Thin shells in general relativity without junction conditions: A model for galactic rotation and the discrete sampling of fields

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(Published 3 October 2019)

Interest in general relativistic treatments of thin matter shells has flourished over recent decades, most notably in connection with astrophysical and cosmological applications such as black hole matter accretion, spherical wormholes, bubble universes, and cosmic domain walls. In the present paper, an asymptotically exact solution to Einstein's field equations for static ultra-thin spherical shells is derived using a continuous matter density distribution $\rho(r)$ defined over all space. The matter density is modeled as a product of surface density μ_0 and a continuous or broadened spherical delta function. Continuity over the full domain $0<r<\infty$ ensures unambiguous determination of both the metric and coordinates across the shell wall, obviating the need to patch interior and exterior solutions using junction conditions. A unique change of variable allows integration with asymptotic precision. It is found that ultra-thin shells smaller than the Schwarzschild radius can be used to model supermassive black holes believed to lie at the centers of galaxies, possibly accounting for the flattening of the galactic rotation curve as described by Modified Newtonian Dynamics (MOND). Concentric ultra-thin shells may also be used for discrete sampling of arbitrary spherical mass distributions with applications in cosmology. Ultra-thin shells are shown to exhibit constant interior time dilation. The exterior solution matches the Schwarzschild metric. General black shell horizons, and singularities are also discussed.

I. INTRODUCTION

A long-standing unsolved problem in astrophysics is the observed discrepancy in the orbital velocity $v(r)$ of the luminous matter of galaxies. This discrepancy, often called the *flattening* of the *galactic rotation curve*, has been ascertained from Doppler shift measurements that indicate the outlying stars and hydrogen clouds of galaxies orbit too fast to be gravitationally bound by baryonic matter alone. In regions outside the luminous disk, $v(r)$ does not fall off as $r^{-1/2}$ as predicted by Newtonian dynamics, but tends toward a constant as r increases. The discrepancy is generally attributed to the presence of dark matter, a hypothetical transparent nonradiating material that has never been independently detected nor reconciled with the standard model of particle physics. The failure to identify this elusive substance has given rise to modified gravity theories that obviate the need for dark matter, such as Mordehai Milgrom's Modified Newtonian mechanics (MOND) [1,2] and others [3,4]. Here, a static spherical thin shell solution to Einstein's field equations is derived that may suggest a new explanation for the galactic rotation curve. A solution for concentric shells is also presented that may be useful for discrete sampling of arbitrary spherical mass distributions with applications in cosmology.

Investigation into the gravitational properties of thin matter shells has flourished over the past few decades, most notably in studies of astrophysical and cosmological structures such as spherical wormholes [5-7], black hole accretion shells, bubble universes as models of cosmic inflation [8,9], false vacuum bubbles [10,11], and cosmic membranes or domain walls that split the universe into distinct spacetime regions [12-14]. The structures may be static, as in the case of spherical wormholes; contracting, as in the case of matter accretion shells around black holes [15] and shells collapsing into wormholes [16,17]; rotating and collapsing [18,19]; or expanding, as in the case of cosmic brane worlds [20], inflationary bubbles or bubble universes [21]. Such shells may split the universe into two domains, an interior and exterior joined by an infinitesimally thin wall of singular mass or pressure [22-26]; or into three domains [27], where the wall of finite thickness is sometimes called the *transient layer* [28]. Various interior and exterior metrics are assumed, including the Friedman-Robertson-Walker [29,30], Schwarzschild, de Sitter [31], anti-de Sitter [32], Minkowski, and Reissner-Nordstrom [33,34] metrics. The metrics are often selected *a priori*, their parameters later fixed by junction conditions at the inner and outer surfaces of the wall, or at the shell radius [35]. Common techniques frequently require patching solutions for inner, outer, and possible transient domains, using separate coordinate systems and metrics for each domain [36,37]. The most widely applied junction conditions, attributed to Israel [38,39], or Darmois and Israel [40], require that both the metric $g_{\mu\nu}$ and the extrinsic curvature $K^{\mu}{}_{\nu}$ be continuous across the shell wall. While these conditions are common in the literature, doubt is raised about their application to certain physical scenarios [41] or in modified theories of gravity [42]. Some authors derive new junction conditions that specify jumps in curvature [43], jumps in the tangential metric components to account for domain wall spin currents [44], or other field behavior [45]. Others avoid junction conditions by use of a confining potential [46].

It may be significant that Israel's original derivation was based on properties of electromagnetic fields rather than on general relativity (GR), although recent derivations, in contexts such as cosmological brane-worlds, address the junction by adding a Gibbons-Hawking term to the standard Einstein-Hilbert action of GR [47]. However, some authors point to contradictions in this method, particularly when applied to infinitely thin shells [48].

While procedures for deriving the Israel junction conditions are well established, their implementation relies on concepts outside the core formalism of GR and other metric gravities, including the notion of *induced metric*, or the D─n dimensional metric in the transient domain; the vector $\boldsymbol{n_i}$ normal to the domain wall; the surface stress-energy tensor $S^{\mu}{}_{\nu}$ for the transient domain; the *extrinsic curvature* $K^{\mu}{}_{\nu}$; the Gibbons-Hawking action term, and so forth. A treatment of thin shells that obviates the need for junction conditions may therefore be useful for its simplicity. Cosmic inhomogeneities using cubic lattices that avoid junction conditions have been studied by some authors [49,50]. Nevertheless, examples in the literature of continuous spherical thin-shell solutions to the gravitational field equations have proven difficult to find.

The purpose of this paper is to derive an *asymptotically exact* continuous solution to Einstein's field equations for static, spherical, *ultra-thin* massive shells without the need for junction conditions, employing a uniform set of coordinates defined over all space, with equation of state $p=w\rho$. Here, *asymptotically exact* means exact in the limit of vanishing thickness (although the solution is undefined for zero thickness), and *ultra-thin* denotes arbitrarily thin but non-vanishing. One advantage to the continuous solution method, in which density $\rho(r)$, pressure $p(r)$, and the metric $g_{\mu\nu}(r)$ are uniformly defined over all space, is that only two boundary conditions are needed to fix the metric:

1. $g_{\mu\nu}$ must be nonsingular at $r=0$,
2. $g_{\mu\nu}$ must match Minkowski space as $r \rightarrow \infty$,

where *Minkowski space* is here defined by the metric $g_{\mu\nu}=diag(1,-1,-r^2,-r^2 sin^2(\Theta))$. The first condition follows from the absence of matter in the interior [51]. This is relaxed in the case of a central mass. The second dictates that space be asymptotically flat, assuming $g_{00} \rightarrow 1$ as $r \rightarrow \infty$, or that the standard laboratory clock rate is the same as that at infinity.

