Symmetry and Asymmetry in Feature Optics

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Abstract

The slab and wedge act as *symmetry* and *asymmetry* transforms on the states of optical features. We interpret these as the *thermodynamic entropy* and the *information-theoretic entropy*, respectively. We show how slabs and wedges affect the beam and the grating.

1 Introduction

Feature optics (FO) was recently introduced[1] as a new framework for describing the beam and the grating, which are simple models of diffraction and interference. Reference 1 discussed both bright and dark features, but considered only one possible state for each type, mentioning only parenthetically that other states exist for both.

This essay considers all possible states for both bright and dark features. This will lead to a discussion of the *transforms* which may act on the states: the slab and wedge transforms. These will also turn out to embody the *symmetries* and *asymmetries* of the features.

The mathematical language of this essay is *group theory*, which is a rich and deep subject on its own (this paper does not give any background to group theory). The important calculations are all implemented in Matlab code, which is available on Github[2].

2 Features and their states

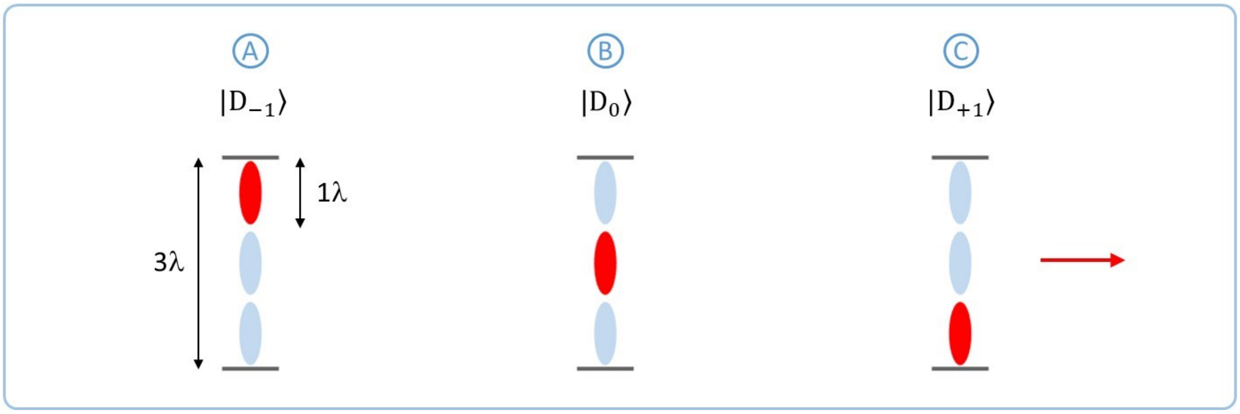
2.1 Dark features

A dark feature of size *n* may exist in any of *n* possible position states. We express these through the quantum-mechanical state vectors $|\mathrm{D_p}\rangle$. In our case of *n*=3, $\mathrm{p} \in \{ -1, 0, 1 \}$ and

$$|D_{-1}\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad |D_0\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad |D_{+1}\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

These three states correspond to the three diagrams in Figure 2.1. For example, $|D_{+1}\rangle$ refers to the state drawn in Figure 2.1c, in which the single bright patch is positioned in the lowest patch of space.

Figure 2.1, States of the dark feature



2.2 Bright features

A bright feature also may exist in any of *n* possible states, but these correspond to different *angles* of propagation. To simplify our discussion we define the *base phase*, which for a feature of size *n* is the number

$$r = e^{2\pi i/n}$$

Note that

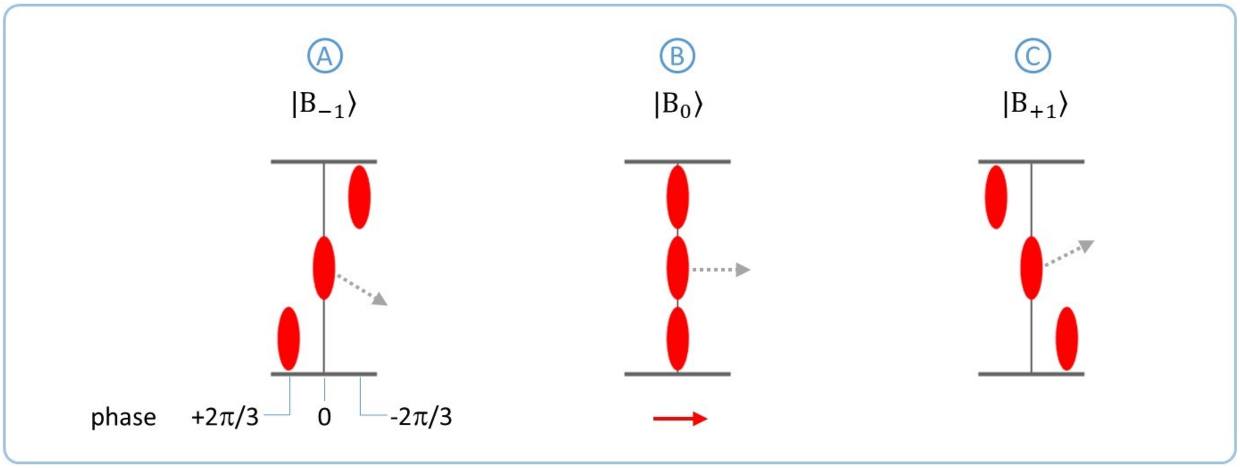
$$r^0 = 1$$
$$r^1 \cdot r^1 = r^2$$
$$r^{+1} \cdot r^{-1} = r^0$$
$$r^1 \cdot r^1 \cdot r^1 = r^2 \cdot r^1 = r^1 \cdot r^2 = r^0 \qquad \textit{(in this line only, n=3)}$$

The state vectors $|B_a\rangle$ are equally distributed in magnitude across the *n* patches. However, they differ in the phase relationships between patches. For our case of *n*=3:

$$|B_{-1}\rangle = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix}r^{-1}\\r^0\\r^{+1}\end{bmatrix}, \qquad |B_0\rangle = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix}r^0\\r^0\\r^0\end{bmatrix}, \qquad |B_{+1}\rangle = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix}r^{+1}\\r^0\\r^{-1}\end{bmatrix}$$

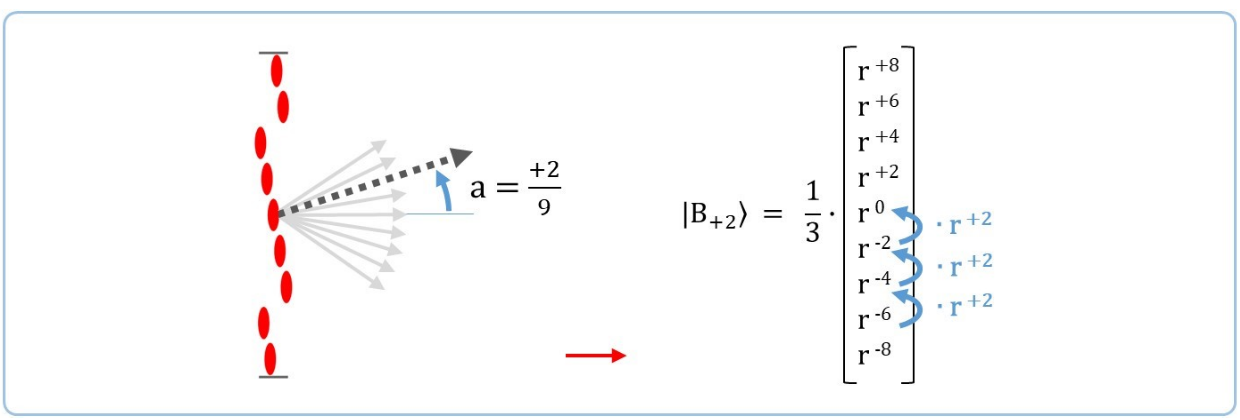
We draw these states below in Figure 2.2. In these diagrams the horizontal position of each patch represents its phase, not its spatial position. Note that the sense of the phase is positive to the left, negative to the right. The light grey arrow shows the angle of propagation.

Figure 2.2, States of the bright feature



The logic of the phase exponents is clearer in a case with a greater number of patches, say *n*=9 (within the scope of this example, r = $e^{2\pi i/9}$ correspondingly). Figure 2.3 shows that in the state vector $|B_{+2}\rangle$, each entry is a factor of r^{+2} greater than the entry below it. For vector $|B_{+3}\rangle$ the factor would be r^{+3}, etc. Each state vector discretely approximates a sinusoid of a different spatial frequency, and each propagates forward at a different angle.

Figure 2.3, Angle of bright feature

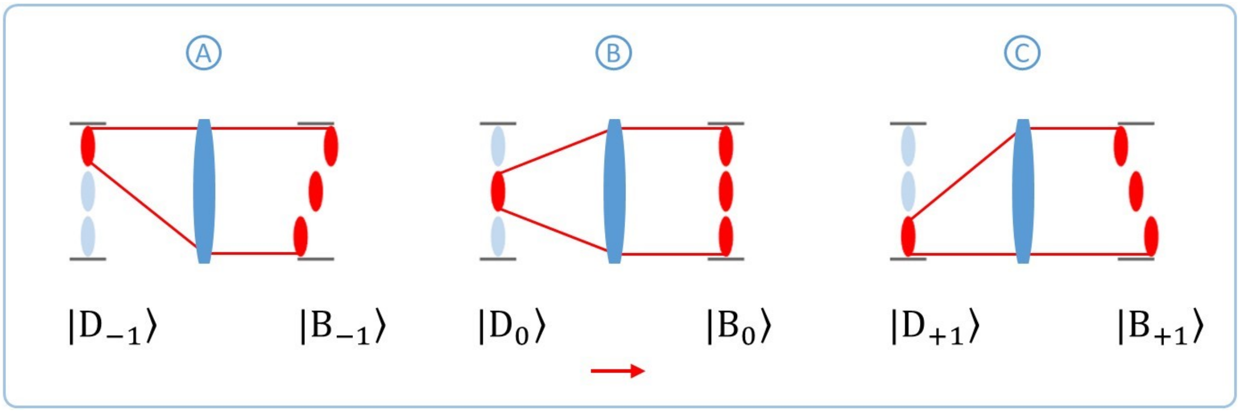


This diagram also shows that the phase may advance through multiple cycles. Once the phase exceeds the maximum of +π, it 'wraps around' to -π and continues to increase.

2.3 The FT of a feature

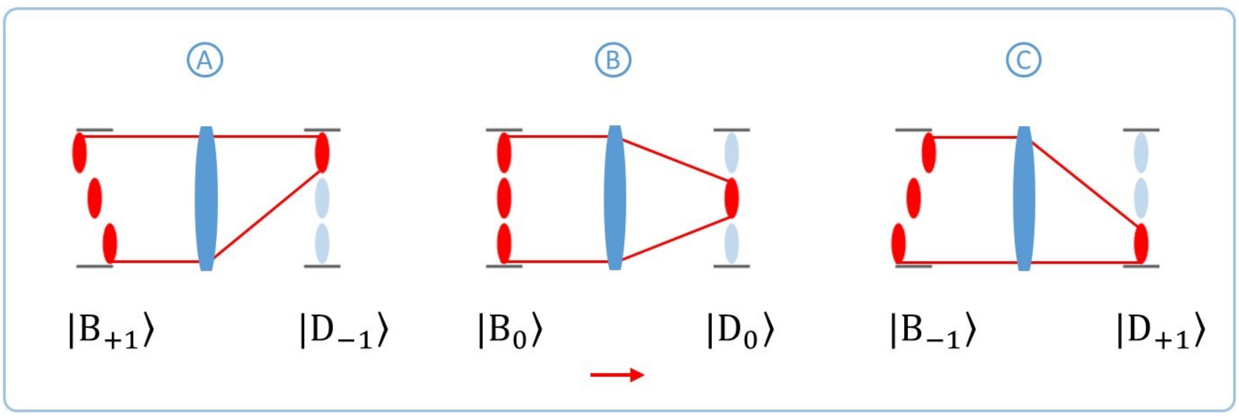
When a dark feature propagates from the input plane of a 2f system to the output plane, it is Fourier-transformed into a bright feature with the corresponding index, see Figure 2.4.

Figure 2.4, FT from dark to bright



Because of the front-back symmetry of the 2f system, the same rule applies in reverse with a small modification: For a bright feature as input, the output index is the *negative* of the input index, see Figure 2.5.

Figure 2.5, FT from bright to dark



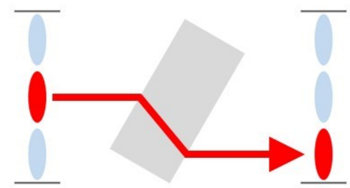
3 Slab and wedge transforms

3.1 Slab and wedge as changes, or asymmetries

The slab and wedge are optical devices used to change light from one state to another. We call such changes *asymmetrical*, because the final state is different from the initial state.

A *slab* is a thick optic with parallel flat faces (see Figure 3.1), placed at an angle in the path of the light. The light refracts at the first surface, propagates at an angle, and then refracts again at the second surface so that it emerges parallel to its original propagation, but displaced laterally.

Figure 3.1, Slab transform on dark state



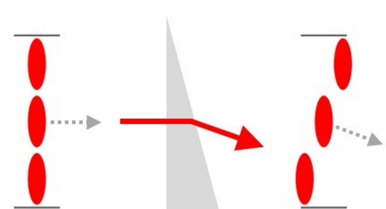
The action of the slab is represented by the matrix S. When it is applied to a dark feature in any position state $|D_p\rangle$, it shifts it to the next position state $|D_{p+1}\rangle$. For example,

$$S\cdot|D_0\rangle = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ r^0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r^0 \end{bmatrix} = |D_{+1}\rangle$$

(Note that this mathematical slab differs from a real physical slab, in that the mathematical slab 'wraps around' the limits of the space discontinuously, shifting the last patch into the first.)

