Tutorial: The Real Powers from Extension of Two Determining Properties of Positive-Integer Powers

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Abstract

This tutorial parlays two determining properties of positive-integer power functions of a positive real variable into closed formulas for the real power functions of a positive real variable. These two determining properties are that any positive-integer power of unity equals unity, and the linear first-order differential equation that a positive-integer power function of a real positive variable satisfies, which is implicit in its derivative. These two determining properties of positive-integer power functions of a positive real variable are extended to arbitrary real values of the positive-integer power. The extended linear first-order differential equations and initial conditions are then used to generate the Taylor expansions of those real power functions of a positive real variable around the zero value of the real power; this can be carried out in at least two different ways. Those Taylor expansions converge for every real value of the power and every positive real value of the variable, and are readily reexpressed entirely in terms of the exponential function and its inverse; one thus has closed formulas for all the real power functions of a positive real variable. Logarithms describe arbitrary positive numbers as real powers of a given positive number; they can expressed entirely in terms of the exponential function's inverse. The value of the particular positive constant whose powers yield the exponential function itself is worked out.

Extending two determining properties of positive-integer powers to real powers

A positive-integer power function b^j of a positive real variable b *is usually implicitly presented as*,

$$b^j \stackrel{\text{def}}{=} \overbrace{b \times b \times \cdots \times b}^{j \text{ times}}.$$

Alternatively, however, b^j *is completely determined by two properties*: (a) $b^j = 1$ when $b = 1$, i.e.,

$$(b = 1)^j = 1,$$

and (b) *the linear first-order differential equation implied by the derivative* $db^j/db = jb^{j-1}$ *of* b^j, namely,

$$db^j/db = (j/b)b^j.$$

We now *extend* these two *determining properties* of b^j to $j = x$, *where x is any real number*,

$$(b = 1)^x = 1, \tag{1a}$$

and,

$$db^x/db = (x/b)b^x. \tag{1b}$$

Eq. (1b) is a *set* of linear first-order differential equations for functions b^x *of the positive real variable b and also of the real power x*. Eq. (1a) specifies *the initial conditions for all of the b^x at $b = 1$*. The Eq. (1b) differential equations *are undefined at $b = 0$*, but standard theorems *assure unique well-defined real solutions for all of the b^x when b is positive*, since their initial conditions are specified at $b = 1$, *which is positive*.

The *product* of any two real power functions b^x and b^y that satisfy Eqs. (1b) and (1a) *is itself the real power function* b^{x+y} that satisfies Eqs. (1b) and (1a), i.e.,

$$b^x b^y = b^{x+y}. \tag{2a}$$

Eq. (2a) is true because Eq. (1b) implies that,

$$d(b^x b^y)/db = ((x/b)b^x)b^y + b^x((y/b)b^y) = ((x + y)/b)(b^x b^y), \tag{2b}$$

and Eq. (1a) implies that,

$$(b^x b^y)_{b=1} = (1 \times 1) = 1. \tag{2c}$$

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The Eq. (2a) relation among the solutions described by Eqs. (1b) and (1a) *enables*, when b is positive, *Taylor expansion around* $x = 0$ *of* b^x; to make such an expansion of course requires the $(d^k b^x/dx^k)_{x=0}$ functions of b for $k = 0, 1, 2, 3, \ldots$. Regarding $k = 0$, Eq. (2a) implies that $b^{x=0}$ satisfies,

$$b^{x=0}\, b^y = b^{0+y} = b^y \textit{ for all } b^y, \text{ and therefore } b^{x=0} = 1. \tag{3a}$$

Eq. (2a) *furthermore implies the following factorization of the derivative* db^x/dx,

$$db^x/dx = \lim_{\delta x\to 0}\big(\big(b^{x+\delta x} - b^x\big)/\delta x\big) = \lim_{\delta x\to 0}\big(\big(b^{\delta x} - 1\big)/\delta x\big)b^x. \tag{3b}$$