To obtain a continuous solution to Einstein's field equations (EFE), *i.e.* a metric composed of continuous analytic functions $g_{\mu\nu}(r)$ defined over all space, one must first define a continuous density $\rho(r)$ spanning the range $0 \leq r \leq \infty$. For an ultra-thin shell, $\rho(r)$ will be modeled here as continuous approximation to the spherical Dirac delta function $\delta(r-r_0)$, where the continuous or *broadened* version of the delta function, to be written $\delta_c(r-r_0)$, will be derived in Section II. According to this model, the mass density distribution is

$$\rho(r) = \mu_0\, \delta_c(r - r_0). \tag{1}$$

Here, μ_0 is the surface density of the shell and has dimensions $[m/r^2]$. Recalling that the δ function has dimensions $[1/r]$, it is clear that the volume density $\rho(r)$ has dimensions $[m/r^3]$, or $[1/r^2]$ in the units $G=c=1$. This density distribution may be substituted into the energy-momentum tensor $T_{\mu\nu}$ on the right-hand side of EFE. The equations are then solved using a unique change of variable that allows integration to arbitrary accuracy. The result is an asymptotically exact continuous metric for an empty ultra-thin shell.

The metric signature (+ - - -) and units $c=G=1$ will be used throughout this paper. Small Greek letters stand for spacetime indices $0,1,2,3$. The symbol $\approx$ denotes *asymptotic equality*, or equality in limit as thickness parameter ε approaches zero, although the formalism is undefined at $\varepsilon=0$. An equation of state (EoS) of the form $p(r)=w\rho(r)$ for w a constant will be assumed. While the method here applies to static shells, it can in principle be generalized to account for expansion or contraction. This is a topic for future research. The presentation is organized as follows. In Section II, the broadened spherical delta function will be derived. Section III shows how to solve EFE for a thin shell using the continuous solution method. In Section IV, the novel properties of *black shells* (those of radius less than or equal to the Schwarzschild radius) will be examined. Section V discusses how the galactic rotation curve might be explained by a supermassive black shell at the galactic core, and Section VI presents the concentric shell solution as a method for discrete sampling. Concluding remarks are found in Section VI.

II. MASS DENSITY: DEFINING THE CONTINUOUS DELTA FUNCTION

Spherical Dirac delta functions as models for mass or charge distributions have appeared in the literature for many decades. Use of the delta function for thin shell solutions to EFE is frequently encountered in such applications as bubble universes and cosmic domain walls. However, the discontinuities in the delta function and its integral, the step function, necessitate piecewise solutions and attendant junction conditions, as noted above. To apply a delta function model uniformly over all space requires that the Dirac delta function $\delta(r-r_0)$ be replaced by a continuous or broadened delta function $\delta_c(r-r_0)$ with similar properties. One such function can be defined as follows:

1) Let $\delta_c(r-r_0)$ be an approximation to a *spherical Dirac delta function* $\delta(r-r_0)$, where the latter is expressed in terms of the *normalized spherical Gaussian*

$$G(r) := (\varepsilon\sqrt{\pi})^{-1} \exp[-(r-r_0)^2/\varepsilon^2]. \qquad (2)$$

Here $G(r)$ is defined over the domain $r\geq 0$, with a peak centered at $r=r_0$ of height $1/(\pi^{1/2}\varepsilon)$ and width proportional to ε. For $\varepsilon << r_0$, $G(r)$ obeys the relation

$$\int_0^\infty dr\, G(r) = \int_0^\infty dr\, (\varepsilon\sqrt{\pi})^{-1} \exp[-(r-r_0)^2/\varepsilon^2] \approx 1,$$

$$\varepsilon << r_0.$$

This relation may be verified by evaluating the integral of a normalized rectangular Gaussian $G(x)$, which for $\varepsilon << x_0$ has the property

$$\int_0^\infty dx\, (\varepsilon\sqrt{\pi})^{-1} \exp[-(x-x_0)^2/\varepsilon^2] \approx \int_{-\infty}^\infty dx\, (\varepsilon\sqrt{\pi})^{-1} \exp[-(x-x_0)^2/\varepsilon^2] = 1.$$

The delta function may thus be written

$$\delta(r-r_0) = \lim_{[\varepsilon\to 0]} (\varepsilon\sqrt{\pi})^{-1} \exp[-(r-r_0)^2/\varepsilon^2]. \qquad (3)$$

2) The *continuous* or *broadened delta function* $\delta_c(r-r_0)$ is obtained as an approximation to $\delta(r-r_0)$ by taking an incomplete limit in Eq. (3), that is, by letting ε become arbitrarily small but nonzero.

3) For n a small integer such that $2n\varepsilon$ approximates the peak width to some selected accuracy, the broadened delta function $\delta_c(r-r_0)$ nearly vanishes in the domains $r<r_0-n\varepsilon$ and $r>r_0+n\varepsilon$. Therefore mass density $\rho(r)$ approaches that of a *near-vacuum* in these regions. By increasing n and decreasing ε, the vacuum can be achieved as closely as desired.

4) The broadened delta function δ_c obeys, to any desired accuracy, the defining properties of the Dirac delta function:

a) $\int_0^\infty dr\, \delta_c(r-r_0) \approx 1$

b) $\int_0^\infty dr\, f(r)\, \delta_c(r-r_0) \approx f(r_0)$

provided that $f(r)$ is slowly varying over the transient layer $r_0-n\varepsilon<r<r_0+n\varepsilon$.

5) The integral $\int_0^r dr\delta_c$, or the *inverse derivative* of the broadened delta function δ_c, is a *continuous* or *broadened step function* $S_c(r;r_0)$ such that

$$\int_0^r dr\, f(r)\, \delta_c(r-r_0) \approx f(r_0)\, S_c(r;r_0), \qquad (4)$$

where $f(r)$ varies slowly over the transient layer, and $S_c(r;r_0)$ has the properties

$$\begin{array}{ll} S_c(r;r_0) \approx 0 & r < r_0-n\varepsilon \\ S_c(r;r_0) \approx 1/2 & r = r_0 \\ S_c(r;r_0) \approx 1 & r > r_0+n\varepsilon. \end{array}$$

(For convenience, the symbol r represents both the dummy variable and the integral limit.) That $S_c\approx 1/2$ for $r=r_0$ can be seen by integrating $G(r)$ from 0 to r_0, and recalling that the integral over all space of a normalized Gaussian is unity. The function S_c, while locally continuous, appears *globally discontinuous* in that its value changes rapidly over the thickness $2n\varepsilon$ of the transient layer.

One advantage to modeling mass density $\rho(r)$ in terms of a broadened delta function is the ease of integration when solving EFE. Many integrals can be read off by simply applying Eq. (4). This technique can be extended to concentric shells, such as those discussed in reference [52], and may be useful for modeling astrophysical objects such as spherical dust accretion clouds surrounding dirty black holes [53], spherical domain walls enclosing the known cosmos, or for a discrete sampling of any continuous spherical mass distribution.

