Sobre los números:

$z(n)\in\mathbb{C} / n z(n)=1+i+(z(n))^3 \ , n\in\mathbb{N}$

Edgar Valdebenito

27/08/2019

Resumen

En esta nota mostramos algunas fórmulas relacionadas con las raíces de la ecuación:

$n z = 1+i+z^3 \ , i=\sqrt{-1} \ , n\in\mathbb{N}, \operatorname{Re}(z)>0, \operatorname{Im}(z)>0$.

Introducción.

Para $n\in\mathbb{N}=\{1,2,3,...\}$, $i=\sqrt{-1}$, sea $z=z(n)\in\mathbb{C}$, tal que:

$$n z = 1+i+z^3 \tag{1}$$

Se tiene

- $n,m\in\mathbb{N}, A_0=1, B_0=C_0=0$

$$z^m = A_m + B_m z + C_m z^2 \tag{2}$$

$$A_{m+1}=-(1+i)C_m \quad , \quad B_{m+1}=A_m+nC_m \quad , \quad C_{m+1}=B_m \tag{3}$$

- $n,m\in\mathbb{N}, a_0=1, b_0=c_0=0$

$$z^{-m}=a_m+b_m z^{-1}+c_m z^{-2} \tag{4}$$

$$a_{m+1}=-\left(\frac{1-i}{2}\right)c_m \quad , \quad b_{m+1}=a_m \quad , \quad c_{m+1}=b_m+n\left(\frac{1-i}{2}\right)c_m \tag{5}$$

En esta nota mostramos algunas fórmulas relacionadas con los números $z = z(n) \in \mathbb{C}$, $n \in \mathbb{N}$, tales que:

$$n\,z(n) = 1 + i + (z(n))^3 \quad ,\mathrm{Re}(z(n)) > 0, \mathrm{Im}(z(n)) > 0 \tag{6}$$

Fórmulas.

- Si $n \in \mathbb{N}, c_0 = 1, c_1 = n\left(\frac{1-i}{2}\right), c_2 = -\frac{n^2 i}{2}$, y

$$c_k = n\left(\frac{1-i}{2}\right)c_{k-1} - \left(\frac{1-i}{2}\right)c_{k-3} \quad ,k \geq 3 \tag{7}$$

entonces

$$\lim_{k\to\infty}\left(\frac{c_{k-1}}{c_k}\right) = z(n) \tag{8}$$

- Para $n \in \mathbb{N} - \{1\}$, se tiene

$$z(n) = \frac{1+i}{n} + \frac{1}{n}\left(\frac{1+i}{n} + \frac{1}{n}\left(\frac{1+i}{n} + \ldots\right)^3\right)^3 \tag{9}$$

- Para $n \in \mathbb{N} - \{1\}$, se tiene

$$z(n,k+1) = \frac{1+i}{n} + \frac{1}{n}(z(n,k))^3 \;\; , z(n,0) = 0 \Rightarrow \lim_{k\to\infty} z(n,k) = z(n) \tag{10}$$

Observación: $(9) \equiv (10)$.

- Para $n \to \infty$, se tiene

$$z(n) = \frac{1+i}{n} + \frac{(1+i)^3}{n^4} + \frac{3(1+i)^5}{n^7} + \frac{12(1+i)^7}{n^{10}} + \frac{55(1+i)^9}{n^{13}} + \frac{273(1+i)^{11}}{n^{16}} + \\ + \frac{1428(1+i)^{13}}{n^{19}} + \frac{7752(1+i)^{15}}{n^{22}} + \ldots \tag{11}$$

- Si $n\in\mathbb{N}-\{1\}$, se tiene

$$z(n)=\frac{1+i}{n}-\frac{2-2i}{n\left(n^3-6i\right)}+\frac{24-24i}{n\left(n^3-6i\right)^3}+\frac{(16+16i)\left(n^3+30i\right)}{n\left(n^3-6i\right)^5}$$
$$-\frac{(960+960i)\left(n^3+12i\right)}{n\left(n^3-6i\right)^7}+\frac{(384+384i)\left(n^6+114in^3-792\right)}{n\left(n^3-6i\right)^9}+O\left(\left(\frac{2-2i}{n^4}\right)^6\right) \tag{12}$$