We evaluate $\lim_{\delta x\to 0}((b^{\delta x} - 1)/\delta x)$, which is a function of b, *via working out its derivative with respect to* b *and also obtaining its value at* $b = 1$. Since from Eq. (1a), $1^{\delta x} = 1$ *for all values of* δx, we see that,

$$\big[\lim_{\delta x\to 0}\big(\big(b^{\delta x} - 1\big)/\delta x\big)\big]_{b=1} = \lim_{\delta x\to 0}\big(\big(1^{\delta x} - 1\big)/\delta x\big) = 0. \tag{3c}$$

We work out *the derivative* of $\lim_{\delta x\to 0}((b^{\delta x} - 1)/\delta x)$ with respect to b by *interchanging* differentiation with respect to b with the $\delta x \to 0$ limit, which permits *first applying* Eq. (1b), *followed by applying* Eq. (3a),

$$d\big(\lim_{\delta x\to 0}\big(\big(b^{\delta x} - 1\big)/\delta x\big)\big)/db = \lim_{\delta x\to 0}\big(\big((\delta x/b)b^{\delta x}\big)/\delta x\big) = \lim_{\delta x\to 0}\big(b^{\delta x}/b\big) = \big(b^{\delta x=0}/b\big) = (1/b). \tag{3d}$$

Since from Eqs. (3d) and (3c), $d(\lim_{\delta x\to 0}((b^{\delta x} - 1)/\delta x))/db = (1/b)$ and $[\lim_{\delta x\to 0}((b^{\delta x} - 1)/\delta x)]_{b=1} = 0$,

$$\lim_{\delta x\to 0}\big(\big(b^{\delta x} - 1\big)/\delta x\big) = \textstyle\int_1^b db'/b', \tag{3e}$$

which is well-defined when b *is positive* and has the standard denotation $\ln b$. Eqs. (3e) and (3b) yield,

$$db^x/dx = \big(\textstyle\int_1^b db'/b'\big)b^x = (\ln b)b^x. \tag{3f}$$

Using Eq. (3f) to repeatedly differentiate b^x with respect to x produces,

$$d^k b^x/dx^k = \big(\textstyle\int_1^b db'/b'\big)^k b^x = (\ln b)^k b^x,\ k = 0, 1, 2, 3, \ldots, \tag{3g}$$

and since $b^{x=0} = 1$,

$$\big(d^k b^x/dx^k\big)_{x=0} = \big(\textstyle\int_1^b db'/b'\big)^k = (\ln b)^k,\ k = 0, 1, 2, 3, \ldots, \tag{3h}$$

which implies that the Taylor expansion of b^x around $x = 0$ is,

$$b^x = \textstyle\sum_{k=0}^{\infty}\big(x\int_1^b db'/b'\big)^k/k! = \sum_{k=0}^{\infty}(x\ln b)^k/k!, \tag{3i}$$

a sum which, when b is positive, converges for all of the real values of x. At $b = 1$, $\int_1^b db'/b'$ vanishes, so Eq. (3i) implies that,

$$1^x = 1, \tag{3j}$$

which accords with Eq. (1a). Moreover, differentiating Eq. (3i) with respect to b yields,

$$db^x/db = (x/b)\textstyle\sum_{k=1}^{\infty}(x\int_1^b db'/b')^{k-1}/(k-1)! = (x/b)\sum_{l=0}^{\infty}(x\int_1^b db'/b')^l/l! = (x/b)b^x, \tag{3k}$$

which accords with Eq. (1b). Thus not only is Eq. (3i) a sum which, when b is positive, converges for all of the real values of x; it is as well, when b is positive, *the unique well-defined real solution of the differential equation set given by* Eqs. (1b) *and* (1a). So Eq. (3i) *by itself* achieves the goal of *extending* the positive-integer powers j of a positive real variable b *to the real powers* x of that positive real variable b.

That said, it is nevertheless of interest *to understand in explicit detail how the infinite sum in* Eq. (3i) *meshes with the integral* $\int_1^b db'/b' = \ln b$ *that occurs in each of its terms to accomplish that goal*.