III. SOLVING EINSTEIN'S FIELD EQUATIONS FOR A THIN SHELL: THE CONTINUOUS SOLUTION METHOD

We will now derive a locally continuous ultra-thin shell solution to EFE, assuming a static spherically symmetric metric $g_{\alpha\beta}$ of the form

$$\begin{aligned} ds^2 &= g_{00}(r)\, dt^2 + g_{11}(r)\, dr^2 - r^2 d\Omega^2 \\ &= e^\nu\, dt^2 - e^\lambda\, dr^2 - r^2 d\Omega^2 \end{aligned}$$

The appropriate gravitational field equations may be found by substituting this metric into Einstein's field equations, given by

$$R^\mu{}_\nu - (1/2)\, g^\mu{}_\nu R = \kappa\, T^\mu{}_\nu \qquad (5)$$

where $R^\mu{}_\nu$ is the curvature or Ricci tensor, R is the scalar curvature, κ is a constant with the value $\kappa=-8\pi G/c^2$ (using Dirac's sign convention [54]), or $\kappa=-8\pi$ for $G=c=1$, and $T^\mu{}_\nu=diag(\rho,-p,-p,-p)$ is the stress energy tensor, with $\rho(r)$ the mass-energy density and $p(r)$ the pressure. After calculating the Christoffel symbols $\Gamma^\mu{}_{\alpha\beta}$ and curvatures R and $R^\mu{}_\nu$, EFE of Eq. (5) simplify to a pair of simultaneous equations [55]

$$\kappa T^0{}_0 = \kappa\rho(r) = e^{-\lambda}/r^2 - 1/r^2 - e^{-\lambda}\lambda'/r \qquad (6a)$$

$$\kappa T^1{}_1 = -\kappa p(r) = e^{-\lambda}/r^2 - 1/r^2 + e^{-\lambda}\nu'/r, \qquad (6b)$$

where primes denote derivatives with respect to r. Eq. (6a) can be solved by rearranging terms to produce a pure differential (see Appendix for details of derivations in this section):

$$\kappa\rho r^2 + 1 = (re^{-\lambda})'.$$

Integrating and solving for e^λ, we obtain

$$e^\lambda = [1 + k_0/r + (\kappa/r)\int_r dr\, \rho(r)\, r^2]^{-1}. \qquad (7)$$

where k_0 is a constant of integration. Substituting $\rho(r)=\mu_0\delta_c(r-r_0)$ and $\mu_0=m_0/4\pi r_0^2$, and applying Eq. (4), this becomes

$$e^\lambda = (1 + k_0/r - 2m_0S_c/r)^{-1}. \qquad (8)$$

For an empty shell, the boundary condition that e^λ be non-singular at $r=0$ requires that $k_0=0$. (If the shell contains a central mass M, an integration constant $k_0=-2M$ is generally assumed.) The rr component of the ultra-thin shell metric is therefore

$$g_{11} = -e^\lambda = -(1 - 2m_0S_c/r)^{-1}. \qquad (9)$$

Outside the shell, where $S_c\approx 1$, we see that g_{11} matches the radial component of the Schwarzschild metric $g^S_{\mu\nu}$, as given by

$$ds^2 = (1-2m/r)dt^2 - (1-2m/r)^{-1}dr^2 - r^2d\Omega^2 \quad (10)$$

for m the central mass. In the interior of the shell, where $S_c\approx 0$, it is clear that g_{11} matches the Minkowski metric.

Next, the tt component $g_{00}=e^\nu$ can be evaluated by subtracting Eq. (6b) from Eq. (6a) to obtain

$$\kappa(\rho+p) = -e^{-\lambda}\lambda'/r - e^{-\lambda}\nu'/r.$$

Solving for ν', substituting e^λ from Eq. (9) and $\rho(r)$ from Eq. (1), and using equation of state $p=w\rho$, the result is

$$\nu' = -\lambda' - \kappa(1+w)\mu_0\delta_c r\,/\,(1-2m_0S_c/r),$$

where δ_c and S_c are abbreviated notations for the broadened delta and step functions. Upon integrating, this becomes

$$\nu = -\lambda + k_1 - \kappa(1+w)\mu_0\int_r dr\,[\delta_c r\,/\,(1-2m_0S_c/r)] \quad (11)$$

with k_1 a constant of integration. Eq. (11) represents an exact solution to EFE for the tt metric component $g_{00}=e^\nu$ of an ultra-thin shell. The integrand, however, contains the spherical Gaussian $G(r)$ and may be difficult to evaluate analytically. For the present, an arbitrarily close approximation can be found using the properties of the broadened step and delta functions. This procedure requires care due to the rapid variation of $S_c(r;r_0)$ in the transient layer $r_0-n\varepsilon<r<r_0+n\varepsilon$. We proceed by writing the integral in Eq. (11) as a function of the upper limit r

$$I(r) = \int_0^r dr\,\delta_c\, r/(1-2m_0S_c/r). \qquad (12)$$

Since $\delta_c(r-r_0)\approx 0$ in the near-vacuum domains $r<r_0-n\varepsilon$ and $r>r_0+n\varepsilon$, the integrand vanishes to any desired accuracy in these domains. (An exception is the case $r_0=2m_0$, where the integrand approaches $0/0$ rather than 0 for $r>>r_0+n\varepsilon$, as will be discussed in Section IV.) Hence in general, r changes by a near infinitesimal amount $2n\varepsilon$ across the non-vanishing domain of the transient layer and may be treated as a constant $r\approx r_0$. Thus we have,

$$I(r) \approx r_0\int_0^r dr\,\delta_c\,/\,(1-2m_0S_c/r_0) \qquad r_0\neq 2m_0. \quad (13)$$

(Here as elsewhere, the symbol $\approx$ denotes asymptotic equality, for which precision increases as ε decreases.) $I(r)$ can now be integrated to asymptotic precision by a unique change of variable. Recalling from Eq. (4) that $S_c=\int_r\delta_c dr$ and therefore $dS_c=\delta_c dr$, the continuous monotonic function S_c can be used as the variable of integration. The limits of integration become 0 and $S_c(r)$, and the integral may be written

$$I(r) \approx r_0\int_0^{S_c(r)} dS_c\,/\,(1-2m_0S_c\,/\,r_0)$$

$$\approx -(r_0^2/2m_0)\,\ln|(1-2m_0S_c\,/\,r_0)|,$$

where the absolute value, arising from the standard integral formula $\int(dx/x)=\ln|x|$, will impact later analysis. Substituting $I(r)$ back into Eq. (11) and evaluating the constants κ and μ_0 yields

$$\nu \approx -\lambda + k_1 - (1+w)\ln|1-2m_0S_c/r_0|.$$

Upon substitution of e^λ from Eq. (8), the result is

$$e^\nu \approx (1-2m_0S_c/r)\, e^{k_1}\,|1-2m_0S_c/r_0|^{-(1+w)}.$$

Since e^ν must obey the Minkowski condition $e^\nu \to 1$ as $r\to\infty$, the integration constant e^{k_1} must cancel the right-hand factor in the outer region where $S_c\to 1$, leaving only the left-hand factor, which is asymptotically Minkowski. Hence the integration constant is