The *wedge* is a prism, or optic whose thickness varies linearly with position, as shown in Figure 3.2. The high optical index of the glass delays the phase of the light, effectively multiplying each patch by a different phase factor proportional to the wedge thickness at that position. When applied to a bright state $|B_a\rangle$, it has the physical effect of tilting the wavefront to $|B_{a-1}\rangle$, the next angle in the negative direction.

Figure 3.2, Wedge transform on bright state



The action of the wedge is represented by the matrix W. Operating on the state vector $|B_0\rangle$, it tilts the light to $|B_{-1}\rangle$.

$$W\cdot|B_0\rangle = \begin{bmatrix} r^{-1} & 0 & 0 \\ 0 & r^0 & 0 \\ 0 & 0 & r^{+1} \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} r^0 \\ r^0 \\ r^0 \end{bmatrix} = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} r^{-1} \\ r^0 \\ r^{+1} \end{bmatrix} = |B_{-1}\rangle$$

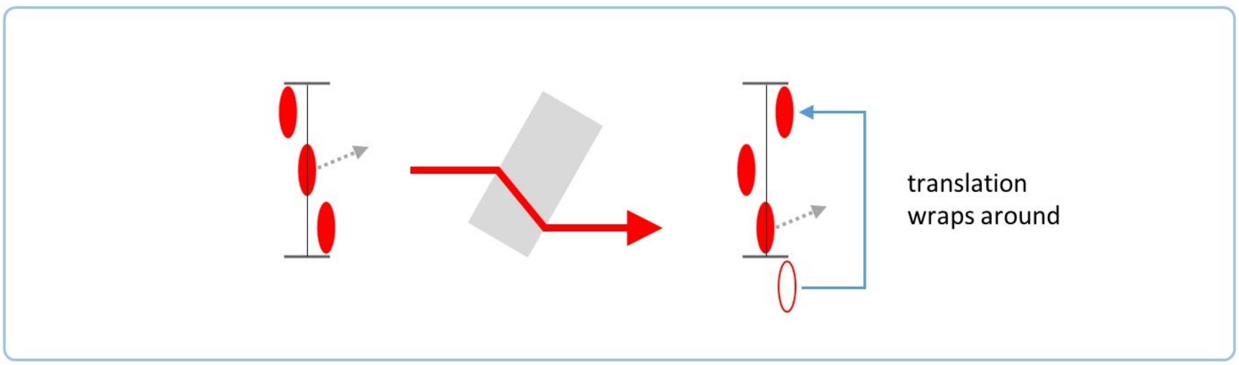
The matrices S and W are also referred to in the literature as *shift* and *clock* matrices, respectively[3].

Note that for both the slab and the wedge, the input and output planes may be arbitrarily close together in Z, limited only by practical constraints. Our model considers the transforms to occur abruptly at a single plane. This is unlike the 2f system, which requires a propagation distance *f* on each side of a lens in order to effect the FT.

3.2 Slab and wedge as phases, or symmetries

When the slab is applied to the bright feature, it translates each patch to the next position, just as it did for the dark feature in the previous section (see Figure 3.3). However, this does not move the feature into the next state.

Figure 3.3, Slab transform on bright state



Rather, the effect is to multiply the state vector by an *scalar phase factor* applied equally to all elements of the vector. For example, applying the transform S to the tilted wavefront $|B_{+1}\rangle$,

$$S\cdot|B_{+1}\rangle = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} r^{+1} \\ r^{0} \\ r^{-1} \end{bmatrix} = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} r^{-1} \\ r^{+1} \\ r^{0} \end{bmatrix}$$

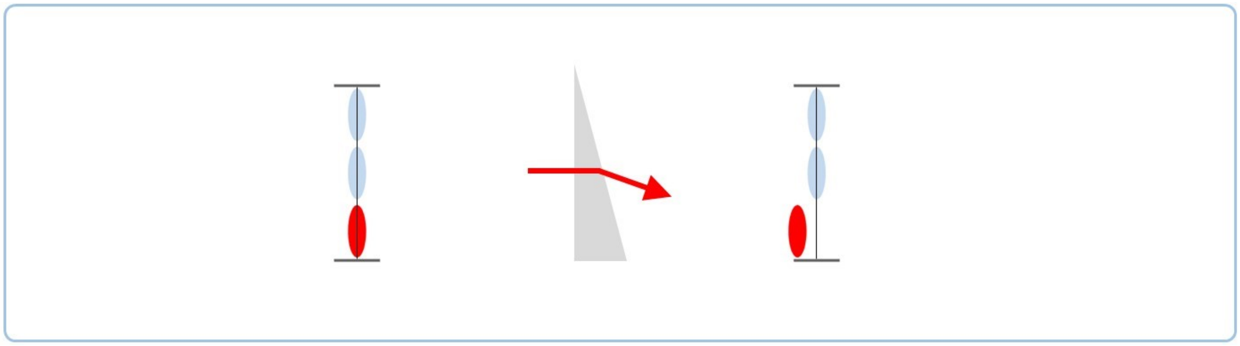
$$= r^{+1} \cdot |B_{+1}\rangle$$

We see that the wavefront $|B_{+1}\rangle$ has undergone a scalar phase change of r^{+1}. If we instead used the wavefront $|B_{-1}\rangle$, the phase change would be r^{-1}, etc.

If the only effect of a matrix on a given vector is to apply a *scalar factor* to the vector, then that vector is an *eigenvector* of the matrix, and the scalar factor is the corresponding *eigenvalue*. We call such changes *symmetrical* changes. The bright angle states are eigenvectors of S.

An analogous situation holds for the wedge transform operating on a dark feature (see Figure 3.4). The position states are eigenvectors of the wedge matrix. A different phase factor is applied at each position because of the varying thickness of the wedge.

Figure 3.4, Wedge transform on dark state



For example, applied to a dark feature in state $|D_{+1}\rangle$, we calculate that

$$W \cdot |D_{+1}\rangle = \begin{bmatrix} r^{-1} & 0 & 0 \\ 0 & r^{0} & 0 \\ 0 & 0 & r^{+1} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ r^{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r^{+1} \end{bmatrix}$$

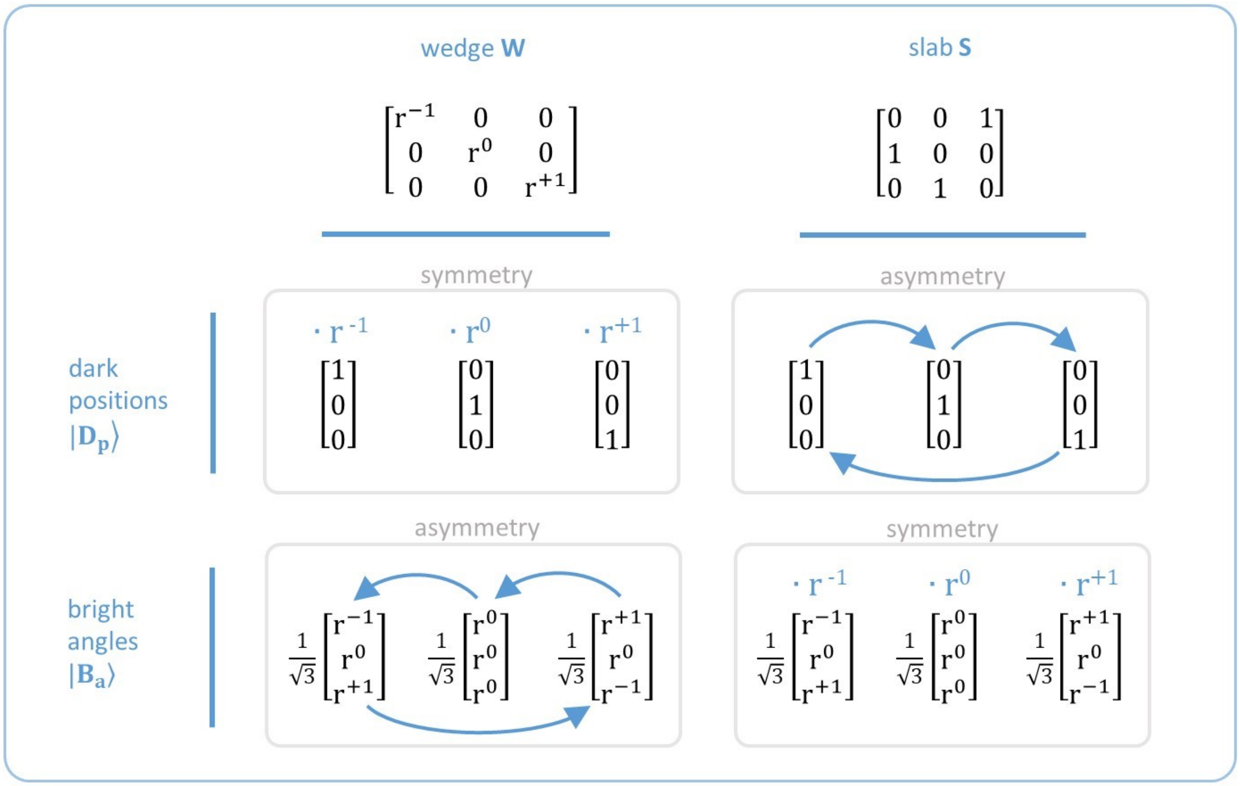
$$= r^{+1} \cdot |D_{+1}\rangle$$

The dark position states are eigenvectors of W.

3.3 Slab and wedge as a complementary pair

Figure 3.5 summarizes the complementary relationship between the matrices S and W and their respective sets of eigenvectors. For dark state vectors $|D_p\rangle$, S is the asymmetry while W is the symmetry. For bright state vectors $|B_a\rangle$, the roles are precisely flipped; W is the asymmetry while S is the symmetry.

Figure 3.5, Wedge and slab as complementary transforms



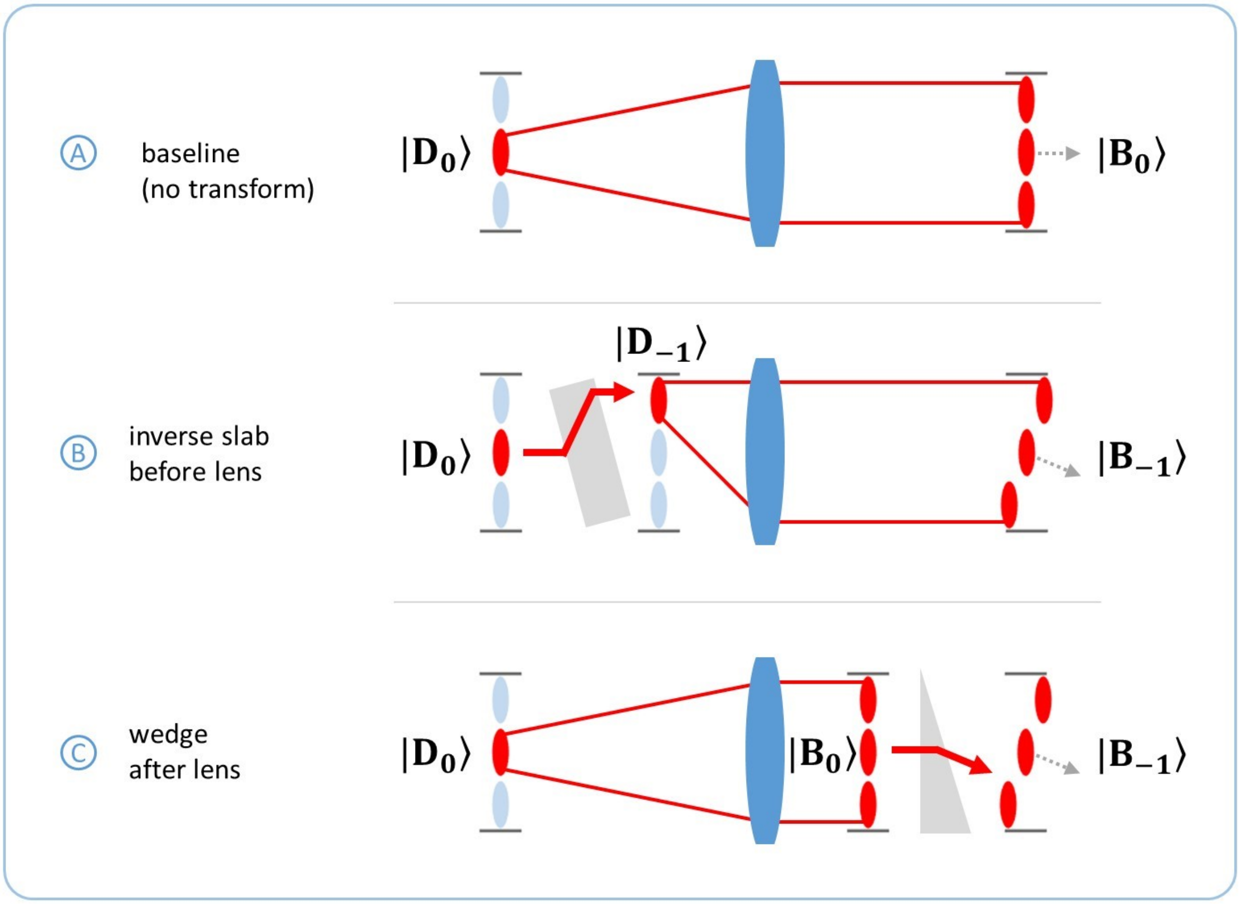
We saw above that the FT transforms a dark state into a bright state, and vice-versa. However, matrices W and S are themselves transforms, not state vectors. Therefore, the FT does *not* transform them by multiplication, but rather *conjugates* them into one another through a similarity transform.

$$\mathrm{FT}^{+1} \cdot \mathrm{W} \cdot \mathrm{FT}^{-1} = \mathrm{S}$$

$$\mathrm{FT}^{-1} \cdot \mathrm{S} \cdot \mathrm{FT}^{+1} = \mathrm{W}$$

A physical interpretation of this conjugation is given in Figure 3.6, where a dark feature lies in the input plane of a 2f system at position 0. Figure 3.6a shows the baseline case of no transform, where the output is a bright feature at angle 0. In Figure 3.6b, an inverse slab transform is applied *before* the lens to move the input to position -1, which transforms to angle -1 at the output. In Figure 3.6c, the light passes through the lens just as in the baseline case, but a wedge is applied to the *output* to tilt it to angle -1. Cases b and c yield the same result; this works for any input.