Relating the power function's infinite sum to the integral in each of its terms

The standard denotation *for the* Eq. (3i) *infinite sum's function structure is the exponential function*,

$$\exp(u) \stackrel{\text{def}}{=} \textstyle\sum_{k=0}^{\infty} u^k/k!, \tag{4a}$$

whose Eq. (4a) defining series *converges for all real* u. We have from Eq. (3i) that,

$$b^x = \exp\big(x\textstyle\int_1^b db'/b'\big) = \exp(x\ln b) \text{ when } b \text{ is positive.} \tag{4b}$$

From Eq. (4a) we note that,

$$\exp(0) = 1, \tag{5a}$$

and,

$$d\exp(u)/du = \textstyle\sum_{k=1}^{\infty} u^{k-1}/(k-1)! = \sum_{l=0}^{\infty} u^l/l! = \exp(u), \tag{5b}$$

so *the derivative of* $\exp(u)$ *is equal to itself*. Eqs. (5a) and (5b) *completely determine* $\exp(u)$ since they imply that $d^k \exp(u)/du^k|_{u=0} = 1,\ k = 0, 1, 2, 3, \ldots$, which immediately yields Eq. (4a) as the Taylor expansion of $\exp(u)$. Eqs. (5a) and (5b) also imply that $\exp(u)$ *has the crucial property*,

$$\exp(v_1)\exp(v_2) = \exp(v_1 + v_2), \tag{5c}$$

because,

$$\begin{gathered}(\exp(v_1 u)\exp(v_2 u))_{u=0} = 1 \ \ \textit{and also} \ \ d(\exp(v_1 u)\exp(v_2 u))/du = \\ (v_1\exp(v_1 u))\exp(v_2 u) + \exp(v_1 u)(v_2\exp(v_2 u)) = (v_1 + v_2)(\exp(v_1 u)\exp(v_2 u)),\end{gathered} \tag{5d}$$

which *taken together* yield that,

$$d^k(\exp(v_1 u)\exp(v_2 u))/du^k|_{u=0} = (v_1 + v_2)^k,\ k = 0, 1, 2, 3, \ldots, \tag{5e}$$

so by Taylor expansion,

$$(\exp(v_1 u)\exp(v_2 u)) = \textstyle\sum_{k=0}^{\infty}(v_1 + v_2)^k u^k/k! = \exp((v_1 + v_2)u) \ \Rightarrow\ \exp(v_1)\exp(v_2) = \exp(v_1 + v_2). \tag{5f}$$

Application of $\exp(v_1)\exp(v_2) = \exp(v_1+v_2)$ to the Eq. (4b) representation $b^x = \exp(x \ln b)$ of b^x immediately yields *the fundamental power relation* $b^x b^y = b^{x+y}$ *that is given by* Eq. (2a). That relation *by itself*, albeit fundamental, *doesn't ensure that* $b^1 = b$, a gap *which is filled by the* $\ln b$ *function*. Since $b^x = \exp(x \ln b)$, $b^1 = \exp(\ln b)$, *so to ensure that* $b^1 = b$, *the* ln *function must be the inverse of the* exp *function*.

This *obligates the* exp *function to have an inverse*, so $\exp(u)$ *must be strictly increasing (or strictly decreasing) at every real value of its argument* u. Since the *derivative* of the exp function *is equal to itself*, $\exp(u)$ *is strictly increasing at all real values* u *where it is positive*. When u is positive, all of the terms of the series for $\exp(u)$ are positive, which makes $\exp(u)$ positive for $u > 0$, as it also is at $u = 0$ since $\exp(u = 0) = 1$. To understand the character of $\exp(u)$ at negative values of u, we note that Eq. (5f) implies that $\exp(u)\exp(-u) = \exp(0) = 1$. Thus $\exp(-u) = (1/\exp(u))$, so the fact that $\exp(u)$ is positive at positive values of u means that it is positive as well at negative values of u, so it is strictly increasing *at all values of* u. Therefore $\exp(u)$ *indeed has an inverse*, but that inverse is *only* required to be well-defined *for the positive values of its argument*. If we denote the inverse of $\exp(u)$ as $\exp^{-1}(w)$, then $\exp^{-1}(w)$ *only* needs to be well-defined on the domain $w > 0$, but of course *it must satisfy*,

$$\exp^{-1}(\exp(u)) = u, \tag{5g}$$

which when $u = 0$ implies that,

$$\exp^{-1}(1) = 0. \tag{5h}$$

Since $d\exp(u)/du = \exp(u)$, differentiation with respect to u of both sides of Eq. (5g) yields,

$$\left[d\big(\exp^{-1}(\exp(u))\big)/d(\exp(u))\right]\exp(u) = 1, \tag{5i}$$

which, upon denoting $\exp(u)$ as w, implies that,

$$d\big(\exp^{-1}(w)\big)/dw = (1/w) \text{ when } w \text{ is positive.} \tag{5j}$$