- Sea $n\in\mathbb{N}, z(n)=a+bi\,,\left(a=a(n),b=b(n)\right)$, se tiene

$$a=\frac{1}{2}R^{1/3}\left(\cos\frac{\theta}{3}+\sqrt{3}\sin\frac{\theta}{3}\right)+\frac{1}{2}\rho^{1/3}\left(\cos\frac{\phi}{3}-\sqrt{3}\sin\frac{\phi}{3}\right) \tag{13}$$
$$b=\frac{1}{2}R^{1/3}\left(\sin\frac{\theta}{3}-\sqrt{3}\cos\frac{\theta}{3}\right)+\frac{1}{2}\rho^{1/3}\left(\sin\frac{\phi}{3}+\sqrt{3}\cos\frac{\phi}{3}\right) \tag{14}$$

donde

$$R=\sqrt{\left(P-\frac{1}{2}\right)^2+\left(Q-\frac{1}{2}\right)^2}\quad,\quad \rho=\sqrt{\left(P+\frac{1}{2}\right)^2+\left(Q+\frac{1}{2}\right)^2} \tag{15}$$

$$\theta=-\tan^{-1}\left(\frac{2Q-1}{1-2P}\right)\quad,\quad \phi=\tan^{-1}\left(\frac{2Q+1}{2P+1}\right) \tag{16}$$

$$P=\sqrt{-\frac{n^3}{54}+\frac{1}{2}\sqrt{\frac{1}{4}+\left(\frac{n^3}{27}\right)^2}}\quad,\quad Q=\sqrt{\frac{n^3}{54}+\frac{1}{2}\sqrt{\frac{1}{4}+\left(\frac{n^3}{27}\right)^2}} \tag{17}$$

- Si $n\in\mathbb{N}, z(n)=a+bi$, se tiene

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{b}{a}\right)-\tan^{-1}\left(\frac{b}{\sqrt{n}-a}\right)+\tan^{-1}\left(\frac{b}{\sqrt{n}+a}\right) \tag{18}$$

- Si $\operatorname{Re}\left(z(1)\right)=a,\operatorname{Im}\left(z(1)\right)=b$, se tiene

$$\frac{\pi}{4}=-\tan^{-1}\left(\frac{a}{b}\right)+\tan^{-1}\left(\frac{1-a}{b}\right)+\tan^{-1}\left(\frac{b}{1+a}\right) \tag{19}$$

- Si $n\in\mathbb{N}-\{1\}, z(n)=a+bi$, se tiene

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{a}{b}\right)+\tan^{-1}\left(\frac{b}{\sqrt{n}-a}\right)-\tan^{-1}\left(\frac{b}{\sqrt{n}+a}\right) \tag{20}$$

- Si $n \in \mathbb{N}, z(n) = a + bi$, se tiene

$$a^3 + 1 = 3ab^2 + na \quad , \quad 3a^2 b + 1 = b^3 + nb \tag{21}$$

- Algunos valores de $z(n)$:

$$z(1) = 0.3589... + i \times 0.7989...$$
$$z(2) = 0.3682... + i \times 0.5327...$$
$$z(3) = 0.3050... + i \times 0.3515...$$
$$z(4) = 0.2415... + i \times 0.2570...$$
$$z(5) = 0.1966... + i \times 0.2030...$$

- Una integral para $z(n)$:

$$I(n) = \int_0^{2\pi} \frac{e^{ix}}{1 + i - e^{ix} + e^{3ix}} dx = \frac{2\pi}{3\left(z(n)\right)^2 - n} \quad , n \in \mathbb{N} \tag{22}$$

$$z(n) = \sqrt{\frac{n}{3} + \frac{2\pi}{3I(n)}} \quad , n \in \mathbb{N} \tag{23}$$

Observación: $\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.141592...$

Referencias

1. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST Handbook of Mathematical Functions. Cambridge University Press, 2010.
2. Ramanujan, S.: Collected Papers, Chelsea, New York, 1962.
3. Ramanujan, S.: The lost notebook and other unpublished papers. Narosa Publishing House, New Delhi , 1988.