Figure 3.6, Wedge as conjugate of inverse slab



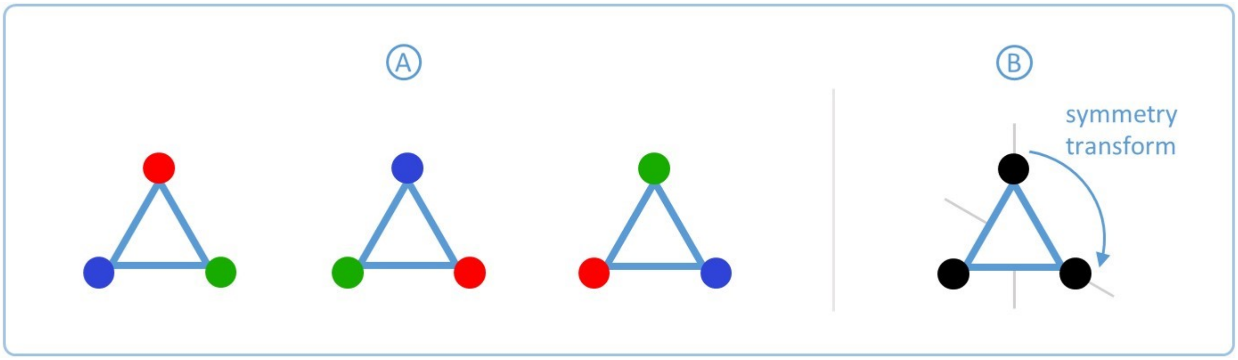
4 Groups, Symmetries and Asymmetries

4.1 Groups and entropy

Muller's remarkable book *Asymmetry: The Foundation of Information*[4] introduces a new way of using group theory as a mathematical language to describe *entropy* – not only in the thermodynamic sense, but also in the information-theoretic sense. This section reviews and summarizes Muller's method by applying it to a simple example.

Consider an equilateral triangle with its 3 vertices colored red, green, and blue respectively. Figure 4.1a shows this triangle in its 3 possible orientations. We assume that the triangle cannot rotate to any other angle, and cannot flip over; therefore, these 3 are the entire state space of the triangle.

Figure 4.1, Symmetries of the triangle



At first, we consider the case in which we can 'see color', i.e. we can easily distinguish the three states from one another by their colors, as in Figure 4.1a. In this case, an observation tells us precisely the state of the triangle.

However, we actually wish to model *imperfect observations* in which some information is unobservable. Now consider the same triangle and a *colorblind observer*. Without the colors to identify the orientations, the 3 states cannot be distinguished. The observer's knowledge is represented in Figure 4.1b: it is a shape with 3-fold rotational symmetry. It has 3 indistinguishable *microstates* and therefore a (thermodynamic) entropy of log(3).

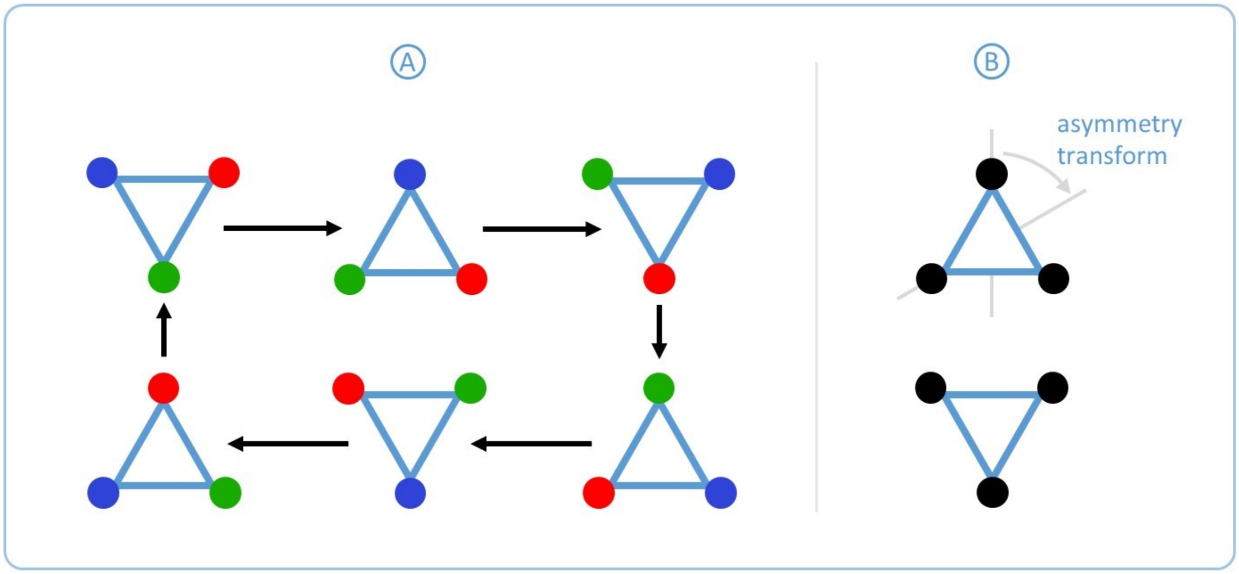
The observer's colorblindness is described by the symmetry transform, which rotates the triangle 1/3 of a full rotation. The transform generates a group of symmetry transforms: 1/3, 2/3, and 3/3 of a full rotation. The transforms in this group change the state from one indistinguishable microstate to another.

The essential point is that in order to discuss entropy, we must specify *both* the system under study *and* the capabilities of the observer (in Muller's terms, the 'observer' may be any IGUS – an Information Gathering and Using System).

Next, consider the symmetry transforms of a hexagon, which is exactly like the triangle but with more vertices. It has a larger symmetry group of 6 elements, a greater entropy of log(6), and the symmetry transform of 1/6 of a rotation.

Now, consider what happens if the hexagon symmetry group is applied to the triangle shape. Figure 4.2a shows the state space of 6 different possible microstates, with all colors visible. Figure 4.2b shows the states as seen by an observer who is colorblind, but *can* distinguish orientation. This observer partitions the state space into two subsets: those with a vertex facing up, and those with a vertex facing down.

Figure 4.2, Symmetries of the hexagon, applied to the triangle



We think of this in terms of group theory: as before, the triangle has the symmetry transform of a 1/3 rotation. However, it also has an *asymmetry* transform of 1/6 rotation, which changes it from up to down, or vice-versa.

The size of the asymmetry group equals the number of different *distinguishable* states, i.e. the number of subspaces into which the observer partitions the state space. This amounts to the same thing as Muller's equation for information (even though that equation is formulated using the different concept of stabilized points in a set):

$$I = \log\left(\frac{1}{|G|} \sum_{g \in G} |S^g|\right)$$

In Figure 4.2b, the asymmetry group is of order 2. The corresponding logarithm log(2) is also referred to as the *entropy* within the field of information theory, even though it has nearly the opposite meaning from entropy in thermodynamics. A typical application deals with the amount of information that a memory device can store. It may have *B* bits total, and each bit can be in two possible states; therefore, the device as a whole has the capability of being in 2^B different distinguishable states. The device 'carries information' by being confined to just one out of those 2^B possibilities. The triangle in the figure can carry one bit of information, and thus has log(2) of information-theoretic entropy.

Whichever distinguishable state the triangle is in, it has 3 possible indistinguishable microstates, a symmetry group of order 3, and thus a thermodynamic entropy of log(3)

4.2 Entropy of features

In this section, we apply the principles of symmetry and asymmetry to optical features.

In QM, the observer is always unable to distinguish two states that differ only by a scalar phase; it has no physical effect and cannot be measured. FO now postulates a new rule: that scalar phase factors must be quantized, appearing only as integer powers of r^1, the base phase (in standard QM, a state vector may have any arbitrary phase factor). This means that there are exactly *n* distinct phase factors that a state may have. For example, these 3 vectors are considered indistinguishable; they represent the same physical state $|D_0\rangle$:

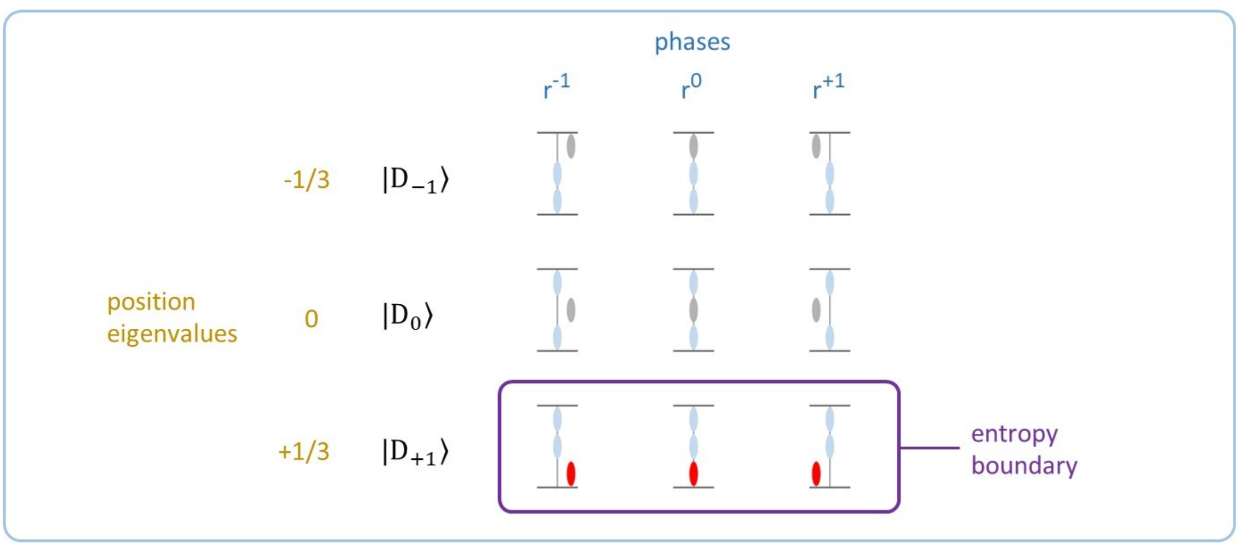
$$\begin{bmatrix} 0 \\ r^0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ r^{+1} \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ r^{-1} \\ 0 \end{bmatrix}$$

These 3 vectors all represent the same physical state $|B_0\rangle$:

$$\frac{1}{\sqrt{3}} \cdot \begin{bmatrix} r^0 \\ r^0 \\ r^0 \end{bmatrix}, \qquad \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} r^{+1} \\ r^{+1} \\ r^{+1} \end{bmatrix}, \qquad \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} r^{-1} \\ r^{-1} \\ r^{-1} \end{bmatrix}$$

We now list and count all possible *microstates*. One microstate exists for every pairing of a state and a phase. Thus, the total number of microstates of a feature is n^2, as shown in Figure 4.3 for the dark feature:

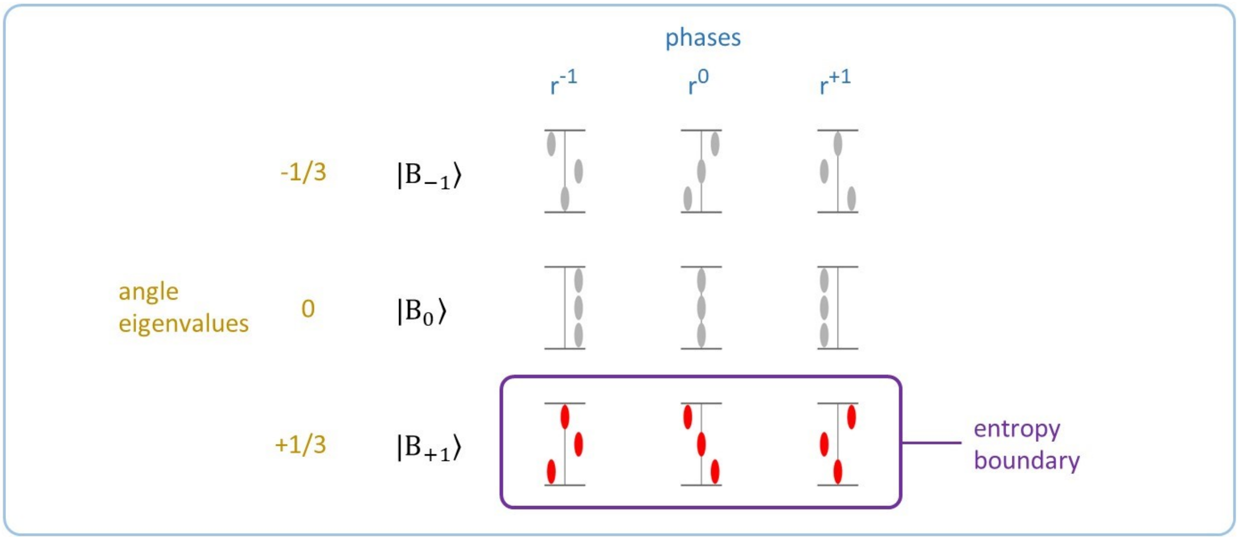
Figure 4.3, All dark microstates



Each row is tagged with a real eigenvalue. That number is physically observable, as we will discuss later in section 4.3. The eigenvalue refers to the position, *scaled* by the feature size; for example, a dark feature of 3 patches observed at eigenvalue +1/3 is located at (+1/3)·3 = the +1 patch.