Eqs. (5j) and (5h) together uniquely determine that when w is positive,

$$\exp^{-1}(w) = \textstyle\int_1^w dw'/w' = \ln w. \tag{5k}$$

so the ln function in the Eq. (4b) expression for b^x *is indeed the inverse of the* exp *function*.

Changing the differential equations to integral relations before Taylor expanding

The quite involved direct Taylor expansion of b^x around $x = 0$ that is carried out in Eqs. (3a) through (3i) is greatly simplified if the set of linear first-order differential equations and initial conditions given by Eqs. (1b) and (1a) *is first converted to a set of integral relations*. To do that we divide Eq. (1b) by b^x to obtain,

$$(1/b^x)(db^x/db) = x(1/b), \tag{6a}$$

and then note that,

$$\begin{aligned} &d\Big(k_1 + \textstyle\int_1^{b^x} dw'/w'\Big)\Big/db = (1/b^x)(db^x/db), \text{ where } k_1 \text{ is an arbitrary constant,} \\ &\text{and also, } d\Big(k_2 + x\textstyle\int_1^{b} db'/b'\Big)\Big/db = x(1/b), \text{ where } k_2 \text{ is an arbitrary constant.} \end{aligned} \tag{6b}$$

Since, as pointed out below Eq. (3e), the standard denotation of $\int_1^b db'/b'$ is $\ln b$, which is well-defined when b is positive, Eq. (6b) implies that integrating both sides of Eq. (6a) with respect to b yields that,

$$k_1 + \ln(b^x) = k_2 + x\ln b. \tag{6c}$$

The Eq. (1a) initial conditions for b^x, namely $(b = 1)^x = 1$, when inserted into Eq. (6c) yield that $k_1 = k_2$, so those initial conditions reduce Eq. (6c) to simply,

$$\ln(b^x) = x\ln b. \tag{6d}$$

It may initially seem puzzling that the entity $\ln(b^x)$ on the left side of Eq. (6d) is well-defined only when b^x is positive. That isn't actually an issue because, although the domain of $\ln(w)$ only encompasses $w > 0$, *the range of* $\ln(w)$ *turns out to encompass all real numbers*. That assertion would be confirmed if the inverse $\ln^{-1}(u)$ of $\ln(w)$ was worked out, and $\ln^{-1}(u)$ was then shown to be positive for all real numbers u. To obtain b^x from Eq. (6d) we *anyway* need to work out the inverse $\ln^{-1}(u)$ of $\ln(w)$ since obviously,

$$b^x = \ln^{-1}(x\ln b). \tag{6e}$$

The fact that $\ln(w)$ *actually has an inverse* is *verified* by noting that $d\ln(w)/dw = (1/w) > 0$ for every w in the domain of $\ln(w)$, which is $w > 0$. The inverse $\ln^{-1}(u)$ of $\ln(w)$ of course *must satisfy*,

$$\ln^{-1}(\ln(w)) = w \text{ when } w \text{ is positive,} \tag{6f}$$

which when $w = 1$ implies that,

$$\ln^{-1}(0) = 1. \tag{6g}$$

Differentiating Eq. (6f) with respect to w yields,

$$\big[d\big(\ln^{-1}(\ln(w))\big)/d(\ln(w))\big](1/w) = 1 \;\Rightarrow\; d\big(\ln^{-1}(\ln(w))\big)/d(\ln(w)) = w = \ln^{-1}(\ln(w)), \tag{6h}$$

where the last equality in Eq. (6h) is simply Eq. (6f). Denoting $\ln(w)$ in Eq. (6h) as u implies that,