$$e^{k_1} = |1-2m_0/r_0|^{(1+w)}$$

and the final result for the tt component of the ultra-thin shell metric is

$$g_{00}\approx(1-2m_0S_c/r)\,|1-2m_0/r_0|^{(1+w)}\,|1-2m_0S_c/r_0|^{-(1+w)}$$

$$r_0\neq 2m_0. \qquad (14)$$

To analyze this result, we evaluate g_{00} for the interior and exterior, obtaining

$$g_{00int} \approx |1-2m_0/r_0|^{(1+w)} \qquad (15a)$$

$$g_{00ext} \approx (1-2m_0/r). \qquad (15b)$$

The exterior component g_{00}, like the exterior component g_{11}, matches the Schwarzschild solution as expected. Note that the quantity $-2m_0$ in the exterior metric arises automatically from the field equations and, unlike for the case of Schwarzschild metric, is not put in as an integration constant. That this quantity is predetermined by EFE further confirms the consistency of general relativity, in that vacuum and non-vacuum solutions agree for regions surrounding a central mass. Thus, solar system tests confirm not just the vacuum equations, where $T^{\mu}{}_{\nu}=0$, but also the massive equations, where $T^{\mu}{}_{\nu}\neq 0$, insofar as a thin shell serves as well as a point mass for modeling a star or planet.

Regarding time dilation, it is significant that the interior metric g_{00int} is a constant not equal to unity, while the exterior metric g_{00ext} asymptotically approaches unity, indicating clocks inside the shell run at different rates than those at infinity. For so-called non-phantom matter, which has an EoS $p(r)=w\rho(r)$ with $w>-1$, we note that $g_{00int}<1$, indicating time inside the shell is dilated with respect to infinity. This result may seem at odds with occasional claims that time does not dilate inside an empty shell. Such claims may arise from piecewise solutions and are often based on two arguments: 1) Minkowski spacetime, with $g_{00}=1$, prevails inside a hollow shell; or 2) according to Birkhoff's theorem, the Schwarzschild metric governs the vacuum in an empty shell, leading to $g_{00}=1$ [51]. These arguments, however, depend on a rescaling of the time coordinate inside the shell. The continuous solution method, in contrast, assumes a uniform time coordinate over the whole space domain $0\leq r<\infty$. It is clear, nevertheless, that no apparent gravitational forces exist inside an empty shell due to the constant value of the interior metric.

For a shell composed of dust, the EoS parameter is $w=0$, and the interior and exterior solutions match at $r=r_0$. Therefore g_{00} and the corresponding clock rates are continuous across the shell wall. The *tt* component for a thin dust shell thus satisfies the first Israel junction condition.

For a shell composed of *stiff matter,* which has an EoS of $w=1$, we see that g_{00} changes abruptly across the shell wall, allowing interior time dilation up to twice that at the outer surface. Thus the continuous solution method predicts time dilation measurements using real non-dust shells would show a violation of the Israel conditions. It seems interesting that the interior metric g_{00int} depends on the EoS of the shell, while the exterior metric g_{00ext} like the Schwarzschild metric, is independent of the EoS. This curious distinction resolves the seeming paradox, mentioned in a previous paper [56], that while non-vacuum solutions to EFE require an EoS, Schwarzschild vacuum solutions do not, even though mass appears in the metric.

IV. BLACK HOLES AND BLACK SHELLS

The ultra-thin shell metric of Eqs. (9) and (11) may be applied to shells of radius equal to or less than the Schwarzschild radius, or shells such that $r_o\leq 2m_0$. To be called *black shells,* these exotic objects would generally appear to a distant observer as a Schwarzschild black hole (although unexpected singularities may occur). At close range, black shells display unique properties with respect to horizons and singularities. To compare black holes and black shells, first recall the properties of the Schwarzschild black hole with metric $g^S{}_{\mu\nu}$ as given by Eq. (10):

1. A coordinate singularity, or *horizon*, exists at $r=2m$, where $g^S{}_{00}=0$ and $g^S{}_{11}\rightarrow-\infty$.
2. Inside the horizon, squared proper time intervals $d\tau^2=g^S{}_{00}dt^2$ are negative, and thus proper time is spacelike, while squared proper radial intervals $dR^2=g^S{}_{00}dr^2$ are positive, and proper radial distance is timelike.
3. A physical singularity is generally assumed to exist at $r=0$, where $g^S{}_{00}\rightarrow-\infty$ and $g^S{}_{11}=0$.
4. There are no finite discontinuities in the domain $r>0$.

To compare the properties of black shells, we consider the metrics for four shell types: ordinary shells with $r_0>2m_0$; *horizon black shells* with $r_0=2m_0$; *subhorizon black shells* with $r_0<2m_0$, and *semi-horizon black shells* with $r_0=m_0$. First, recall the interior and exterior thin shell metrics of Section III:

$$g_{00int}\approx(1-2m_0/r_0)^{(1+w)} \qquad r_0\neq 2m_0 \qquad (16a)$$

$$g_{00ext}\approx 1-2m_0/r \qquad r_0\neq 2m_0 \qquad (16b)$$

$$g_{11int}\approx -1 \qquad (16c)$$

$$g_{11ext}\approx -(1-2m_0/r)^{-1}. \qquad (16d)$$

In the case of ordinary shells ($r_0>2m_0$), the Schwarzschild radius $r_s=2m_0$ lies inside the shell where the metric is constant. Thus there is no horizon at $r=r_s$. In addition, no singularity exists at $r=0$. Although the metric is *locally* continuous everywhere, comparison of Eqs. (16c) and (16d) reveals a *global* discontinuity or *jump* across r_0 in the component g_{11}. For non-dust models, for which $w\neq 0$, there is also a jump across r_0 in the component g_{00}, in apparent violation of the Israel junction conditions. However when $w=0$ as in the case of dust, g_{00} remains unchanged across r_0, in agreement with the Israel conditions.

For *horizon black shells* ($r_0=2m_0$), the shell radius is equal to the Schwarzschild radius $r_s=2m_0$, and as noted earlier, the approximation $r\rightarrow r_0$ in the integrand of $I(r)$ of Eq. (13) is no longer valid. Deriving the properties of g_{00} would require computing the exact integral of Eq.

(12) using the spherical Gaussian. Such a calculation is not attempted here. If, however, we naively allow the approximation $r \rightarrow r_0$ and apply Eq. (14), the apparent properties of horizon black shells suggest such objects may be nonphysical. To illustrate, recall the full equations for the metric:

$$g_{00} \approx (1-2m_0S_c/r)|1-2m_0/r_0|^{(1+w)}|1-2m_0S_c/r_0|^{-(1+w)} \quad (17)$$

$$g_{11} \approx -1 / (1-2m_0S_c/r) \quad (18)$$

Setting $r_0=2m_0$ in the first equation and assuming $w>-1$, it is clear that $g_{00}(r)=0$ for $0<r<\infty$. This can be seen by noting that the middle factor in g_{00} vanishes identically, while the right-hand factor (denominator) is non-vanishing for all finite r due to the property $S_c(r)<1$, and the left-hand factor is finite for all $r>0$. The vanishing of g_{00} suggests that a horizon black shell would stop all clocks in the universe, a physical impossibility and a violation of the asymptotic Minkowski condition. Whether this nonphysical result can be avoided by evaluating g_{00} analytically using the function $G(r)$, by applying numerical methods, or by redefining δ_c in terms of a function other than $G(r)$, is a question for future research.