The corresponding microstates for the bright feature are shown in Figure 4.4. The observable eigenvalues represent the angle in radians.

Figure 4.4, All bright microstates



The purple box is the *entropy boundary*. It represents the two opposite aspects of observability: We can observe *which row* the feature is in, i.e. we can observe its eigenvalue. But we cannot observe where it is *within the row*.

Slab S is the symmetry transform of the state $|B_{+1}\rangle$, because it moves the microstates around *within the entropy boundary*. S applies a phase factor, an imaginary eigenvalue such as r^2, to each microstate. This changes it to an indistinguishable microstate. S applies different eigenvalues to different rows.

Wedge W is the asymmetry transform of $|B_{+1}\rangle$. In principle, it is the transform that *would* move the entropy boundary from one row to the next, from one *observable eigenvalue* to the next. But 'knowing information' about an eigenvalue necessarily entails that the entropy boundary is remaining constant over time. This means that W is *not* acting to transform it.

Observation can not tell us whether or not S acts. However, we treat the system as though S has scrambled it in the past, such that we have no information about its phase in the present. That scrambling, or loss of information, is how thermodynamic entropy is created in the first place.

Finally, each of these transforms also generates its own *group*. If a slab transform of size *n* is applied *n* times to any state, dark or bright, it returns it to the starting state. In other words, S^n = S^0 = the identity matrix. Equivalently, the transforms S^1, S^2,… S^n form a commutative group which we call G_{Slab}. An analogous group G_{Wedge} also exists. For the case of *n*=3:

$$G_{Slab} = \{ S^0, S^{+1}, S^{-1} \}$$

$$G_{Wedge} = \{ W^0, W^{+1}, W^{-1} \}$$

4.3 Observation

Observation is built into the structure of QM. An observable takes the form of a self-adjoint matrix, meaning that it has a set of orthogonal eigenvectors along with a corresponding set of real eigenvalues. When we *observe* a system, i.e. make a measurement, the system is found in one of the eigenstates and the corresponding real eigenvalue is the measurement *result*.

To calculate the observable matrix, we start with the eigenvectors and their corresponding eigenvalues. Then, for each eigenvector we calculate the *projector* by taking the outer product of the eigenvector with itself, then scaling it by its eigenvalue. The sum of all the projectors is the observable matrix.

The position of a dark feature is observed with this matrix:

$$P_{observ} = \frac{1}{3} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The angle of a bright feature is observed with this matrix:

$$A_{observ} = \frac{1}{3\sqrt{3}} \cdot \begin{bmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{bmatrix}$$

The observable manifests physically as the interaction Hamiltonian between the light and the measurement apparatus[5,6] used by the observer. The apparatus is not made of light, but of matter. Each mode of the matter has a different energy, which is its eigenvalue. The spatial eigenstates of the light are coupled 1:1 to the energy eigenstates of the apparatus. A measurement result is manifested when a particle is detected at some known energy, from which we can infer the original spatial eigenvalue.

The energy of the detected particle determines its unitary evolution over time, following Schroedinger's equation. The principle is that each eigenstate of the Hamiltonian is incremented by a scalar phase factor at a rate proportional to its energy. To calculate the evolution over a given time interval Δt, we form the *time-evolution operator* by taking the matrix exponential of the Hamiltonian times $-i \cdot 2\pi \cdot \Delta t/h$. We find that when $\Delta t = -h$, the time-evolution operators are in fact the Wedge and Slab matrices. In other words, the measurement apparatus effectively applies S or W to the state, causing it to evolve over time.

$$e^{-2\pi i \cdot \Delta t/h} \cdot P_{observ} = \begin{bmatrix} r^{-1} & 0 & 0 \\ 0 & r^{0} & 0 \\ 0 & 0 & r^{1} \end{bmatrix} = W$$

$$e^{-2\pi i \cdot \Delta t/h} \cdot A_{observ} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = S$$

For the apparatus to yield a stable result, it must not change the property it is measuring; it must *preserve* it. So while we have been discussing the symmetry transforms that the apparatus causes, observability is actually defined by the *asymmetry* transforms that the apparatus does *not* cause. The symmetry transforms are not essential to the operation of the apparatus. Rather, they are the transforms that don't matter, and don't pose a problem for the apparatus.

4.4 Irreducible representations

The slab and wedge matrices commute with the angle and position observables, respectively. This is written as

$$S^{-1} \cdot A_{observ} \cdot S^{1} = A_{observ}$$

$$W^{-1} \cdot P_{observ} \cdot W^{1} = P_{observ}$$

Correspondingly, G_{Wedge} and G_{Slab} are the symmetry groups of their respective Hamiltonians. Following QM, each eigenstate of an observable is a basis for an *irreducible representation* (irrep) of its symmetry group. We can see the details in the character table[7] for G_{Slab}, shown in Figure 4.5 for a feature of size 3. Each row corresponds to an irreducible representation of G_{Slab}. For example the state vector $|B_{+1}\rangle$ carries a representation with the group of S^0, S^{+1}, S^{-1} represented by r^0, r^{+1}, and r^{-1}. While these appear to be scalars, they are actually best thought of as 1x1 matrices.

Figure 4.5, Character table for slab on bright feature

irreducible representations	group transforms: S^0	S^{+1}	S^{-1}
$\lvert \mathbf{B_0} \rangle$	r^0	r^0	r^0
$\lvert \mathbf{B_{+1}} \rangle$	r^0	r^{+1}	r^{-1}
$\lvert \mathbf{B_{-1}} \rangle$	r^0	r^{-1}	r^{+1}

It is tempting to assume that the character table is just like the microstate figures in section 4.2. However, there is a crucial difference: in these character tables, the columns are the group transforms. When S acts, it moves all 3 states 1 'click' to the right in the character table. But in the microstate figures, each row is transformed in a different way, because each one is incremented by a different phase factor.

Naturally, all these observations can be extended to the wedge, shown in Figure 4.6.

Figure 4.6, Character table for wedge on dark feature

irreducible representations	group transforms: W^0	W^{+1}	W^{-1}
$\lvert \mathbf{D_0} \rangle$	r^0	r^0	r^0
$\lvert \mathbf{D_{+1}} \rangle$	r^0	r^{+1}	r^{-1}
$\lvert \mathbf{D_{-1}} \rangle$	r^0	r^{-1}	r^{+1}

Each column is a different element of the group. In character tables generally, each column corresponds to a conjugacy class of the group. But since G_S is commutative, each element of the group is an entire conjugacy class.

5 The Beam in 2f

5.1 Outer products of transforms

A beam in FO is the outer product of 2 component features of size *m* and *n* respectively. We can write this outer product as a 2-d array of size *m* x *n*. Or, we can write it as a 1-d vector *m·n* long, with replicates of the low feature nested inside each patch of the high feature, which matches how it appears in physical space.

The 2-d array (or multidimensional array, for larger systems) is a better format for taking the FT. But for applying slab and wedge matrices, the 1-d nested vector is better because we can easily make outer products of matrices (the matrices are *m·n* x *m·n* 2-d arrays; they act on *m·n*–length 1-d vectors). Each basic transform is formed from the slab or wedge matrix of one feature, times the identity matrix I (no change) of the other feature(s). For the beam, there are four such matrices:

$$\mathrm{W_{High}} \otimes \mathrm{I_{Low}} = \begin{bmatrix} r^{-2} & 0 & 0 \\ 0 & r^{0} & 0 \\ 0 & 0 & r^{+2} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & r^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & r^{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & r^{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & r^{+2} & 0 \\ 0 & 0 & 0 & 0 & 0 & r^{+2} \end{bmatrix}$$

$$\mathrm{S_{High}} \otimes \mathrm{I_{Low}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathrm{I_{High}} \otimes \mathrm{W_{Low}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} r^{-3} & 0 \\ 0 & r^{0} \end{bmatrix} = \begin{bmatrix} r^{-3} & 0 & 0 & 0 & 0 & 0 \\ 0 & r^{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & r^{-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & r^{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & r^{-3} & 0 \\ 0 & 0 & 0 & 0 & 0 & r^{0} \end{bmatrix}$$

$$\mathrm{I_{High}} \otimes \mathrm{S_{Low}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

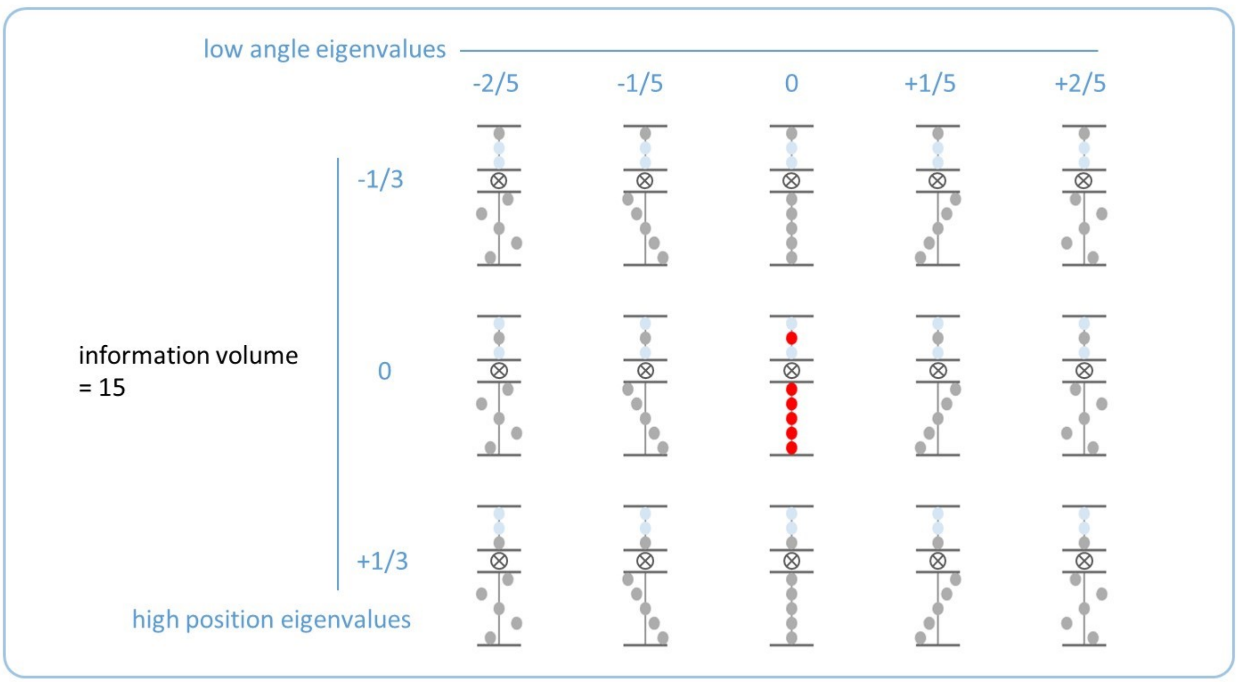
The matrices have a nested structure, just like the vectors that they act on. For example, the last of these examples shows the low slab, which is a 2x2 matrix with 1's in the off-diagonal positions. It is nested inside a 3x3 identity matrix with 1's along the diagonal. In the outer product we see a repeated series of 2x2 blocks along the diagonal, one at the position of each diagonal element of the high identity matrix.

Note that in this example r = $e^{2\pi i/6}$, because there are a total of $m \cdot n$ = 6 patches in the combined space.

5.2 Asymmetry transforms of the beam

Figure 5.1 shows the *information volume* of the beam. These are the $m \cdot n$ possible distinguishable states, and the beam is in just one of them. Each combined state is a combination of one distinguishable state from each feature.

Figure 5.1, Information volume of the beam



The asymmetry transforms of the beam change it to a distinguishably different state. This encompasses all practical applications of slab and wedge devices, since there is no practical need for symmetry transforms.

Figure 5.2a shows the action of the low wedge. The purple drawing shows the system before the wedge acts; the red drawing shows the system afterwards. Figure 5.2b shows the same action using feature diagrams. We see that the low wedge tilts the beam angle.

