Definition Let G be a group and $Z(G)$ be the center of G.

Theorem 1 If $|G| = p^n$, p a prime number, and $H \neq G$ is a subgroup of G, then there exists an $x \in G, x \notin H$ such that $x^{-1}Hx = H$.

Proof.

The proof is by induction on n. The statement is true for $n = 1$. Suppose the result is true for $n - 1$. Let G be a group of order p^n and $H \neq G$ be its subgroup. By Theorem 2.11.2 in [1], $Z(G) \neq (e)$. Since $|Z(G)| > 1$ and $Z(G)$ is a subgroup of G, $|Z(G)| = p^k, 1 \leqq k \leqq n$. If $Z(G)$ is not a subset of H, then $\{g \in G | g^{-1}Hg = H\} \neq H$. Suppose not. Then $Z(G) \subset \{g \in G | g^{-1}Hg = H\} \subset H$, a contradiction. Now assume that $Z(G) \subset H$. Since p divides $|Z(G)|$, by Cauchy's theorem, $Z(G)$ has an element $b \neq e$ of order p. Let B be the subgroup of G generated by b. So $|B| = p$. Since $b \in Z(G), B$ must be normal in G. Consider the quotient group G/B and its subgroup H/B. Since $|G/B| = p^{n-1}$, by the induction hypothesis, there is an element $X \in G/B, X \notin H/B$ such that $X^{-1}(H/B)X = H/B$. Since $X \in G/B, X = Bx$ for some $x \in G$. Thus $(Bx^{-1})(H/B)(Bx) = H/B$. Certainly $x \notin H$. Suppose $x \in H$. Then $X = Bx \in H/B$, a contradiction. Let $a \in x^{-1}Hx$. So $a = x^{-1}hx$ for some $h \in H$. Since $(Bx^{-1})(Bh)(Bx) \in (Bx^{-1})(H/B)(Bx)$ and $(Bx^{-1})(H/B)(Bx) \subset H/B$, it follows that $(Bx^{-1})(Bh)(Bx) \in H/B$. Hence $(Bx^{-1})(Bh)(Bx) = Bh'$ for some $h' \in H$. But $a \in (Bx^{-1})(Bh)(Bx)$ and $(Bx^{-1})(Bh)(Bx) \subset Bh'$. Thus $a \in Bh'$. So $a = b'h'$ for some $b' \in B$. To conclude $a \in H$ since $B \subset Z(G) \subset H$. Finally since $x^{-1}Hx \subset H$ and $|x^{-1}Hx| = |H|, x^{-1}Hx = H$.

References

[1] I. N. Herstein, *Topics in Algebra*, John Wiley & Sons, New York, 1975.

[2] I. M. Isaacs, *Algebra: A Graduate Course*, American Mathematical Society, Providence, 1994.