Concerning the *rr* metric component, we see from Eq. (16c) that $g_{11} \approx -1$ inside the shell, implying no interior singularities exist. To check this result, note that by Eq. (18), no singularity can exist unless there is an r such that $2m_0S_c(r)/r=1$, or $r/r_0=S_c(r)$. Since it is always true that $S_c(r)<1$, any such singularity can only reside at $r<r_0$. It will be stated without proof that since $S_c(r)\approx 1/2$ when $r/r_0=1$, and since $S_c(r)$ falls to zero more rapidly than r/r_0, there can be no $r>0$ such that $2m_0S_c(r)/r=1$, and hence no singularity in the domain $0<r<r_0$. Moreover, by L'Hopital's rule it is found that

$$\lim_{[r \rightarrow 0]} 2m_0 S_c(r)/r = 0,$$

ruling out a singularity at the origin. Thus a horizon black shell, unlike a Schwarzschild black hole, manifests no singularities in g_{11}.

Subhorizon black shells ($r_0<2m_0$), in contrast, appear at close range like Schwarzschild black holes, with a horizon at $r\approx 2m_0$. Subhorizon black shells also have approximate Schwarzschild behavior for $r>2m_0$. However, a new singularity in g_{00} may arise due to the vanishing of $1-2m_0S_c/r_0$ in the right-hand factor (denominator) of Eq. (17). To locate this singularity, recall that $S_c(r)$ increases monotonically over the range $0<S_c<1$. Thus g_{00} becomes singular at some unique r such that $S_c(r)=r_0/2m_0$. Since $S_c(r)$ traverses nearly all of its range within a distance $n\varepsilon$ of r_0, such singularities usually fall within $r_0-n\varepsilon<r<r_0+n\varepsilon$, or in the transient layer of the wall itself. However if $r_0=2m_0-\varsigma$, where ς is some extremely small quantity, a singularity may occur at some large radius $r=R_0$ where $S_c(R_0)=r_0/2m_0\approx 1$. This means subhorizon black shells could in principle cause singularities in g_{00} at cosmological distances. Such models may have astrophysical applications related to the composition of galactic cores (the topic of Section V), or cosmological interpretations with respect to Hubble redshift, bubble universes or spherical domain walls, to be addressed in a later paper.

In the unique case of a *semi-horizon black shell* for which the radius $r_0=m_0$ is half the Schwarzschild radius, one might expect a singularity in $g_{00}(r)$ at $r=r_0$, where $S_c(r)\approx 1/2$. However, it turns out that $g_{00}(r)$ has a finite discontinuity rather than a singularity at $r=r_0$. This can be shown as follows. Setting $w=0$ and $r_0=m_0$, Eq. (17) simplifies to

$$g_{00}(r) \approx [1-2r_0S_c(r)/r] \,/\, |1-2S_c(r)|,$$

which, as r tends to r_0, approaches the improper limit 0/0. Applying L'Hopital's rule yields the ratio H of the derivatives of numerator and denominator:

$$\begin{aligned} H &= \partial_r [1-2r_0S_c/r] \,/\, \partial_r |1-2S_c| \\ &= (2r_0S_c/r^2 - 2r_0\delta_c/r) \,/\, (-/+ 2\delta_c) \\ &= -/+ (r_0/r)(S_c/r\delta_c - 1) \end{aligned}$$

where the sign ambiguity springs from the absolute value. Taking the limit $r \rightarrow r_0$, the term $S_c/r\delta_c$ approaches $\pi^{1/2}\varepsilon/2r_0<<1$, and H tends to positive or negative unity, with the positive case corresponding to approach from $r>r_0$ and the negative to $r<r_0$. Thus for $r_0=m_0$, the limit is not unique, leaving g_{00} undefined at $r=r_0$. Whether the semi-horizon black shell discontinuity arises as an artifact of the approximation is not known.

V. BLACK SHELLS, MOND, AND THE GALACTIC ROTATION CURVE

Can supermassive black shells in the cores of galaxies explain the discrepancy in the galactic rotation curve? If so, it would obviate the need for postulating a dark matter halo. The discrepancy in orbital velocity $v(r)$, as noted earlier, arises from observations of differential Doppler shift, which indicate the outer stars and hydrogen clouds of galaxies orbit too fast to be gravitationally bound by luminous or baryonic matter alone. Thus, outside the bright galactic disk, $v(r)$ does not fall off as $r^{-1/2}$, as would be expected from Newtonian dynamics [57], but tends toward a constant as r increases. This anomaly was noted by Fritz Zwicky in 1933 [58] and first quantified observationally by Vera Rubin [59].

The flattening of the galactic rotation curve can be described by an effective potential $\varphi_m(r)$ that depends only baryonic mass and increases with r at large distances. The potential φ_m, for reasons evident below, will be called the *MOND potential*. The goal is to show that a subhorizon black shell (SBS), or similar exotic black object, located in the galactic core, could theoretically account for the observed excess orbital velocities, or equivalently, that an SBS potential $\varphi_{SBS}(r)$ can be made consistent with the MOND potential $\varphi_m(r)$ in outlying regions. The MOND potential will be derived first, followed by the SBS potential. The two will then be equated to show, by a redefinition of the broadened delta function, a close correspondence in the metrics.

The MOND potential can be calculated from *Modified Newtonian Dynamics* (MOND), a formalism developed in 1983 by Mordehai Milgrom [60] to account for the discrepancy in the rotation velocity of galaxies. Although the excess velocity is usually attributed to the presence of an unseen dark matter halo, the MOND formalism, relying on baryonic matter alone, has proven accurate in predicting orbital motion [61], and thus provides a means for testing theories.