Figure 5.2, Changing beam by low wedge

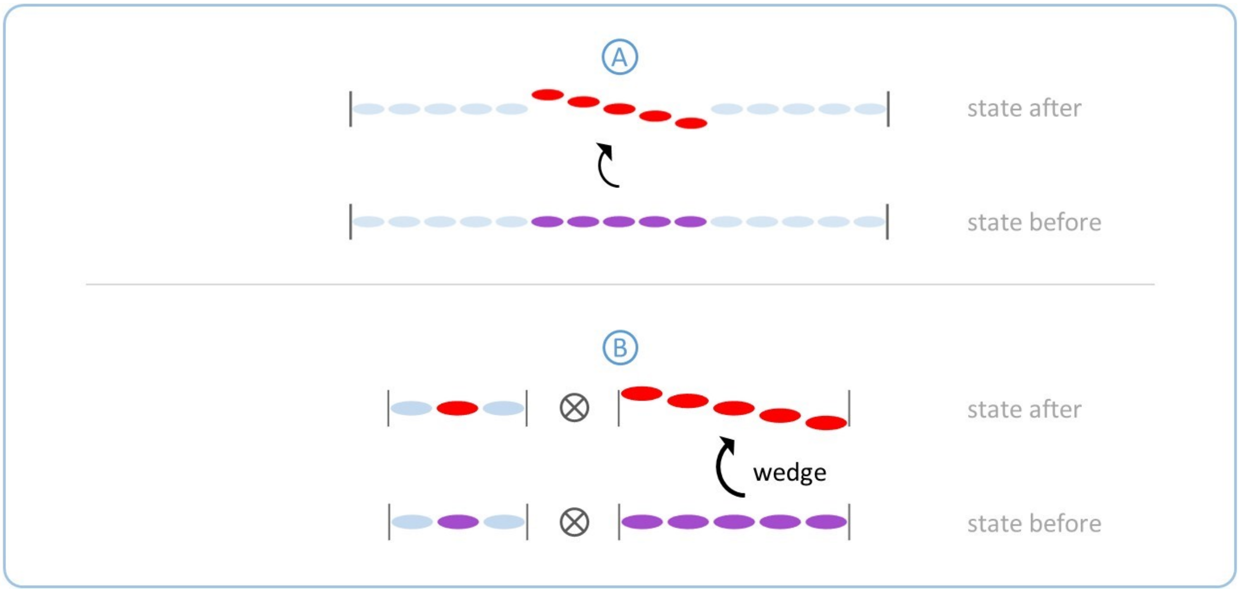
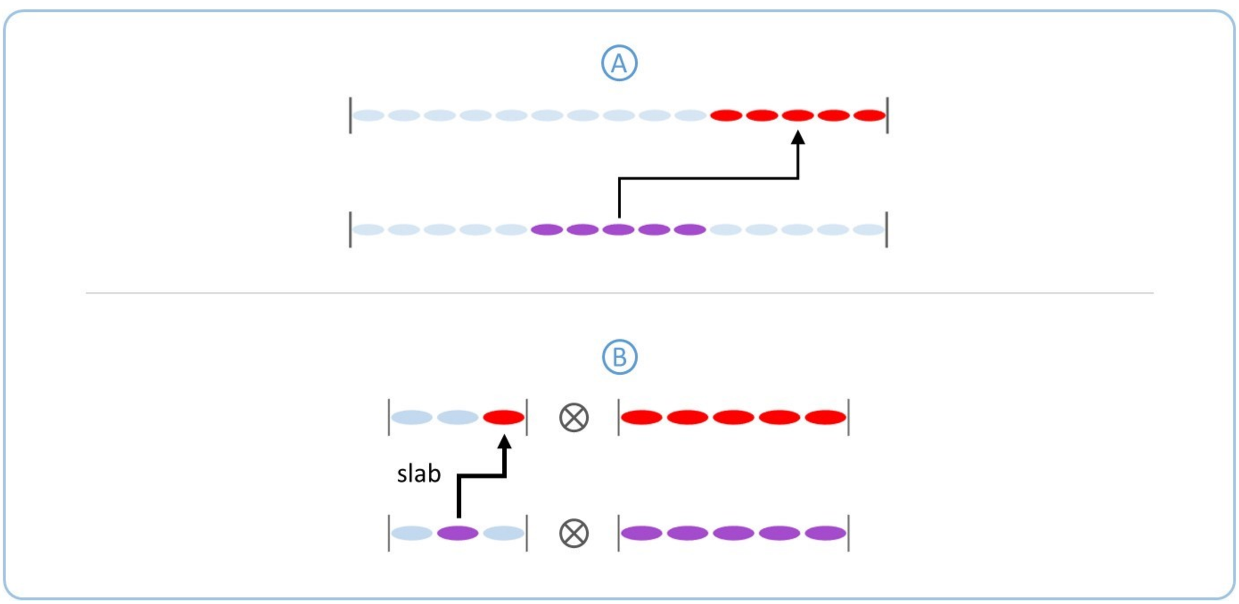


Figure 5.3 shows the action of the high slab. We see that it translates the beam to a different position.

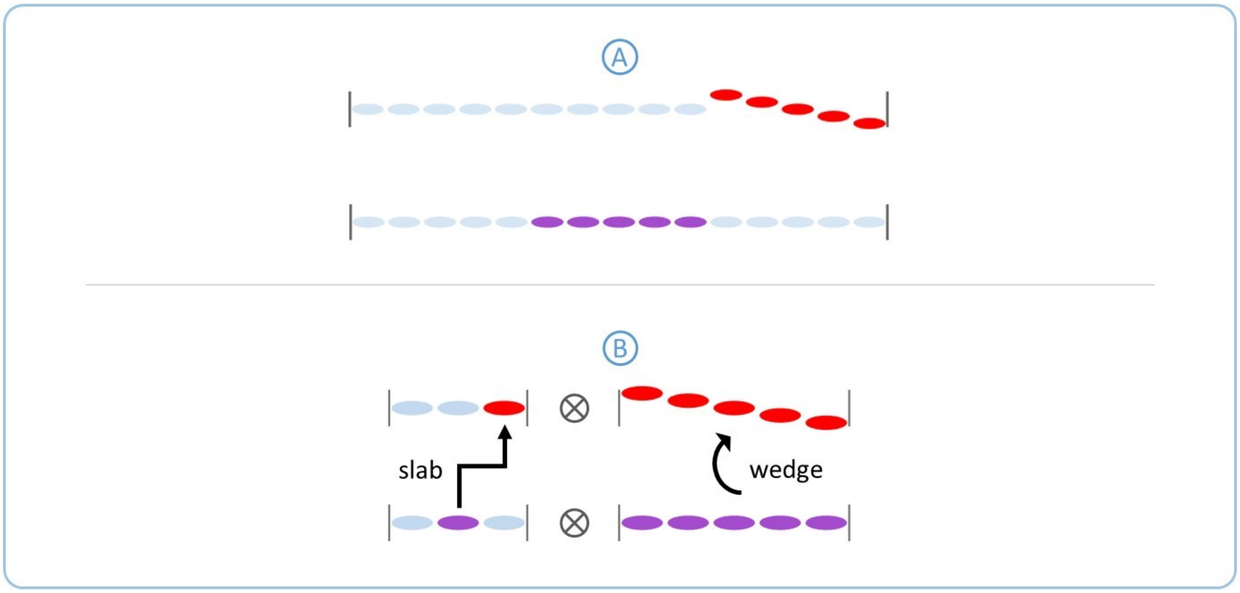
Figure 5.3, Changing beam by high slab



We remark that the slab is only permitted to translate intervals of one full high patch. While a smaller translation is physically possible, it takes the system outside the narrow and simple parameter space that is the scope of this exposition. Similarly, the wedge can only tilt to one of 3 permitted angles.

Figure 5.4 shows that multiple transforms can be combined. A matrix nested at one rank commutes with any matrix at a different rank.

Figure 5.4, Changing beam by two transforms



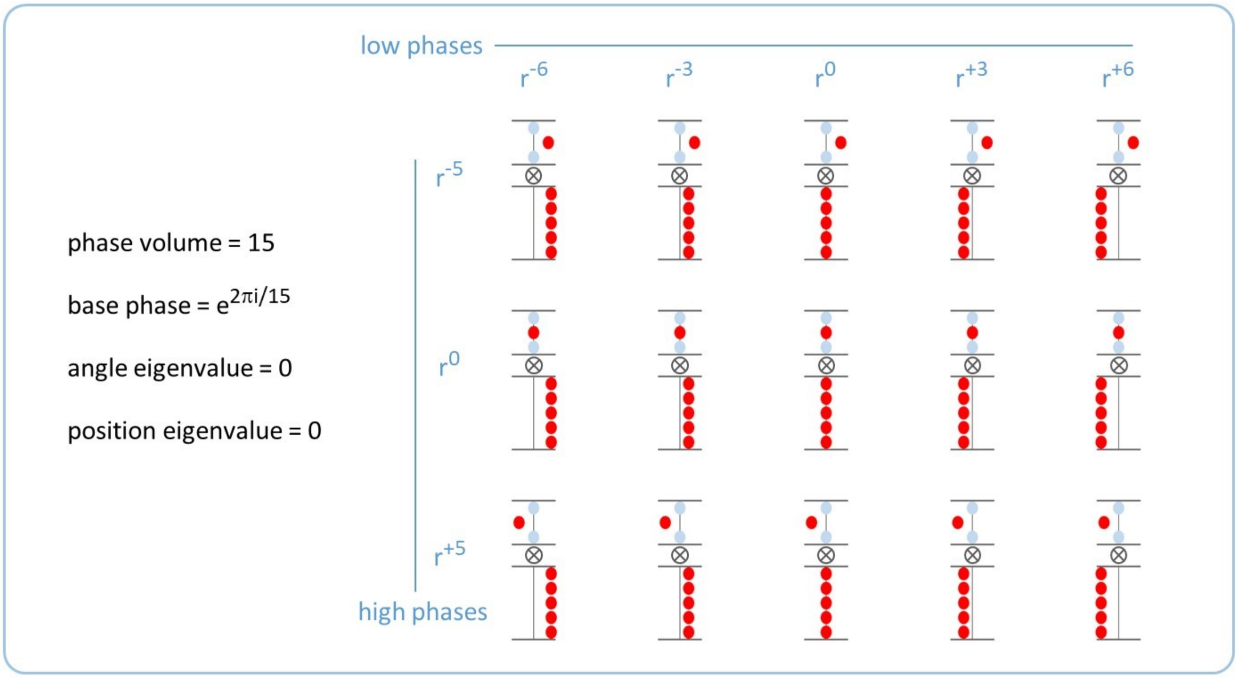
Each basic transform generates a group, and the direct product of these two is the full asymmetry group of the beam.

$$G_{Beam\_Asymmetry} = G_{Wedge\_Low} \otimes G_{Slab\_High}$$

5.3 Symmetry transforms of the beam

We saw earlier that in FO, the possible phase factors are restricted to the *n* powers of the base phase $e^{2\pi i/n}$. For a composite system of two factors *m* and *n* such as the beam, one power may be chosen independently for each feature. The two component phases multiply to form the combined phase. This yields a total of *m·n* phase factors, namely all the powers of the base phase $e^{2\pi i/(m\cdot n)}$. These are shown in
Figure 5.5. The value *m·n* is called the *phase volume*. It is the number of possible microstates that the system might be in – the exponential of the thermodynamic entropy.

Figure 5.5, Phase volume of the beam



The symmetry transforms of the beam are those which move the state of the beam *within* the phase volume. Equivalently, they are transforms for which the beam is an eigenstate. Figure 5.6 shows the low slab transform.

Figure 5.6, Phase of beam by low slab

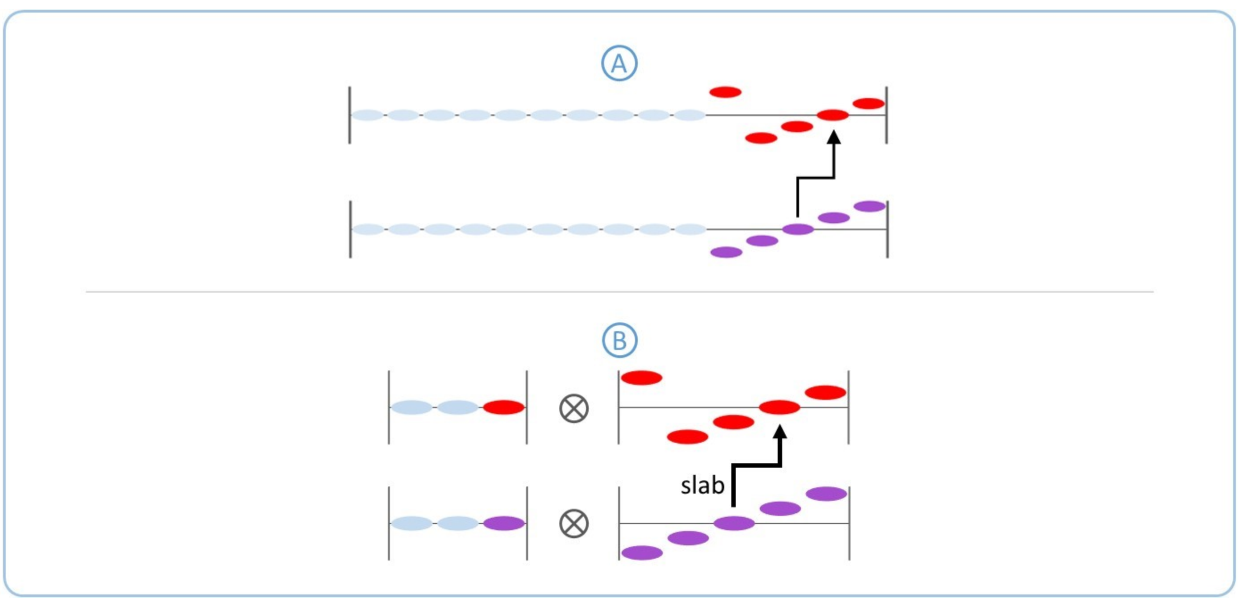
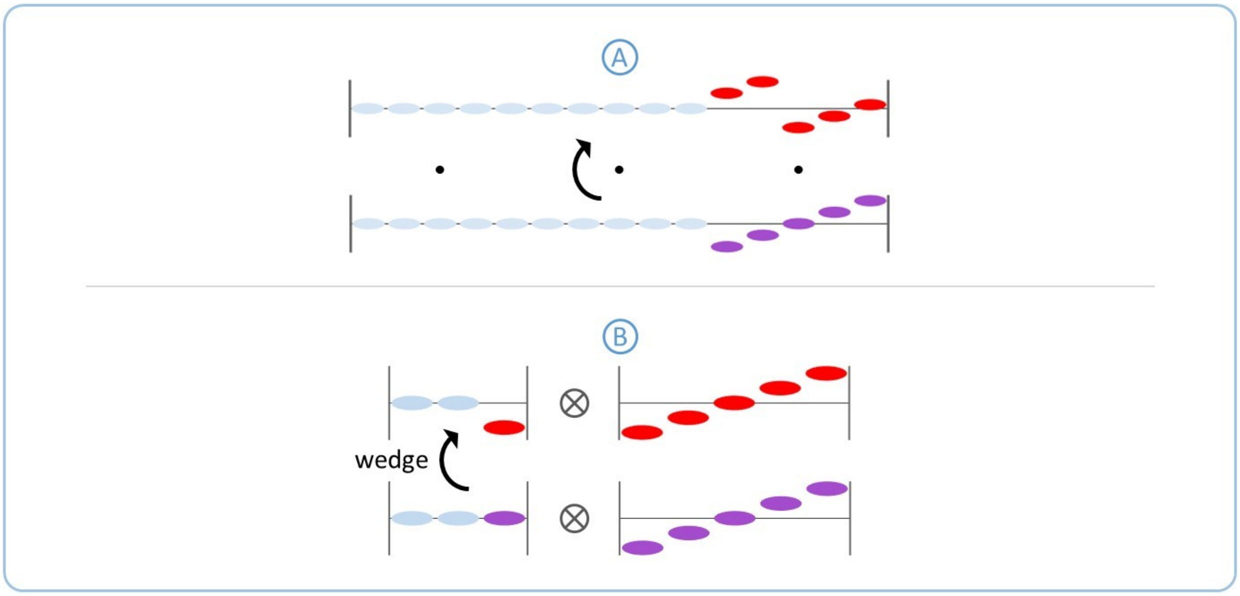


Figure 5.7 shows the high wedge transform. The high wedge applies a scalar phase factor to all patches of the beam waist. It does not appear as a tilt, because it does not affect the relative phases of the patches.

