Some Fourier Series - Identities

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ABSTRACT
We give some Fourier Series - Identities

1. Introduction

Entry 1. Define a,b,c by

$$a=2-\sqrt{4-\cfrac{2}{\sqrt{4-\cfrac{2}{\sqrt{4-\dots}}}}} \tag{1}$$

$$b=2-\left\{\frac{1}{2}+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{4}\left(\frac{1}{2}+\dots\right)^3\right)^3\right\} \tag{2}$$

$$c=2+\sqrt[3]{2+4\sqrt[3]{2+4\sqrt[3]{2+\dots}}} \tag{3}$$

then

$$a^3-6a^2+8a-2=0 \tag{4}$$

$$b^3-6b^2+8b-2=0 \tag{5}$$

$$c^3-6c^2+8c-2=0 \tag{6}$$

Entry 2.

$$a=2-\frac{4}{\sqrt{3}}\cos\left(\frac{\pi}{6}+\frac{1}{3}\sin^{-1}\frac{3\sqrt{3}}{8}\right) \tag{7}$$

$$b=2-\frac{4}{\sqrt{3}}\sin\left(\frac{1}{3}\sin^{-1}\frac{3\sqrt{3}}{8}\right) \tag{8}$$

$$c=2-\frac{4}{\sqrt{3}}\cos\left(\frac{5\pi}{6}+\frac{1}{3}\sin^{-1}\frac{3\sqrt{3}}{8}\right) \tag{9}$$

Entry 3.

$$\pi=3\sin^{-1}\left(\frac{\sqrt{3}(2-a)}{4}\right)+\sin^{-1}\frac{3\sqrt{3}}{8} \tag{10}$$

$$\pi = 6\sin^{-1}\left(\frac{\sqrt{3}\left(2-b\right)}{4}\right) + 2\sin^{-1}\frac{\sqrt{37}}{8} \tag{11}$$

$$\pi = 3\sin^{-1}\left(\frac{\sqrt{3}\left(c-2\right)}{4}\right) - \sin^{-1}\frac{3\sqrt{3}}{8} \tag{12}$$

2. Fourier Series - Identities

Entry 4.

$$\pi\sum_{n=1}^{\infty}\frac{\cos\left(n\pi a\right)}{n^2} = \sum_{n=1}^{\infty}\frac{\sin\left(n\pi a\right)}{n^3} \tag{13}$$

$$\pi\sum_{n=1}^{\infty}\frac{\cos\left(n\pi b\right)}{n^2} = \sum_{n=1}^{\infty}\frac{\sin\left(n\pi b\right)}{n^3} \tag{14}$$

$$\pi\sum_{n=1}^{\infty}\frac{\cos\left(n\pi c\right)}{n^2} = \left(\frac{120}{299}c^2 - \frac{12c}{13} - \frac{121}{299}\right)\sum_{n=1}^{\infty}\frac{\sin\left(n\pi c\right)}{n^3} \tag{15}$$

Entry 5.

$$\begin{aligned}\pi\sum_{n=1}^{\infty}\frac{2n+1}{n^2\left(n+1\right)^2}\sin\left(\frac{n\pi a}{2}\right)\cos\left(\frac{\left(n+1\right)\pi a}{2}\right) &= \\ &= \sum_{n=1}^{\infty}\frac{3n^2+3n+1}{n^3\left(n+1\right)^3}\sin\left(\frac{n\pi a}{2}\right)\sin\left(\frac{\left(n+1\right)\pi a}{2}\right)\end{aligned} \tag{16}$$

$$\begin{aligned}\pi\sum_{n=1}^{\infty}\frac{2n+1}{n^2\left(n+1\right)^2}\sin\left(\frac{n\pi b}{2}\right)\cos\left(\frac{\left(n+1\right)\pi b}{2}\right) &= \\ &= \sum_{n=1}^{\infty}\frac{3n^2+3n+1}{n^3\left(n+1\right)^3}\sin\left(\frac{n\pi b}{2}\right)\sin\left(\frac{\left(n+1\right)\pi b}{2}\right)\end{aligned} \tag{17}$$

Entry 6.

$$\pi\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n^2}\cos\left(\frac{n\pi c}{2}\right)\cos\left(\frac{n\pi\left(a-b\right)}{2}\right) = \sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n^3}\sin\left(\frac{n\pi c}{2}\right)\cos\left(\frac{n\pi\left(a-b\right)}{2}\right) \tag{18}$$

$$\pi\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n^2}\sin\left(\frac{n\pi c}{2}\right)\sin\left(\frac{n\pi\left(b-a\right)}{2}\right) = \sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n^3}\cos\left(\frac{n\pi c}{2}\right)\sin\left(\frac{n\pi\left(a-b\right)}{2}\right) \tag{19}$$

Entry 7.

$$\pi \sum_{n=1}^{\infty} \frac{\cos\left(n\pi\left(a+c\right)\right)}{n^2} = \sum_{n=1}^{\infty} \frac{\sin\left(n\pi b\right)}{n^3} \tag{20}$$

$$\pi \sum_{n=1}^{\infty} \frac{\cos\left(n\pi\left(b+c\right)\right)}{n^2} = \sum_{n=1}^{\infty} \frac{\sin\left(n\pi a\right)}{n^3} \tag{21}$$

$$\pi \sum_{n=1}^{\infty} \frac{\cos\left(n\pi b\right)}{n^2} = -\sum_{n=1}^{\infty} \frac{\sin\left(n\pi\left(a+c\right)\right)}{n^3} \tag{22}$$

$$\pi \sum_{n=1}^{\infty} \frac{\cos\left(n\pi a\right)}{n^2} = -\sum_{n=1}^{\infty} \frac{\sin\left(n\pi\left(b+c\right)\right)}{n^3} \tag{23}$$

References

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