The MOND formalism is based on the empirical relation

$$\mu(a/a_0)a = a_N \qquad (19)$$

which connects observed radial acceleration a to predicted Newtonian acceleration $a_N=GM/r^2$ using an interpolating function $\mu(a/a_0)$, where

$$a_0 = 1.2x10^{-8} cm/sec^2 \cong H_0/2\pi = c^2/R = c^2/(\Lambda/3)^{-1/2}$$

is a universal constant with dimensions of acceleration, H_0 is the Hubble parameter [62], and R is roughly 2π times the radius of the visible universe or the de Sitter radius corresponding to cosmological constant Λ [63]. The interpolating function runs smoothly from the inner galaxy, where the field falls off as roughly $1/r^2$, to the region outside the bright galactic disk, called the *deep MOND region,* where the field tends to fall off as $1/r$. Using the simple interpolating function

$$\mu(a/a_0) = (a/a_0) / (1+a/a_0)$$

proposed by Zhao, Famaey and Binney [64, 65], the MOND relation of Eq. (19) becomes a quadratic equation with solution,

$$a = -(GM/2r^2) [1 + \surd(1+4r^2/R_m{}^2)]. \qquad (20)$$

The radius R_m, to be called the *MOND radius*, lies near the edge of the bright galactic disk and has the value

$$R_m = \surd(GMR/c^2) = \surd(GM/a_0).$$

In the domain of interest $2R_m<r<<R$, which is roughly the region outside the luminous disk, the observed radial acceleration a of Eq. (20) can be approximated as

$$a \cong -GM/2r^2 - GM/R_m r. \qquad (21)$$

The potential in this domain can be expressed as

$$\varphi_m = -\int a(r)\, dr \cong -GM/2r + (GM/R_m) \ln (r/R_m).$$

The factor $1/2$ in the first term on the right does not appear in some presentations of MOND, where different interpolating functions apply and where the potential covers all space [66]. However, since the second term increases with r and becomes dominant near R_m, we can neglect the first term and construct an effective metric for the deep MOND region [67]

$$g_{00} \cong 1 + 2\varphi_m/c^2 \cong 1 + (2GM/c^2R_m) \ln (r/R_m), \qquad (22)$$

which is accurate in the domain $nR_m<r<<R$, for n a small integer on the order of 4 or 5. Note that $g_{00}\rightarrow\infty$ as $r\rightarrow\infty$. Hence the effective metric violates the asymptotic Minkowski condition and cannot, in the form of Eq. (22), be consistent with a black shell metric. Consistency will be attained through a later approximation.

Next, to calculate the SBS potential $\varphi_{SBS}(r)$, we assume the galaxy is centered on a supermassive ultra-thin SBS of radius

$$r_0 = 2m_0 - \varsigma = (1-\sigma)\, r_s, \qquad (23)$$

where $\varsigma>\varepsilon$ is a small distance on the order of meters, r_s is the Schwarzschild radius $2m_0$, shell mass m_0 is a large fraction of galactic mass M, and parameter $\sigma=\varsigma/r_s$ measures the small difference between shell size and Schwarzschild radius. Such an SBS would induce a singularity in g_{00} at some cosmic-scale radius R_0, at which clocks would theoretically run at an infinite rate. In realistic scenarios, no remote singularity can occur due to disturbance of the mass density by other fields. Nevertheless, a remote *virtual singularity* implies a modification of the field in the neighborhood of the galaxy.

The distance to the singularity at R_0 is inversely related to ς and increases with step width $2n\varepsilon$. More specifically, from Eq. (17), R_0 must satisfy

$$1 - 2m_0 S_c(R_0)/r_0 = 0,$$

or, upon substiting $2m_0=r_0+\varsigma$ and rearranging,

$$S_c(R_0) = 1 / (1 + \varsigma / r_s) \cong 1 - \varsigma / r_s. \qquad (24)$$

To calculate the impact of the distant singularity on the field in the galactic neighborhood, we start by introducing a new function $\eta(r)$ and expressing the broadened step function as $S_c(r)=1-\eta(r)$, where $\eta(r)<<1$ in the deep MOND region. This and Eq. (23) are then substituted into the thin shell metric of Eq. (17), and the result is evaluated for the shell's far exterior $r_0<<r<R_0$, yielding

$$g_{00SBS} \approx [1 - r_s(1-\eta)/r]\, \sigma / |\eta(r) - \sigma|$$

$$\cong (1 - r_s/r)\, \sigma / |\eta(r) - \sigma|. \qquad (25)$$

From Eq. (24), we see that $\eta(R_0)\cong\sigma=\varsigma/r_s$, and the denominator of g_{00SBS} vanishes near R_0 as expected.

To match g_{00SBS} to the MOND metric of Eq. (22), we first write a an approximation to the latter which repositions the singularity from infinity to a remote finite distance $r=R_0$ as follows:

$$g_{00MOND} \cong 1 + 2\varphi_m/c^2 \cong 1 + (r_s/R_m)\ ln\ |r/(R_0 - r)|. \quad (26)$$

This approximation can be checked by calculating acceleration a from potential φ_m

$$a \cong -\varphi_m' \cong -GM/R_m r - GM/R_m(R_0 - r).$$

It is clear that for r in the neighborhood of the galaxy, $(R_0 - r)$ is large enough that the right-hand term can be neglected. The remaining term matches the MOND acceleration of Eq. (21). Hence we see that g_{00MOND} of Eq. (26) adequately approximates the MOND metric in the deep MOND region.

The MOND and SBS metric*s* may now be equated, giving

$$1 + (r_s/R_m)\ ln\ |r/(R_0 - r)| = (1 - r_s/r)\ \sigma\ /\ [\eta(r) - \sigma].$$

By solving for $\eta(r)$, a new form $S_m(r)$ of the broadened step function is obtained that is consistent with the MOND metric as follows:

$$S_m(r) = 1 - \eta(r) = 1 - \sigma\ /\ [1+(r_s/R_m)\ ln\ |r/(R_0 - r)|] - \sigma.$$

Simple calculation shows that $S_m(r)$, while different from the broadened step function $S_c(r)$ derived in Section II, has like properties in the domain $r_0<<r<R_0$. To wit, $S_m(r)$ is slightly less than one and increases monotonically to the near-unity value $1-\sigma$ as r approaches the near-infinite distance R_0. Thus, it is possible to derive a MOND-compatible step function $S_m(r)$ by replacing the Gaussian $G(r)$ with some appropriate function $F(r)$ in the definition of the broadened delta function δ_c, thus obtaining a new delta function δ_m. An SBS modeled on δ_m, embedded in the galactic core, would then account for the anomalous orbital velocities. The exact function $F(r)$ is unknown. Questions also remain about SBS formation and stability. What is important is the implication that an exotic black object, possibly a subhorizon black shell, could in principle cause the observed galactic rotation curve without the need for a dark matter halo.

VI. CONCENTRIC SHELLS AND DISCRETE DENSITY SAMPLING

The continuous solution method is easily generalized to n concentric shells of arbitrary mass and radius. This technique provides a formalism for solving EFE for any continuous static spherical density distribution $\rho(r)$, where $\rho(r)$ is modeled by a discrete sampling at

$$r = \{r_0, r_1 \ldots r_{n-1}\}.$$

The method for concentric shell solutions will be illustrated for the simple case of two shells with EoS parameter $w=0$. Assuming surface densities μ_0 and μ_1, radii r_0 and r_1, and masses $m_0=4\pi\mu_0 r_0^2$ and $m_1=4\pi\mu_1 r_1^2$, the mass density can be expressed in terms of broadened delta functions as

$$\rho(r) = \mu_0\delta_0 + \mu_1\delta_1$$

where $\delta_j=\delta_c(r - r_j)$ denotes a broadened delta function at radius r_j. Substituting $\rho(r)$ into Eq. (7) and setting the integration constant to zero yields

$$g_{11} = -e^{\lambda} = -[1 + (\kappa/r)\textstyle\int_r dr\ \rho r^2]^{-1}$$

$$= -[1 + (\kappa\mu_0/r)\textstyle\int_r dr\ r^2\delta_0 + (\kappa\mu_1/r)\int_r dr\ r^2\delta_1]^{-1}.$$