$$d\big(\ln^{-1}(u)\big)/du = \ln^{-1}(u), \tag{6i}$$

for all real u which are in the domain of $\ln^{-1}(u)$. Eqs. (6g) and (6i) for $\ln^{-1}(u)$ *exactly correspond to* Eqs. (5a) *and* (5b) *for* $\exp(u)$, which we have pointed out *completely determine* $\exp(u)$. Eqs. (6g) and (6i) *likewise completely determine* $\ln^{-1}(u)$ since they imply that $d^k\ln^{-1}(u)/du^k|_{u=0} = 1,\; k = 0, 1, 2, 3, \ldots$, which yields that the Taylor expansion of $\ln^{-1}(u)$ around $u = 0$ is,

$$\ln^{-1}(u) = \textstyle\sum_{k=0}^{\infty} u^k/k!, \tag{6j}$$

which converges for all real u; *thus the domain of* $\ln^{-1}(u)$ *encompasses all real* u. Eqs. (6j) and (4a) imply,

$$\ln^{-1}(u) = \exp(u), \tag{6k}$$

so the fact that $\exp(u)$ is positive for all real u, which is demonstrated in the discussion below Eq. (5f), implies that $\ln^{-1}(u)$ is positive for all real u. This, together with the Eq. (6e) fact that $b^x = \ln^{-1}(x\ln b)$, removes the possible puzzlement from the implication of Eq. (6d) that b^x is necessarily positive. The upshot of Eqs. (6e), (6j) and (6k) is that,

$$b^x = \ln^{-1}(x\ln b) = \textstyle\sum_{k=0}^{\infty}(x\ln b)^k/k! = \exp(x\ln b), \tag{6l}$$

which together with the fact that $\ln b = \int_1^b db'/b'$ reproduces the formulas for b^x given by Eqs. (4b) and (3i). By first converting the set of linear first-order differential equations and initial conditions given by Eqs. (1b) and (1a) to the Eq. (6d) set of integral relations, which immediately imply Eq. (6e), the quite involved direct Taylor expansion of b^x around $x = 0$ carried out in Eqs. (3a) through (3i) is replaced by the simple Taylor expansion of $\ln^{-1}(u)$ around $u = 0$ given by Eq. (6j).

The logarithm is the power description. The exponential as a power function.

The positive power function b^x allows an arbitrary positive real number *w to be described by the power x which satisfies $b^x = w$*. Since $b^x = \exp(x \ln(b))$, $\exp(x \ln(b)) = w$ is solved for that power x, yielding,

$$x = \exp^{-1}(w)/\ln(b) = \ln(w)/\ln(b) \text{ (since ln } \textit{is the same as } \exp^{-1}\text{) when } w{>}0,\ b{>}0\ \&\ b{\neq}1.$$

The function symbol $\log_b(w)$ *is the standard denotation* for the power x which satisfies $b^x = w$, i.e.,

$$\log_b(w) = \ln(w)/\ln(b) \text{ when } w{>}0,\ b{>}0\ \&\ b{\neq}1;\ \log_b(w) \text{ satisfies } b^{\log_b(w)} = w. \tag{7}$$

The *particular b* for which $\exp(x)$ *is equal to the power function b^x* has traditionally been of interest, as has *the particular b* for which $\exp^{-1}(w) = \ln(w)$ is equal to the logarithm $\log_b(w)$ for positive w. Since $b^x = \exp(x \ln b)$ and $\log_b(w) = \ln(w)/\ln(b)$ for positive w, *the particular b for both cases satisfies* $\ln(b) = 1$. That particular b *has the standard denotation e*, and since $\ln(e) = 1$, $e = \exp(1)$, i.e.,

$$\begin{gathered}\{\ln(e) = 1\} \Rightarrow \{\exp(x) = e^x \ \ \&\ \ \ln(w) = \log_e(w) \text{ for positive } w\} \Rightarrow \\ \{e = \exp(1) = \textstyle\sum_{k=0}^{\infty} 1/k! = (\exp(-1))^{-1} = \left(\sum_{k=0}^{\infty}(-1)^k/k!\right)^{-1} = 2.718281828459045235\ldots\}.\end{gathered} \tag{8}$$