Tutorial: Generating All of the Real Powers from Two Simple Properties of Positive-Integer Powers

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Abstract

This tutorial parlays two simple properties of positive-integer powers of positive numbers into a closed formula for arbitrary real-valued powers of positive numbers. The two properties are the very familiar result for the derivative of a positive-integer power of a positive real variable, and the fact that positive-integer powers of unity equal unity. Extended to real values of the powers of positive numbers, these two properties comprise a set of first-derivative linear differential equations for functions of a positive real variable, together with their initial conditions, so it isn't surprising that a unique solution is in fact forthcoming via Taylor expansion methods. The additive property of multiplied powers emerges immediately from the two properties of powers that are initially assumed, and enters prominently into working out the closed-formula result, which turns out to be a creature of the exponential function and its inverse. Logarithms are defined in terms of the inverse operation of taking powers in the familiar way, and the special base whose powers reproduce the exponential function and whose logarithms reproduce the exponential function's inverse is pointed out and developed.

Two simple properties that fully describe powers

Positive-integer powers $j = 1, 2, 3, \ldots$ of a positive real-number base b, denoted b^j and defined as,

$$b^j \stackrel{\text{def}}{=} \overbrace{b \times b \times \cdots \times b}^{j\text{ times}},$$

have two simple properties whose extension generates all of the real powers: (a) powers of unity equal unity,

$$1^j = \overbrace{1 \times 1 \times \cdots \times 1}^{j\text{ times}} = 1,$$

and (b) derivatives of such powers with respect to their base b are given by the familiar simple result,

$$db^j/db = \lim_{\delta b\to 0}\big[\big((b+\delta b)^j - b^j\big)/\delta b\big] = \lim_{\delta b\to 0}\big[\big(b^j + \big(jb^{j-1}\big)\delta b + O\big((\delta b)^2\big) - b^j\big)/\delta b\big] = (j/b)b^j.$$

Extended from positive-integer j *to any real power* x of the positive real base b, these properties are,

$$1^x = 1, \tag{1a}$$

and,

$$db^x/db = (x/b)b^x, \tag{1b}$$

which is a set of linear first-derivative differential equations in the independent variable b for *real* power b^x solutions, together with the initial conditions $(b = 1)^x = 1$ of those solutions. Although the Eq. (1b) differential equations *are pathological at* $b = 0$, standard theorems guarantee well-behaved unique *real* power b^x solutions in the region $b > 0$ where their initial conditions have been specified by Eq. (1a).

The solutions b^x and b^y *of different equations* of the set specified by Eqs. (1b) and (1a) also have *a crucial fundamental relation* to the solution b^{x+y} *of still another equation* of that set, i.e.,

$$b^x b^y = b^{x+y}. \tag{2a}$$

Eq. (2a) is true because Eq. (1b) implies that,

$$d(b^x b^y)/db = ((x/b)b^x)b^y + b^x((y/b)b^y) = ((x+y)/b)(b^x b^y), \tag{2b}$$

and Eq. (1a) implies that,

$$(b^x b^y)_{b=1} = (1 \times 1) = 1. \tag{2c}$$

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The fundamental Eq. (2a) relation among solutions *of different equations* of the set enables a Taylor expansion around $x = 0$ *which encompasses all of the solutions* b^x *of the set*; that collective expansion of all of the solutions b^x of the set turns out to be well-defined and convergent for any specified *positive* value of b and any specified *real* value of x.