Upon integration, the double thin-shell solution becomes

$$g_{11} = -e^{\lambda} = -[1 - 2m_0S_0/r - 2m_1S_1/r]^{-1} \quad (27)$$

where $S_0=S_c(r;r_0)$ and $S_1=S_c(r;r_1)$. The interior $(r<r_0 - n\varepsilon)$, middle $(r_0+n\varepsilon<r<r_1 - n\varepsilon)$, and exterior $(r>r_1+n\varepsilon)$ solutions are therefore

$$g_{11int} \approx -1 \quad (28a)$$

$$g_{11mid} \approx -(1 - 2m_0/r)^{-1} \quad (28b)$$

$$g_{11ext} \approx -[1 - 2(m_0 + m_1)/r]^{-1}, \quad (28c)$$

displaying Minkowski properties inside the smaller shell, Schwarzschild behavior between shells, and combined Schwarzschild behavior outside the larger shell.

To solve for the time component $g_{00} = e^{\nu}$ the method of Section III will be applied. From Eqs. (11) and (27), we have

$$\nu = -\lambda + k_1 - \kappa\textstyle\int_r dr\ \rho(r)\ e^{\lambda}\ r$$

$$= -\lambda + k_1 - \kappa\textstyle\int_r dr(\mu_0\delta_0+\mu_1\delta_1)r\ /\ [1 - 2(m_0S_0+m_1S_1)/r]. \quad (29)$$

The integral may be expressed as a sum of two terms:

$$I(r) = \mu_0\textstyle\int_r dr\ \delta_0 r\ /\ [1 - 2(m_0S_0+m_1S_1)/r]$$

$$+ \mu_1\textstyle\int_r dr\ \delta_1 r\ /\ [1 - 2(m_0S_0+m_1S_1)/r].$$

Since r is slowly varying over the two transient layers, it can be approximated by r_0 and r_1 in the two respective integrands, yielding

$$I(r) \approx \mu_0 r_0\textstyle\int_r dr\ \delta_0\ /\ [1 - 2(m_0S_0+m_1S_1)/r_0]$$

$$+ \mu_1 r_1\textstyle\int_r dr\ \delta_1\ /\ [1 - 2(m_0S_0+m_1S_1)/r_1]$$

Note that in the first integral, the outer step function $S_1(r)$ varies slowly over the nonzero domain of the inner delta function δ_0, and hence may be set to a constant $S_1 \approx 0$. Analogously, in the second integral, $S_0(r)$ varies slowly over the nonzero domain of δ_1 and may be set to a constant $S_0 \approx 1$. The total integral then simplifies to

$$I(r) \approx \mu_0 r_0 \int_r dr\, \delta_0 / (1 - 2m_0 S_0 / r_0)$$
$$+ \mu_1 r_1 \int_r dr\, \delta_1 / (1 - 2m_0 / r_1 - 2m_1 S_1 / r_1).$$

Following the method of Section III, a change of variable from r to $S_0(r)$ and $S_1(r)$ in the respective integrals gives, upon integration,

$$I(r) \approx (\mu_0 r_0^2 / 2m_0)\, ln\, |1 - 2m_0 S_0 / r_0|$$
$$+ (\mu_1 r_1^2 / 2m_1)\, ln\, |1 - 2m_0 / r_0 - 2m_1 S_1 / r_1|.$$

Multiplying $I(r)$ by κ and substituting back into Eq. (29) then yields

$$\nu = -\lambda + k_1 - ln\, |1 - 2m_0 S_0\, / r_0|$$
$$- ln\, |1 - 2m_0 / r_0 - 2m_1 S_1 / r_1|.$$

and hence

$$g_{00} = e^{\nu} = [1 - 2(m_0 S_0 + m_1 S_1)/r]\ \ e^{k_1}\ |1 - 2m_0 S_0 / r_0|^{-1}$$
$$X\ |1 - 2m_0 / r_0 - 2m_1 S_1 / r_1|^{-1} \qquad (30)$$

where X denotes multiplication. Again, to meet the asymptotic Minkowski condition, the integration constant must be

$$e^{k_1} = |1 - 2m_0 / r_0|\ |1 - 2m_0 / r_0 - 2m_1 / r_1| \qquad (31)$$

The constant e^{k_1} is then substituted back into Eq. (30), yielding the g_{00} component of the double concentric shell metric. Taken together, Eqs. (27), (30) and (31) represent a complete continuous asymptotically exact solution to EFE for two concentric ultra-thin dust shells of arbitrary mass and radius.

It is straightforward to extend this result to n concentric shells of mass m_i, radius r_i, and thickness ε_i, as long as $\varepsilon_i << (r_{i+1} - r_i)$. Such a set of *locally* continuous thin shells may be viewed as a discrete sampling, at arbitrary radii r_i, of a *globally* continuous mass density distribution $\rho(r)$. The concentric shell formalism thus provides a discrete method for approximating the solution to EFE for any static, spherically symmetric mass-energy density. Hence Einstein's equations can be readily solved for complicated scenarios such as a star surrounded by spherical dust clouds embedded in cosmic bubbles, and so forth. The impact of discreteness on accuracy is a topic for future discussion.

VII. CONCLUSION

We have derived an asymptotically exact solution to Einstein's field equations for individual and multiple concentric ultra-thin shells of arbitrary mass and radius using a continuous solution method that does not require junction conditions. The single shell solution is given by Eqs. (9) and (14), and the double shell solution by Eqs. (27), (30) and (31). These solutions are fixed by two boundary conditions: asymptotic flatness at infinity and non-singularity at the origin. The interior of a thin shell is found to manifest no effective gravitational forces. However, interior clocks run at different rates from those at infinity. For non-phantom matter $(w > -1)$, time in the interior of the shell is dilated with respect to infinity, while for phantom matter, time is contracted.

Exterior to the shell, the field generally matches that of the Schwarzschild metric. Exceptions are found for black shells, *i.e.* shells of radius less than or equal to the Schwarzschild radius. The method breaks down for equal radii, and an asymptotically exact solution was not attempted. However, approximations suggest such objects may be unphysical. Subhorizon black shells, which have a radius smaller than the Schwarzschild radius, are more easily analyzed, and were shown in general to appear as Schwarzschild black holes everywhere outside the shell. This holds with one key exception. When the radius of a supermassive black shell is less than its Schwarzschild radius by a very small distance on the order of meters, a singularity may occur in the time component of the metric at cosmological distances. It was then shown that this singular metric approximates an effective MOND metric, where the latter is expressed in terms of an effective potential that accounts for the observed galactic orbital velocities. Thus, a supermassive subhorizon ultra-thin black shell or similar exotic black object, located at the center of a galaxy, could theoretically explain the flattening of the galactic rotation curve without the need for dark matter.