Figure 5.7, Phase of beam by high wedge



The two component transforms each generate a group, and their direct product is the total symmetry of the beam.

$$G_{\text{Beam\_Symmetry}} = G_{\text{Slab\_Low}} \otimes G_{\text{Wedge\_high}}$$

We have considered the case in which *both* features' phases contribute to the thermodynamic entropy. However, it is unresolved in FO whether this actually occurs physically. That would seem to lead to a physical absurdity, because it means that the physical entropy depends upon how large the empty adjacent space is taken to be, or equivalently how far away a putative lens is imagined to be. Neither of these could physically affect the thermodynamic entropy of the beam waist in any plausible way. This presents a challenge to FO.

An alternative possibility is that only the low feature contributes to the phase volume, but that leaves the status of the high feature unresolved. If it is not part of the symmetry group, it cannot have gone over to the asymmetry group either, because that would imply that it is observable and we cannot observe it. Nevertheless, the groups in this case would be

$$G_{\text{Beam\_Symmetry\_Alt}} = G_{\text{Slab\_Low}}$$

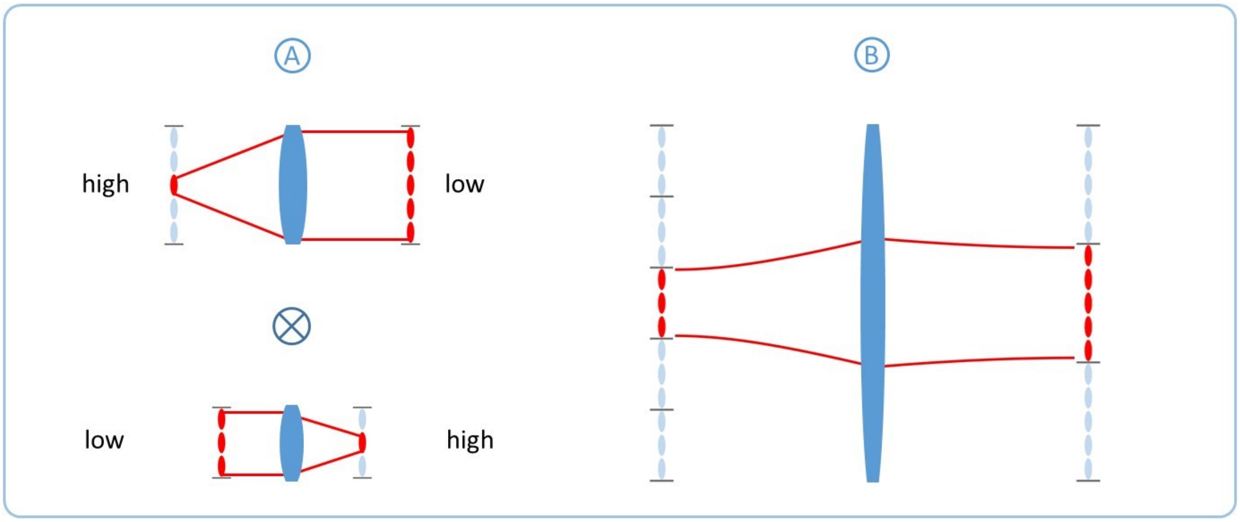
$$G_{\text{Beam\_Asymmetry\_Alt}} = G_{\text{Wedge\_Low}} \otimes G_{\text{Slab\_High}} \otimes G_{\text{Wedge\_high}}$$

This question is difficult to resolve by experiment, since it cannot be observed, by definition.

5.4 FT of the beam

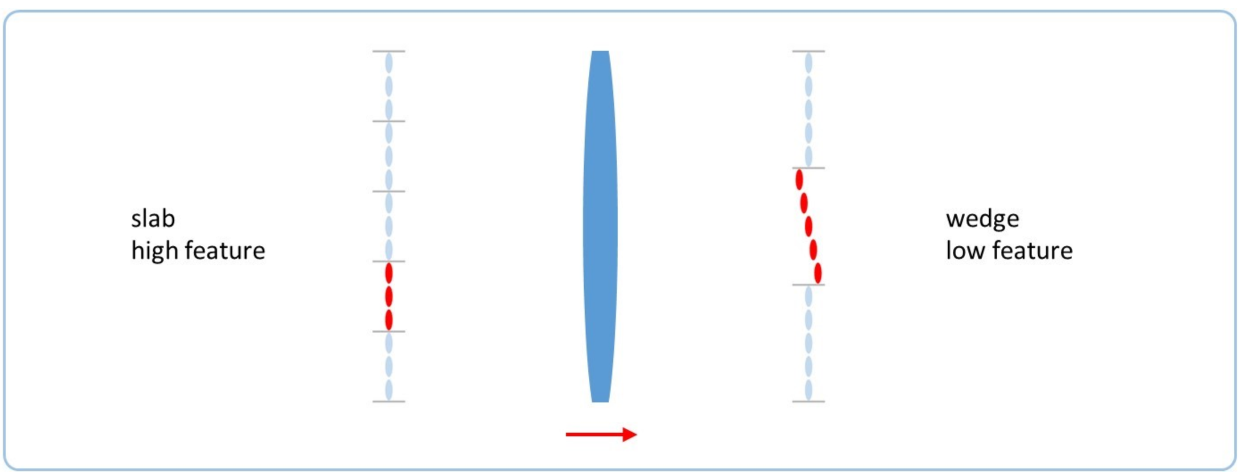
Recalling reference 1, Figure 5.8 shows that the beam can be understood as the direct product of two features. When we take the FT as in Figure 5.8b, the features exchange nesting rank and each one changes from bright to dark, or vice-versa.

Figure 5.8, Beam and component features in 2f system



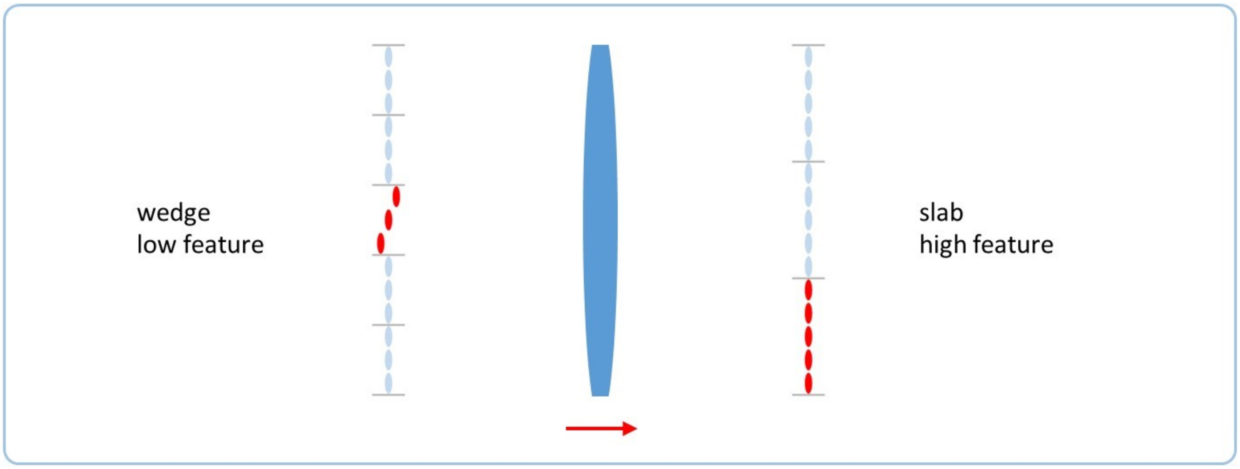
The FT effectively acts to conjugate a slab or wedge input. Figure 5.9 shows that a slab translating the input beam results in a wedge-like tilt of the output beam. This pure slab motion occurs when the beam source is translated relative to the lens while maintaining a constant angle.

Figure 5.9, High slab FTs to become low wedge



Pure wedge (see Figure 5.10) occurs when the input beam is tilted, while holding its lateral position constant at the front focal plane of the lens. It is virtually the same as the previous example, with input and output exchanged (there is also a factor of -1).

Figure 5.10, Low wedge FTs to become high slab



This section does not show the effect of the FT on the symmetry transforms. However, they are conjugated by the FT, following the same rule as the asymmetry transforms. They become symmetries of the output.

6 The Grating in 2f

6.1 Asymmetry transforms of the grating

Figure 6.1 shows the spatial and feature diagrams of a typical compound grating[1]. It is formed by taking the direct product of four features' states. As in the beam, each individual feature has both a symmetry and an asymmetry transform. This section examines the asymmetry transforms and their effect on the composite system.

Figure 6.1, Compound grating in spatial and feature diagrams

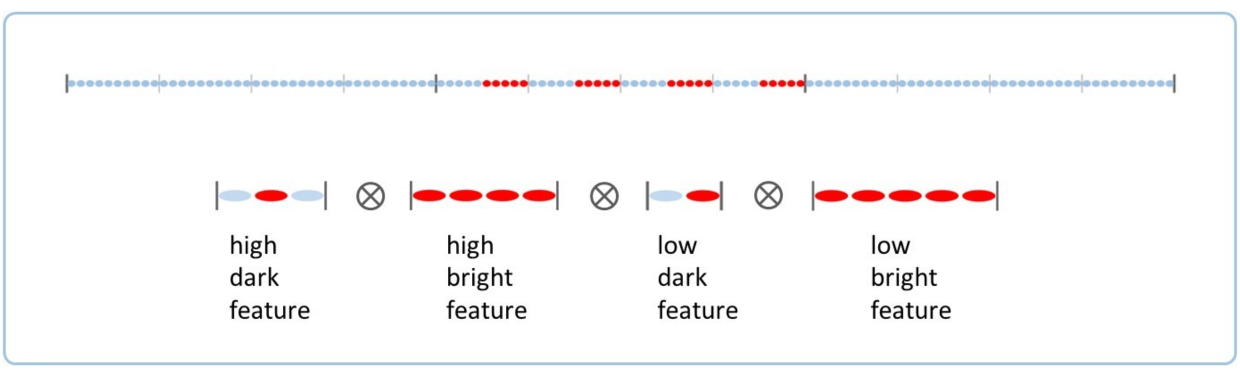


Figure 6.2 shows the wedge applied to the low bright feature.

Figure 6.2, Wedge on low bright feature

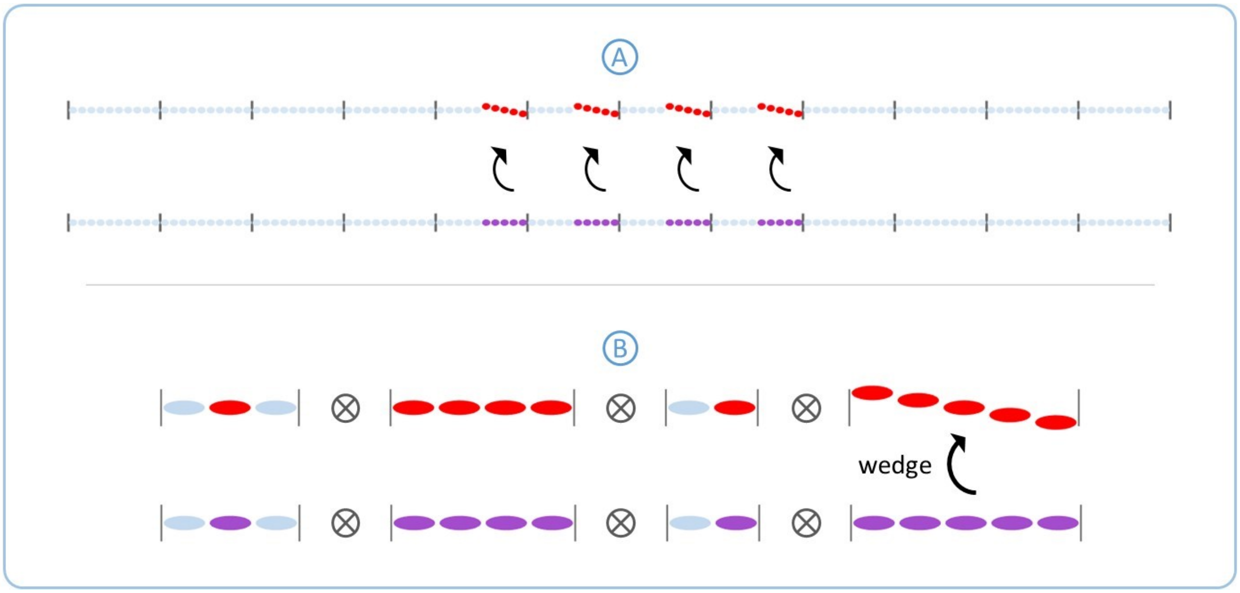


Figure 6.3 shows the slab applied to the low dark feature.