Such a Taylor expansion around $x = 0$ requires the coefficients $((d^k b^x/dx^k)_{x=0}/k!)$, $k = 0, 1, 2, 3, \ldots$, which themselves are not-yet known functions of b. From Eq. (2a), however, we see that,

$$b^{x=0}\, b^y = b^{0+y} = b^y \text{ for all } b^y, \text{ so } b^{x=0} = 1 \text{ is in fact } \textit{already known}. \tag{3a}$$

Eq. (2a) also implies *a crucial simplification of the derivative* db^x/dx,

$$db^x/dx = \lim_{\delta x\to 0}\big((b^{x+\delta x} - b^x)/\delta x\big) = \lim_{\delta x\to 0}\big((b^{\delta x} - 1)/\delta x\big)\, b^x. \tag{3b}$$

We evaluate $\lim_{\delta x\to 0}((b^{\delta x} - 1)/\delta x)$, which is a function of b only, via working out its derivative with respect to b and obtaining its value at $b = 1$. Since from Eq. (1a), $1^{\delta x} = 1$ for all values of δx, we see that,

$$\big[\lim_{\delta x\to 0}\big((b^{\delta x} - 1)/\delta x\big)\big]_{b=1} = \lim_{\delta x\to 0}\big((1^{\delta x} - 1)/\delta x\big) = 0. \tag{3c}$$

We calculate the derivative with respect to b of $\lim_{\delta x\to 0}((b^{\delta x} - 1)/\delta x)$ by interchanging the $\delta x \to 0$ limit with the differentiation with respect to b and then applying Eq. (1b) to carry out that differentiation,

$$d\big(\lim_{\delta x\to 0}\big((b^{\delta x} - 1)/\delta x\big)\big)/db = \lim_{\delta x\to 0}\big(((\delta x/b)b^{\delta x})/\delta x\big) = \lim_{\delta x\to 0}\big(b^{\delta x}/b\big) = (1/b). \tag{3d}$$

Since from Eqs. (3d) and (3c), $d(\lim_{\delta x\to 0}((b^{\delta x} - 1)/\delta x))/db = (1/b)$ and $[\lim_{\delta x\to 0}((b^{\delta x} - 1)/\delta x)]_{b=1} = 0$,

$$\lim_{\delta x\to 0}\big((b^{\delta x} - 1)/\delta x\big) = \textstyle\int_1^b db'/b', \tag{3e}$$

which is well-defined for positive b, and is universally denoted as $\ln b$. Eqs. (3e) and (3b) thus yield,

$$db^x/dx = \big(\textstyle\int_1^b db'/b'\big)\, b^x = (\ln b)\, b^x, \tag{3f}$$

which implies,

$$d^k b^x/dx^k = (\ln b)^k\, b^x,\ k = 0, 1, 2, 3, \ldots, \tag{3g}$$

and since $b^{x=0} = 1$,

$$\big(d^k b^x/dx^k\big)_{x=0} = (\ln b)^k,\ k = 0, 1, 2, 3, \ldots, \tag{3h}$$

yielding the Taylor expansion of b^x around $x = 0$ as,

$$b^x = \textstyle\sum_{k=0}^\infty (x \ln b)^k/k! = \sum_{k=0}^\infty \big(x \int_1^b db'/b'\big)^k/k!, \tag{3i}$$

a sum which is well-defined and convergent for any specified positive value of the base b and any specified real value of the power x. At $b = 1$ we see that $\ln b = \int_1^b db'/b'$ vanishes, so Eq. (3i) yields,

$$1^x = 1, \tag{3j}$$

in accord with Eq. (1a). Moreover, upon differentiating b^x with respect to b Eq. (3i) yields,

$$db^x/db = (x/b) \textstyle\sum_{k=1}^\infty (x \ln b)^{k-1}/(k-1)! = (x/b) \sum_{l=0}^\infty (x \ln b)^l/l! = (x/b)b^x, \tag{3k}$$

in full accord with Eq. (1b). Thus not only is Eq. (3i) a sum which is well-defined and convergent for any specified positive value of b and any specified real value of x; it is as well the complete solution of the differential equation set given by Eqs. (1b) and (1a). So Eq. (3i) by itself fully achieves the goal of extending the positive-integer powers j, $j = 1, 2, 3, \ldots$ of the positive numbers to any real-valued power x whatsoever of the positive numbers.

That said, it is nevertheless interesting *to understand in more explicit detail how the function* $\ln b = \int_1^b db'/b'$ *meshes with the infinite sum in* Eq. (3i) to achieve that goal.

