [x_0, x]$.

Direct estimation of *the integral rendition of the error in* Eq. (2i) may be inconvenient in practice. We therefore work out an integral-free mean-value-theorem style of presentation of that error term. To do so *we of course assume that* $F^{(n+1)}(x')$ *is continuous for all* x' *in the closed interval* $[x_0, x]$.

Direct application to the error-rendition integral $(1/n!)\int_{x_0}^{x}(x-x')^n F^{(n+1)}(x')dx'$ in Eq. (2i) of the integral mean-value theorem is inhibited by the presence of the factor $(x - x')^n$ in the integrand of that integral. Therefore we attempt to "flatten" that factor *so that it effectively becomes constant* by applying a suitable change of integration variable *before* we apply the integral mean-value theorem itself.

A technical issue which complicates finding a suitable change of integration variable is that although $(x - x') \geq 0$ when $x > x_0$, $(x - x') \leq 0$ when $x_0 > x$, *so it is best to make two different changes of integration variable in the two cases* $x > x_0$ *and* $x_0 > x$.

If $x > x_0$, we change the integration variable from x' to $U(x') = (x - x')^{n+1}$, so $dU(x')/dx' = -(n + 1)(x - x')^n$. Thus the factor $(x - x')^n$ we wish to "flatten" equals $-(1/(n + 1))dU(x')/dx'$, which implies that $(x - x')^n dx' \propto dU$. Additional relevant facts are that $U(x' = x) = 0$, $U(x' = x_0) = (x - x_0)^{n+1}$ and $x' = x - (U(x'))^{1/(n+1)}$. Thus for this $x > x_0$ case,

$$(1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')\,dx' = (1/(n + 1)!)\int_{0}^{(x-x_0)^{n+1}} F^{(n+1)}(x - U^{1/(n+1)})\,dU. \tag{3a}$$

Eq. (3a) shows the $x' \to U$ variable change "flattened" the factor $(x - x')^n$ because $(x - x')^n dx' \propto dU$. Thus there exists a "mean value" $F^{(n+1)}(c)$ of $F^{(n+1)}(x - U^{1/(n+1)})$ which can be taken *outside* the integration over U between 0 and $(x - x_0)^{n+1}$—where $(x - U^{1/(n+1)}) \in [x_0, x]$ and $F^{(n+1)}(x - U^{1/(n+1)})$ *is continuous*. With this function *outside* the U integration, that integration becomes simple. Thus for $x > x_0$ we obtain,

$$(1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')\,dx' = \big(F^{(n+1)}(c)/(n + 1)!\big)\int_{0}^{(x-x_0)^{n+1}} dU = F^{(n+1)}(c)(x - x_0)^{n+1}/(n + 1)!, \tag{3b}$$

where $c \in [x_0, x]$.

If $x_0 > x$, we change the integration variable from x' to $V(x') = (x' - x)^{n+1}$, so $dV(x')/dx' = (n + 1)(x' - x)^n$. Thus the factor $(x - x')^n$ we wish to "flatten" equals $(-1)^n(1/(n+1))dV(x')/dx'$, which implies that $(x - x')^n dx' \propto dV$. Additional relevant facts are that $V(x' = x) = 0$, $V(x' = x_0) = (x_0 - x)^{n+1}$ and $x' = x + (V(x'))^{1/(n+1)}$. Thus for this $x_0 > x$ case,

$$(1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')\,dx' = (-1)^{n+1}(1/(n + 1)!)\int_{0}^{(x_0-x)^{n+1}} F^{(n+1)}(x + V^{1/(n+1)})\,dV. \tag{3c}$$

Eq. (3c) shows the $x' \to V$ variable change "flattened" the factor $(x - x')^n$ because $(x - x')^n dx' \propto dV$. Thus there exists a "mean value" $F^{(n+1)}(c)$ of $F^{(n+1)}(x + V^{1/(n+1)})$ which can be taken *outside* the integration over V between 0 and $(x_0 - x)^{n+1}$—where $(x + V^{1/(n+1)}) \in [x_0, x]$ and $F^{(n+1)}(x + V^{1/(n+1)})$ *is continuous*. With this function *outside* the V integration, that integration becomes simple. Thus for $x_0 > x$ we obtain,

$$(1/n!) \int_{x_0}^{x} (x - x')^n F^{(n+1)}(x')\, dx' = (-1)^{n+1} \big(F^{(n+1)}(c)/(n+1)!\big) \int_0^{(x_0 - x)^{n+1}} dV =$$
$$(-1)^{n+1} \big(F^{(n+1)}(c)/(n+1)!\big)(x_0 - x)^{n+1} = F^{(n+1)}(c)(x - x_0)^{n+1}/(n+1)!, \tag{3d}$$

where $c \in [x_0, x]$. This result for the $x_0 > x$ case *has come out to be exactly the same as the* Eq. (3b) *result for the* $x > x_0$ *case*. Thus *we can replace the integral error term in the* Eq. (2i) *Taylor expression by the considerably simpler result obtained here*, which produces,

$$\big(F(x) - F(x_0)\big) = \theta(n - \tfrac{1}{2}) \sum_{k=1}^{n} F^{(k)}(x_0)(x - x_0)^k/k! + F^{(n+1)}(c)(x - x_0)^{n+1}/(n+1)!, \tag{3e}$$

where $c \in [x_0, x]$. Of course, $F^{(n+1)}(x')$ *is assumed to be continuous for all* x' *in the closed interval* $[x_0, x]$. The $n = 0$ case of the Eq. (3e) Taylor theorem is the Eq. (1f) differential form of the mean value theorem.