Figure 6.3, Slab on low dark feature

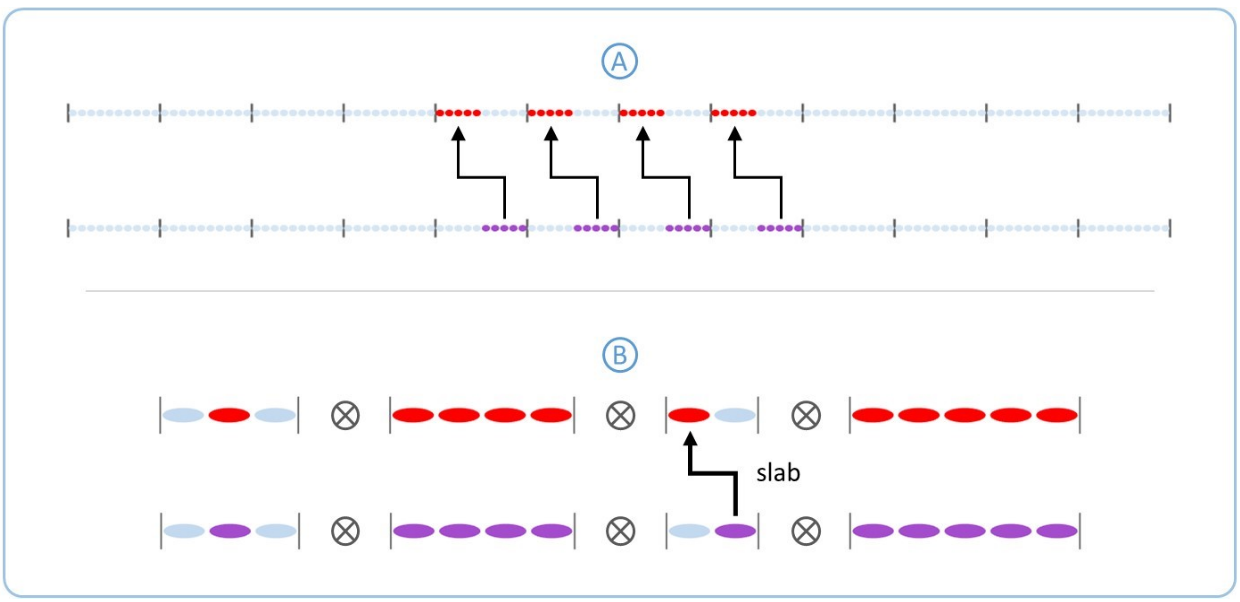


Figure 6.4 shows the wedge applied to the high bright feature. While the wedge introduces a tilt, that tilt has a single value at each patch of the high bright feature. When the outer product is formed this yields a single factor for each slit, so that the phase of the entire slit is shifted as a

single unit. Contrast this with Figure 6.2 above, which yields an actual tilt of the patches *within* each slit.

Figure 6.4, Wedge on high bright feature

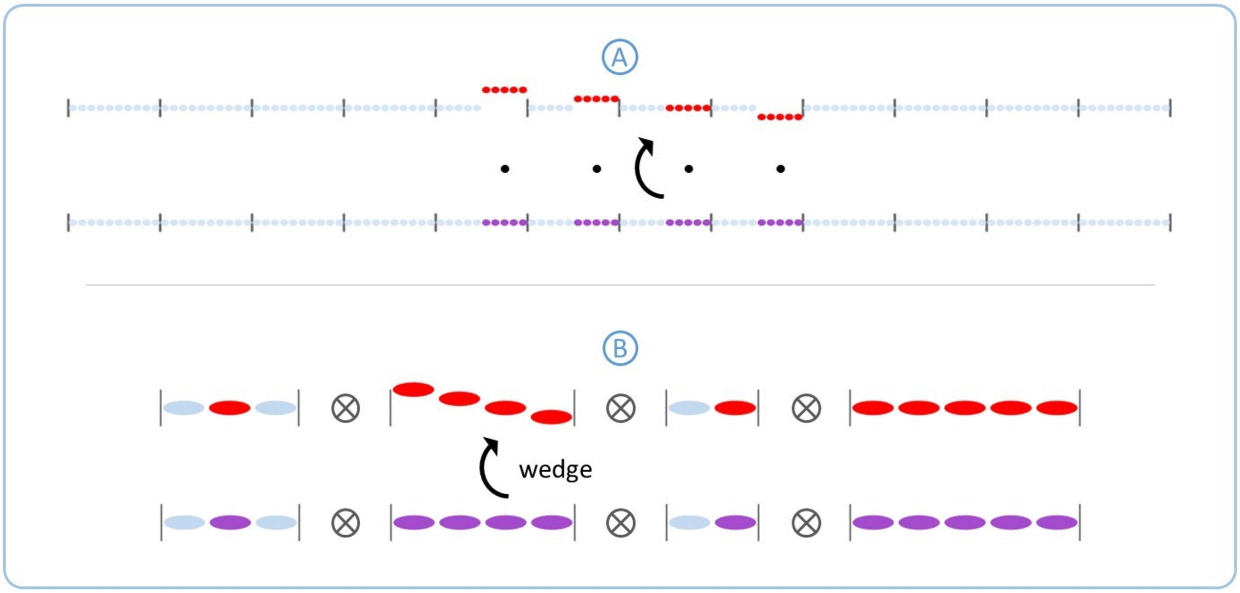
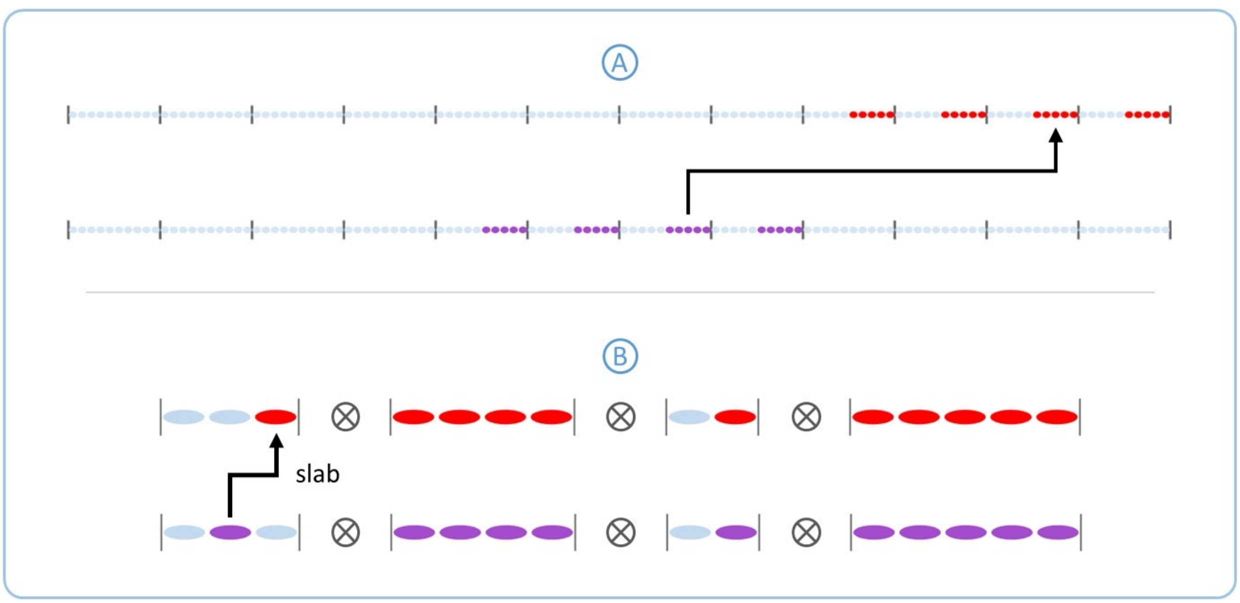


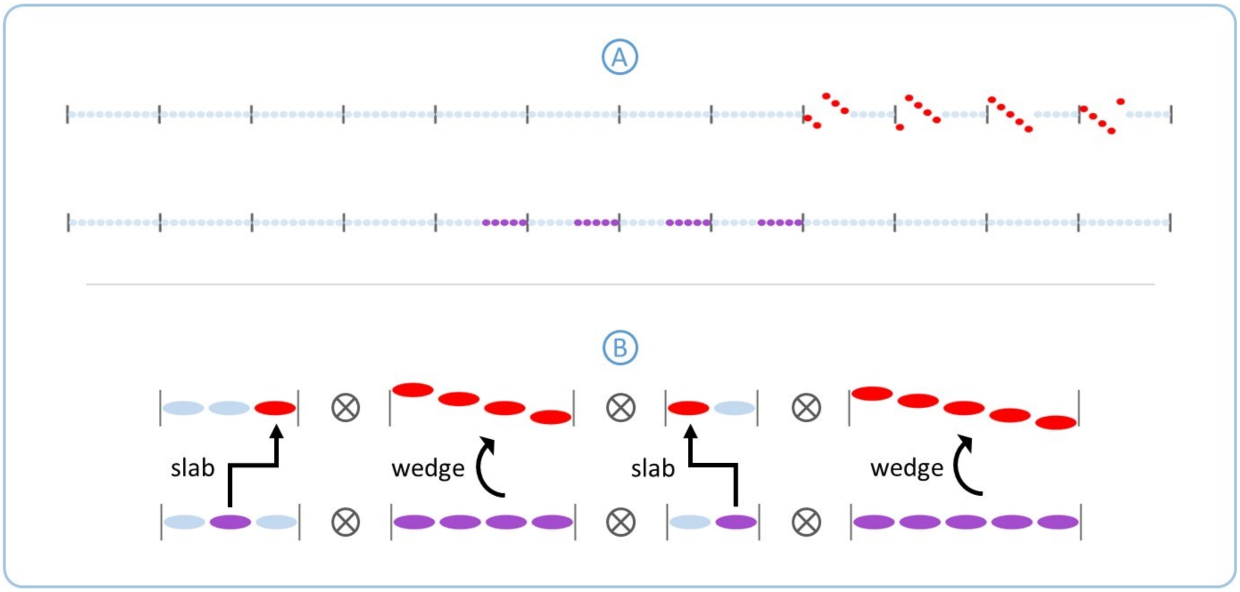
Figure 6.5 shows the slab applied to the high dark feature.

Figure 6.5, Slab on high dark feature



Finally, Figure 6.6 shows that the 4 asymmetry transforms can be applied in arbitrary combination with one another. Because each acts on a separate feature, they commute with one another.

Figure 6.6, Applying all grating asymmetries



6.2 Symmetry transforms of the grating

The grating has 4 symmetry transforms: 2 slabs which act on the low and high bright features, and 2 wedges which act on low and high dark features. The direct product of these 4 is the total symmetry group.

We confront the same uncertainty as in the beam; it is unknown whether *all* symmetry generators are truly part of the thermodynamic entropy.

6.3 FT in the compound grating, Gaussian model

In the 2f system, applying a transform in the input plane is equivalent to applying the conjugate transform to the output plane. This section will illustrate this principle for the asymmetry transforms of the compound grating.

This section will not use FO-type spatial diagrams but rather *Gaussian gratings*, a more conventional representation of the electromagnetic field. A 1-d vector of complex numbers represents the magnitude and phase at each patch. The magnitude is formed from three components: the array, the texture, and the form. The *array* is a set of periodically-spaced delta functions; the *texture* is a small Gaussian beam convolved with the array; the *form* is a large Gaussian beam which is multiplied by the array and texture. We refer to reference 1 for more details.

Figure 6.7 shows an FO-type spatial diagram (Figure 6.7a) and its corresponding Gaussian grating (Figure 6.7b) for comparison's sake. Note that the sense conventions for phase may differ between the two different types of calculations that produced these diagrams.