Relating the power formula's infinite sum to the integral in each term

The *structure of the infinite sum* in Eq. (3i) is what is universally known as *the exponential function*,

$$\exp(u) \stackrel{\text{def}}{=} \textstyle\sum_{k=0}^{\infty} u^k/k!. \tag{4a}$$

whose Eq. (4a) series converges for all real u. We see from Eq. (3i) that,

$$b^x = \exp(x \ln b) = \exp\big(x \textstyle\int_1^b db'/b'\big). \tag{4b}$$

From Eq. (4a) we note that,

$$\exp(u = 0) = 1, \tag{5a}$$

and,

$$d\exp(u)/du = \textstyle\sum_{k=1}^{\infty} u^{k-1}/(k-1)! = \sum_{l=0}^{\infty} u^l/l = \exp(u), \tag{5b}$$

so *the derivative of* $\exp(u)$ *is equal to itself*. Eqs. (5a) and (5b) *completely determine* $\exp(u)$ since they imply that $d^k\exp(u)/du^k|_{u=0} = 1,\ k = 0, 1, 2, 3, \ldots$, which immediately yields Eq. (4a). They also allow us to show that the exp function *has the crucial property*,

$$\exp(v_1)\exp(v_2) = \exp(v_1 + v_2), \tag{5c}$$

because,

$$\begin{gathered}(\exp(v_1u)\exp(v_2u))_{u=0} = 1 \ \text{ and } \ d(\exp(v_1u)\exp(v_2u))/du = \\ (v_1\exp(v_1u))\exp(v_2u) + \exp(v_1u)(v_2\exp(v_2u)) = (v_1 + v_2)(\exp(v_1u)\exp(v_2u)),\end{gathered} \tag{5d}$$

which implies that,

$$d^k(\exp(v_1u)\exp(v_2u))/du^k|_{u=0} = (v_1 + v_2)^k,\ k = 0, 1, 2, 3, \ldots, \tag{5e}$$

and therefore,

$$(\exp(v_1u)\exp(v_2u)) = \textstyle\sum_{k=0}^{\infty}(v_1 + v_2)^k u^k/k! = \exp((v_1 + v_2)u), \tag{5f}$$

so $\exp(v_1)\exp(v_2) = \exp(v_1 + v_2)$ follows from the $u = 1$ case. Application of this identity to $b^x = \exp(x \ln b)$ immediately yields the crucial fundamental power relation $b^x b^y = b^{x+y}$ that is given by Eq. (2a). This relation *by itself*, albeit crucial, *doesn't ensure that* $b^1 = b$, a gap *which is filled by the* $\ln b$ *function*. Since $b^x = \exp(x \ln b)$, $b^1 = \exp(\ln b)$, *so to ensure that* $b^1 = b$*, the* $\ln$ *function must be the inverse of the* exp *function*.

This raises the issue that the exp function *is obligated to in fact have an inverse*, so $\exp(u)$ must be strictly increasing (or strictly decreasing) *at every real value of its argument* u. Since the *derivative* of the exp function *is equal to itself*, $\exp(u)$ *is strictly increasing at all real values* u *where it is positive*. All positive values of u are in this category because all of the terms of the series for $\exp(u)$ are positive when u is positive, and $\exp(u)$ is positive as well at $u = 0$, where its value equals unity. To understand the nature of $\exp(u)$ at the negative values of u, we note that Eq. (5f) implies that $\exp(u)\exp(-u) = \exp(0) = 1$. Thus $\exp(-u) = (1/\exp(u))$ so the fact that $\exp(u)$ is positive at positive values of u means that it is positive as well at all negative values of u, so it is strictly increasing *at all values of* u. Therefore $\exp(u)$ *definitely does have an inverse*, albeit that inverse is *only* required to be a well-defined *at the positive values of its own argument*. If we denote the inverse of $\exp(u)$ as $\exp^{-1}(w)$, then $\exp^{-1}(w)$ need *only* be well-defined on the domain $w > 0$, *but it definitely must satisfy*,

