First Integrals and Lagrangian Analysis of Nonlinear Differential Equations

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Abstract

We propose in this paper first integrals and Lagrangian analysis of nonlinear differential equations.

Theory

In a recent paper [1], the first order differential equation

\[ I(x, \dot{x}) = ax^l g^n(x)e^{\int_{x}^{\dot{x}} h(x)dx} + bx^q \int_{x}^{\dot{x}} f^n(x)e^{\int_{x}^{\dot{x}} \phi(x)dx} \]  

(1)

has been used in the context of Riccati transformation of equations. In the present theory, let

\[ I(x, \dot{x}) = k \]  

(2)

be a first integral of a differential equation where \( k \) is a time-independent constant. In this respect, using (2) the Lagrangian may be written as [2]

\[ L(x, \dot{x}, t) = \frac{a}{l-1} g^n(x)e^{\int_{x}^{\dot{x}} h(x)dx} \dot{x}^l + \frac{b}{q-1} \int_{x}^{\dot{x}} f^n(x)e^{\int_{x}^{\dot{x}} \phi(x)dx} \dot{x}^q \]  

(3)

Therefore one may secure the Euler-Lagrange equation

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \]  

(4)

in the form

\[ \frac{\dot{x}}{\dot{x}^2} \left[ al g^n(x)e^{\int_{x}^{\dot{x}} h(x)dx} \dot{x}^l + bqf^n(x)e^{\int_{x}^{\dot{x}} \phi(x)dx} \dot{x}^q \right] + ag^n(x)e^{\int_{x}^{\dot{x}} h(x)dx} \left[ m \frac{g'(x)}{g(x)} + a h(x) \right] \dot{x}^l + \]  

\[ hf^n(x)e^{\int_{x}^{\dot{x}} \phi(x)dx} \left[ n \frac{f'(x)}{f(x)} + \beta \dot{\phi}(x) \right] \dot{x}^q = 0 \]  

(5)
Now, putting $l = q$, one may ensure the interesting equation

\[
\ddot{x} + \left\{a g^m (x) e^{\int h(x) dx} \left[ m \frac{g'(x)}{g(x)} + \alpha h(x) \right] + b f^n (x) e^{\int \varphi(x) dx} \left[ n \frac{f'(x)}{f(x)} + \beta \varphi(x) \right] \right\} \times \frac{1}{k^2} = 0
\]

where the quadratic term $\dot{x}^2$ is eliminated using (2).

References