Figure 6.7, FO idealization vs Gaussian grating

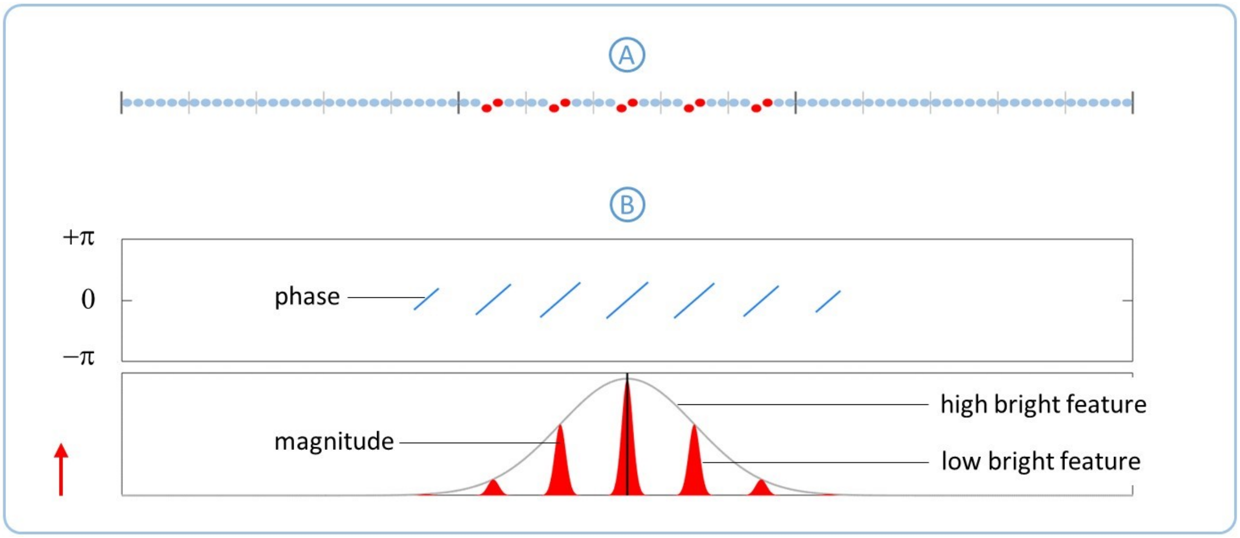
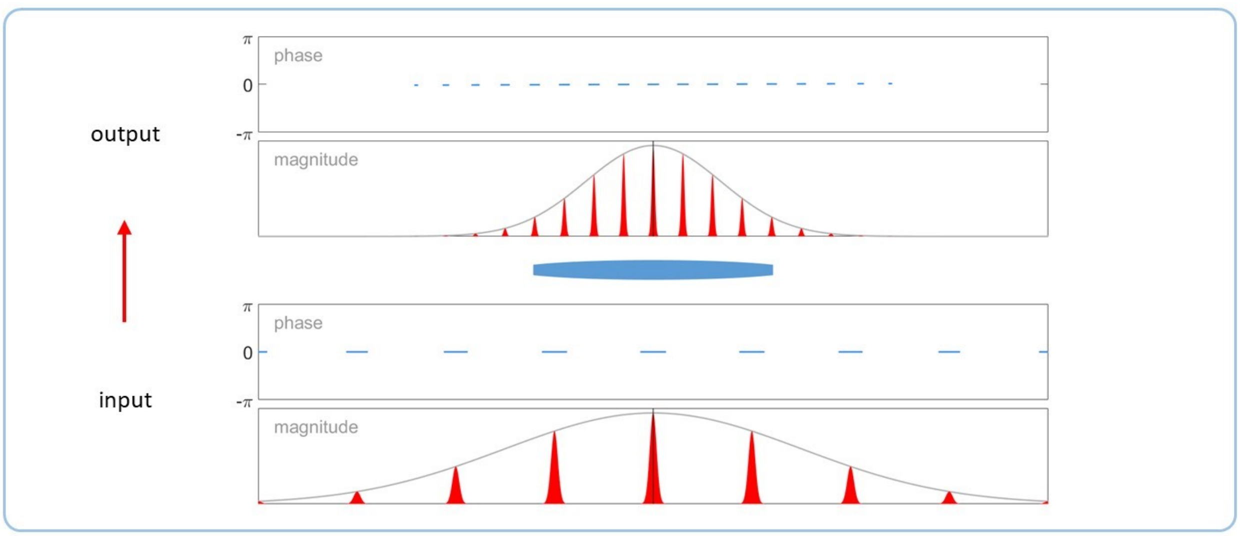


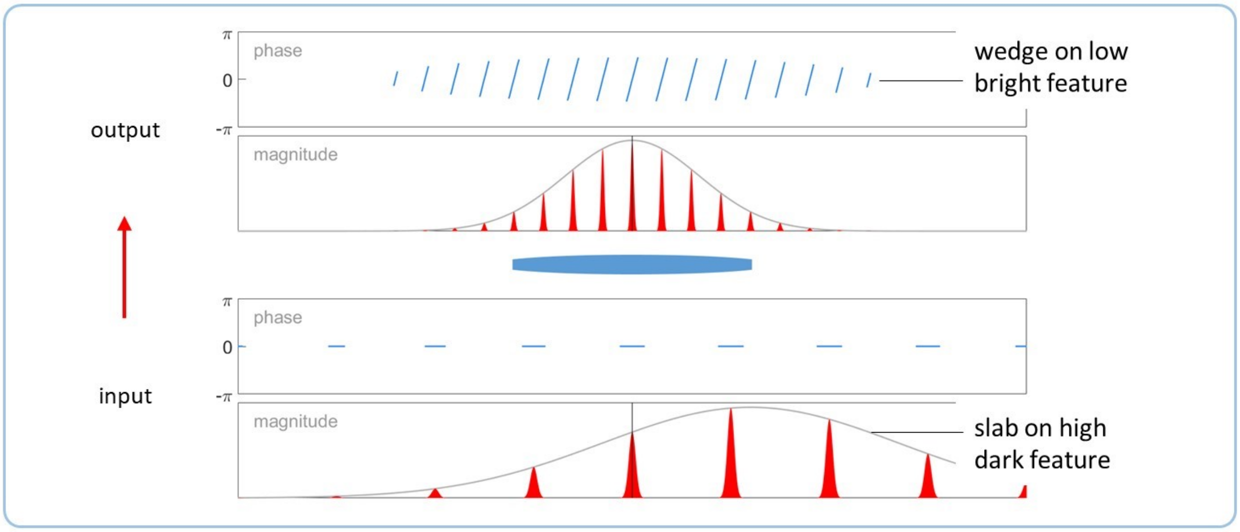
Figure 6.8 shows the Gaussian grating in the zero-order state, i.e. its default state before applying any asymmetry transforms. We show input and output planes; in both, the pattern is centered on the lens, and all wavefronts are flat (phase = 0).

Figure 6.8, Zero-order state



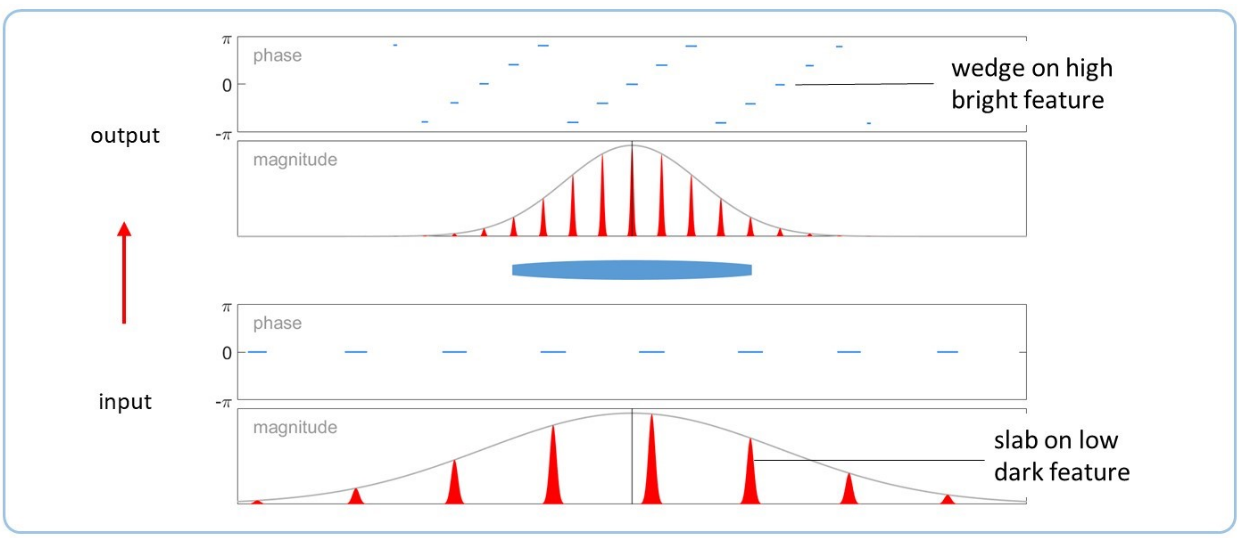
First, we study the slab acting on the high dark feature, drawn in Figure 6.9. This is achieved practically by sliding the illuminating beam while leaving the grating optic in place. The effect in the output is to apply the wedge to the low bright feature. To the eye or to a detector, it appears to have 'no effect' on the output, because changes in phase are not directly detectable.

Figure 6.9, FT of slab on high dark feature



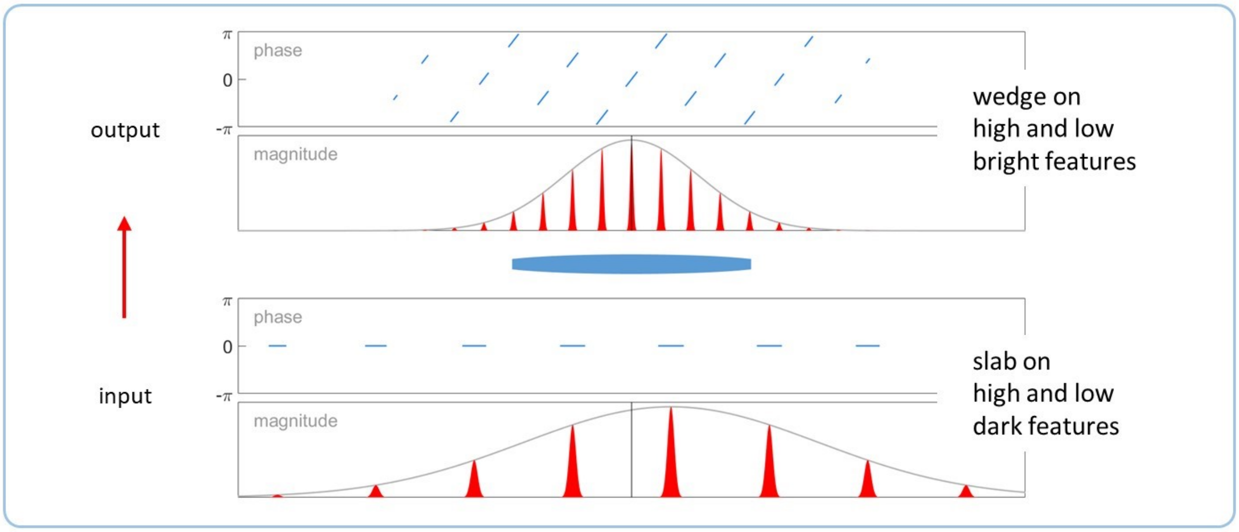
Next we study the slab acting on the *low* dark feature, drawn in Figure 6.10. Practically, this is achieved by sliding the grating optic while leaving the illuminating beam stationary. The output effect is equivalent to a wedge on the high bright feature, i.e. to tilt the angle of the high bright feature with discrete offsets between slits.

Figure 6.10, FT of slab on low dark feature



We have compared the cases of two slab transforms acting independently. However, it often occurs in practice that both the illuminating beam and the grating optic are moved together as a single unit. Figure 6.11 shows that this results in an analogous coupling of the output angle, with the angles of the low and high features changed in fixed proportion to appear as a single degree of freedom.

Figure 6.11, Both slabs together



We have used slab translations as inputs because it is simple to understand intuitively, and easy to achieve practically. The outputs have been wedge angles. But because the 2f system works in reverse, it is also possible to use wedge angles as inputs and get slab translations as outputs. This is more difficult technically, because it requires changing phase in very specific ways, but it can be done with a spatial light modulator or with specialized optics.

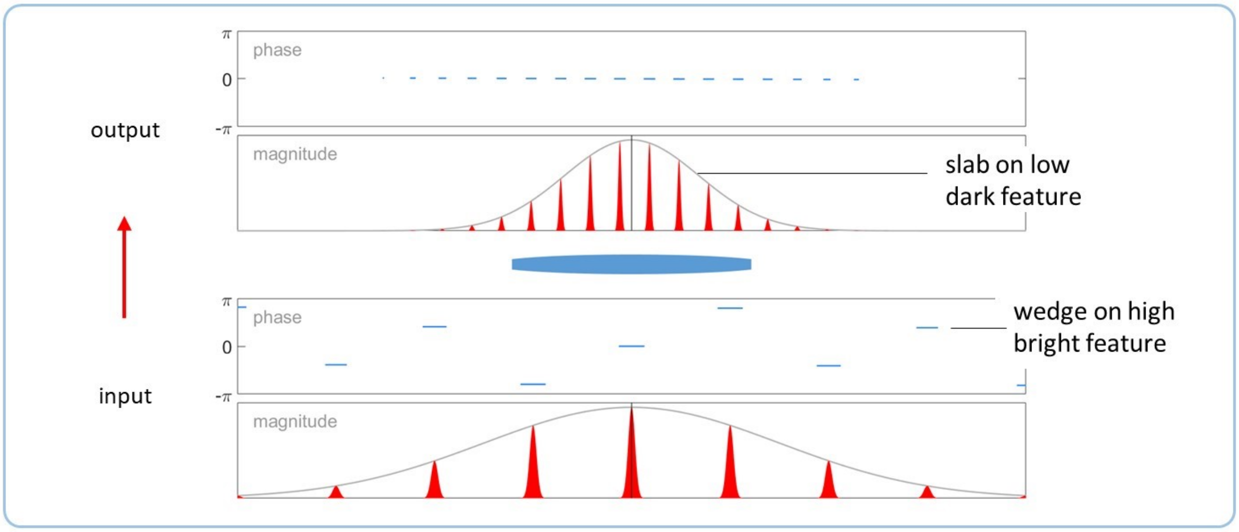
The figures in this section may be better appreciated in animated form; videos are available[8].

6.4 The double-slit experiment

Young's double-slit experiment, which provided early evidence for the wave nature of light, is also a paradigm of the observer effect in QM. In that famous experiment, the light from the two slits overlaps and forms an interference pattern in the far field. However, when any means is used to measure which slit each photon passes through, the interference pattern is washed out. (The experiment is actually more practical using *material* particles such as electrons rather than photons, since photons are typically absorbed when their path is measured).

We can use the compound grating to model this phenomenon. The double slit is simply a diffraction grating in which the high bright feature has only 2 patches. When a particle interacts with one slit during the measurement process, it is coupled to other local degrees of freedom and necessarily experiences random fluctuation in phase, which changes the relative phase between the two slits. It is natural to model this effect using the high wedge transform, drawn in Figure 6.12.

Figure 6.12, FT of wedge on high bright feature



The effect appears in the output plane as random fluctuations of the low slab, which changes the position of the low dark feature. This washes out the interference pattern, leaving only the single-slit diffraction pattern, i.e. the beam that would propagate from just one slit if the other slit were blocked.

7 References

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8. YouTube links:
   https://youtu.be/5Eg4wrTJN38
   https://youtu.be/y5AZVvRBl7Y
   https://youtu.be/294chpIkQXY