$$\exp^{-1}(\exp(u)) = u, \tag{5g}$$

which implies *in particular* that,

$$\exp^{-1}(w = 1) = 0. \tag{5h}$$

Moreover, since $d\exp(u)/du = \exp(u)$, the chain-rule differentiation of both sides of Eq. (5g) with respect to u produces,

$$d\big(\exp^{-1}(\exp(u))\big)/d\exp(u) = (1/\exp(u)), \tag{5i}$$

which, when more transparently expressed in terms of positive w in lieu of $\exp(u)$, reads,

$$d\big(\exp^{-1}(w)\big)/dw = (1/w),\ \text{when } w \text{ is positive.} \tag{5j}$$

When w is positive, Eqs. (5j) and (5h) together uniquely determine that,

$$\exp^{-1}(w) = \int_1^w dw'/w', \tag{5k}$$

so the Eq. (5k) inverse $\exp^{-1}$ of the exp function *is precisely the* ln *function which enters into the* Eq. (3i) *expression for* b^x *when* b *is positive*; note that in Eq. (3i) $\ln b$ *is specifically pointed out to be* $\int_1^b db'/b'$.

The power content is the logarithm. The exponential function's inherent base.

Once the power idea is extended from the positive integers *to all real numbers*, it is the case that for a given positive base other than unity, an arbitrary positive number *uniquely corresponds to the real-valued power of that base which is equal to that positive number*. Such a mapping of positive numbers *to the powers which produce them with a given positive base* is the *inverse* of the power function b^x we have so far been discussing: b^x uses the given positive base b to map real-valued powers x *into positive numbers* b^x, where (see Eq. (4b)),

$$b^x = \exp(x \ln b). \tag{6a}$$

It is straightforward to *invert* Eq. (6a) by noting that the inverse exponential function $\exp^{-1}$ is exactly the same as the ln function, so,

$$x = \ln(b^x)/\ln(b).$$

However, the positive entities b^x *are now switched to the role of arbitrary positive inputs* w, whereas *the consequent real-valued powers* x *of the base* b *are of course switched to the role of the dependent outputs*, which circa 1616 were in Latinate fashion christened "logarithms", and therefore are written $\log_b(w)$. Thus *the above inappropriate presentation of the inverse*, namely $x = \ln(b^x)/\ln(b)$, *is properly rectified to*,

$$\log_b(w) = \ln(w)/\ln(b) \ \text{ when } w > 0. \tag{6b}$$

Eq. (6b) makes it *apparent* that the base $b = 1$ *isn't viable for logarithms*.

The exponential function $\exp(u)$ and its *inverse* $\exp^{-1}(w) = \ln w = \int_1^w dw'/w'$, which is a well-defined real-valued function only when $w > 0$, *are the sole function ingredients* which enter into *the mutually inverse functions* that take a base b to a real-valued power x, namely $b^x = \exp(x \ln b)$, and that give the real-valued *power* of a base b that corresponds to a positive number w, namely its logarithm $\log_b(w) = \ln(w)/\ln(b)$.

The particular value of the base b *for which* $b^x = \exp(x)$ has traditionally been of interest, as has *the particular value of the base* b *for which* $\log_b(w) = \ln(w)$ *when* $w > 0$. Since $b^x = \exp(x \ln b)$ and $\log_b(w) = \ln(w)/\ln(b)$ when $w > 0$, *the particular value of the base* b *in both cases is such that* $\ln(b) = 1$; therefore that base's value b equals $\exp(1)$, *which is universally denoted* e,

$$\begin{gathered}\ln(e) = 1 \ \text{ implies that } \ e^x = \exp(x) \ \text{ and also implies that } \ \log_e(w) = \ln(w) \text{ when } w > 0; \ \text{ moreover,} \\ e = \exp(1) = \textstyle\sum_{k=0}^{\infty} 1/k! = (\exp(-1))^{-1} = \left(\sum_{k=0}^{\infty}(-1)^k/k!\right)^{-1} = 2.718281828459045235\ldots.\end{gathered} \tag{7}$$