It was also shown that the solution for a series of concentric shells provides a discrete sampling method for calculating the approximate gravitational field of any spherical static mass distribution. Applications might include detailed scenarios such as spherical accretion shells around black holes embedded in a constant background density enclosed by a cosmic bubble.

The method developed here applies to static scenarios. It can in principle be generalized to dynamic configurations such as colliding shells in anti-deSitter spacetime [68] or black holes embedded in expanding bubble universes described by the Friedman-Robertson-Walker metric. These are topics for future research. Other questions also remain concerning:

1. Multiple concentric shell techniques for discrete sampling of cosmological mass distributions,
2. The impact of discreteness on accuracy,

3. Comparison of ultra-thin shell boundary properties to Israel junction conditions under a general EoS,
4. Collapsing ultra-thin shells and black shell formation,
5. Whether possible nonphysical features of horizon black shells interfere with black shell formation,
6. The nature and stability of rotating or charged ultra-thin shells,
7. Stability of ultra-thin shells, particularly of subhorizon black shells in galactic cores, and
8. The mathematical properties of functions $F(r)$ and δ_m compatible with MOND and the galactic rotation curve.

APPENDIX

Using the line element

$$ds^2 = g_{00}(r)\, dt^2 + g_{11}(r)\, dr^2 - r^2 d\Omega^2$$
$$= e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2 d\Omega^2$$

with $\kappa = -8\pi$ and a diagonal stress-energy tensor of the form $T^{\mu}{}_{\nu} = diag(\rho, -p, -p, -p)$, Einstein's field equations simplify to

$$\kappa T^0{}_0 = \kappa\rho(r) \; = \; e^{-\lambda}/r^2 - 1/r^2 - e^{-\lambda}\lambda'/r \qquad \text{(a1)}$$

$$\kappa T^1{}_1 \; = -\kappa p(r) = e^{-\lambda}/r^2 - 1/r^2 + e^{-\lambda}\nu'/r, \qquad \text{(a2)}$$

where primes denote derivatives with respect to r. Eq. (a1) can be solved by rearranging terms

$$\kappa\rho r^2 + 1 = e^{-\lambda}\,(1 - \lambda' r)$$
$$= (re^{-\lambda})'.$$

Integration then yields

$$re^{-\lambda} = k_0 + \int_r dr\, (\kappa\rho r^2 + 1)\, .$$

Here, $\int_r$ denotes the inverse derivative and k_0 is a constant of integration. Solving for e^{λ}, we obtain

$$e^{\lambda} = [1 + k_0/r + (\kappa/r)\textstyle\int_r dr\, \rho(r)\, r^2]^{-1}.$$

Substitution of $\rho(r) = \mu_0\delta_c(r - r_0)$ and application of Eq. (4) gives

$$e^{\lambda} = (1 + \kappa\mu_0 r_0{}^2 S_c/r + k_0/r)^{-1}.$$

Using surface density $\mu_0 = m_0/4\pi r_0{}^2$, this becomes

$$e^{\lambda} = (1 - 2m_0 S_c/r + k_0/r)^{-1}. \qquad \text{(a3)}$$

The boundary condition that e^{λ} be nonsingular at $r=0$ requires that $k_0=0$. The rr component of the ultra-thin shell metric is therefore

$$g_{11} = -e^{\lambda} = -(1 - 2m_0 S_c/r)^{-1}. \qquad \text{(a4)}$$

The tt component $g_{00}=e^{\nu}$ can be evaluated by subtracting Eq. (a2) from Eq. (a1) to obtain

$$\kappa(\rho+p) = -e^{-\lambda}\,\lambda'/r - e^{-\lambda}\,\nu'/r.$$

Solving for ν' yields

$$\nu' = -\lambda' - \kappa(\rho+p)\, e^{\lambda}\, r.$$

If we now substitute $\rho(r)$ and e^{λ} from Eq. (a4), and apply the equation of state $p=w\rho$ for w a constant, the result is

$$\nu' = -\lambda' - \kappa(1+w)\mu_0\delta_c r\, /\, (1 - 2m_0 S_c/r).$$

Upon integrating, this becomes

$$\nu = -\lambda + k_1 - \kappa(1+w)\mu_0 \textstyle\int_r dr\, \delta_c r\, /\, (1 - 2m_0 S_c/r) \qquad \text{(a5)}$$

with k_1 a constant of integration. Eq. (a5) represents an exact solution to Einstein's field equations for the tt metric component $g_{00}=e^{\nu}$ of an ultra-thin shell. To approximate the integral, we use the properties of the broadened step and delta functions. The integral may be written

$$I(r) = \textstyle\int_0{}^r dr\, \delta_c\, r/(1 - 2m_0 S_c/r).$$

Since r changes by the near infinitesimal amount $2n\varepsilon$ across the transient layer, it may be treated as a constant $r \approx r_0$, hence

$$I(r) \approx r_0 \textstyle\int_0{}^r dr\, \delta_c/(1 - 2m_0 S_c/r_0) \qquad r_0 \neq 2m_0.$$

$I(r)$ can be integrated by a change of variable $dS_c=\delta_c dr$, with limits of integration 0 and $S_c(r)$:

$$I(r) \approx r_0 \textstyle\int_0{}^{S_c(r)} dS_c/(1 - 2m_0 S_c/r_0)$$
$$\approx -(r_0{}^2/2m_0)\, ln|\,(1 - 2m_0 S_c/r_0)|_0{}^{S_c(r)}$$
$$\approx -(r_0{}^2/2m_0)\, ln|\,(1 - 2m_0 S_c/r_0)|,$$

Substituting $I(r)$ into Eq. (a5) yields,

$$\nu \approx -\lambda + k_1 + [\kappa(1+w)\mu_0 r_0{}^2/2m_0]\; ln\, |1 - 2m_0 S_c/r_0|.$$

Evaluating the constants κ and μ_0, this simplifies to

$$\nu \approx -\lambda + k_1 - (1+w)\; ln\, |1 - 2m_0 S_c/r_0)|,$$

with the result

$$e^{\nu} \approx e^{-\lambda}\, e^{k_1}\, |1 - 2m_0 S_c/r_0|^{-(1+w)}$$

$$\approx (1 - 2m_0 S_c/r)\, e^{k_1}\, |1 - 2m_0 S_c/r_0|^{-(1+w)}.$$

Since e^{ν} must obey the Minkowski condition $e^{\nu} \rightarrow 1$ as $r \rightarrow \infty$, the integration constant e^{k_1} must cancel the right-hand factor in the outer region where $S_c \approx 1$, Hence the integration constant is

$$e^{k_1} = |1 - 2m_0/r_0|^{(1+w)}$$

and the tt component of the ultra-thin shell metric becomes

$$g_{00} \approx (1-2m_0 S_c/r) | 1-2m_0/r_0 |^{(1+w)} | 1-2m_0 S_c/r_0 |^{-(1+w)}$$

$$r_0 \neq 2m_0.$$

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