Improved approximation of Sommerfeld's fine structure constant as a series representation in e and π

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Abstract: In 2017 we published a heuristic approximation of the fine structure constant of physics, represented by the symbol α, in a series whose terms were comprised of the product e•π raised to successive powers, along with divisors as the sequence of odd numbers multiplied by successive powers of 10. Feynman most famously has conjectured that it might be possible to account for α in some type of series or product expression in "e", the base of natural logarithms, and "π" the familiar circular constant. Here, in this communication, we propose an infinite series in the product e•π, that is slightly modified from the 2017 series, by removing successive odd number divisors and replacing them with a quotient multiplier for each term that contains the sequence of odd numbers in successive ratios and raised to even powers. The improved series converges, within a few terms, to better than 99999 parts in 100,000 of the true value of α.

Keywords: fine structure constant α, Sommerfeld, infinite series, e, π

The fine structure constant, α, was identified in 1916 by Sommerfeld, in the course of working out the relativistic theory of the H atom under the old quantum theory of Bohr. [1] As shown by inspection of Equation 1 and the natural physical constants shown in Table 1, the constant is a pure number. Thus α is dimensionless, and it was employed by Sommerfeld to support his analysis of the fine structural detail of the emission spectrum of the H atom.

$$\alpha^2 = \frac{(k_c)(q)^2}{\left(\frac{h}{2\pi}\right)(c)} \tag{1a}$$

$$\alpha^{-2} = 137.036\ldots \tag{1b}$$

$$\alpha = 0.0854245\ldots \tag{1c}$$

Table 1

In 2017, Bucknum et al. introduced a heuristic approximation of the fine structure constant, α. [2] This approximation was comprised of a sequence of terms in successive powers of the product $e \cdot \pi$, and successive divisors consisting of the sequence of odd numbers, multiplied by powers of 10. This so-called "2017 series" is shown in Equation 2. The approximation results it yields for α are thus shown in Table 2 and Table 3. The agreement between the experimental value of α (see Equation 1c) and the calculated value of α, by Equation 2b, and out to 3rd order and beyond, is better than 9999 parts in 10,000. Equation 2a and 2b are hereafter identified as the "2017 series".

$$\alpha' = \sum_{n=0}^{\infty} \frac{(e \cdot \pi)^{n+1}}{(2 \cdot n+1) \cdot (10^{4n+2})} \tag{2a}$$

$$\alpha' = \frac{(e \cdot \pi)^1}{1 \cdot 10^2} + \frac{(e \cdot \pi)^2}{3 \cdot 10^6} + \frac{(e \cdot \pi)^3}{5 \cdot 10^{10}} + \cdots \tag{2b}$$

Table 2

Table 3

Upon introducing a slight modification to Equation 2, as shown in Equation 3 below, in which the sequence of odd number divisors, given by the expression $(2n + 1)^{-1}$; where "n" is the nth integer in the sequence of integers; this is then replaced by the sequence of quotient multipliers, $\left(\frac{2n+1}{2n+3}\right)$, taken to even powers "2n", where "n" is the nth integer in the sequence of integers.

And this series is hereafter known as the “2019 series”. Table 4 thus shows the agreement between the experimental value of α (Equation 1c), and the value of α calculated using the “2019 series”; where the agreement at 3rd order and thereafter, in the “2019 series” is beyond 99999 parts in 100,000 thus.

$$\alpha'' = \sum_{n=0}^{\infty} \left(\frac{2n+1}{2n+3}\right)^{2n} \cdot \frac{(e\cdot\pi)^{n+1}}{10^{4n+2}} \qquad (3a)$$

$$\alpha'' = \left(\frac{1}{3}\right)^{0} \cdot \frac{(e\cdot\pi)^{1}}{1\cdot 10^{2}} + \left(\frac{3}{5}\right)^{2} \cdot \frac{(e\cdot\pi)^{2}}{1\cdot 10^{6}} + \left(\frac{5}{7}\right)^{4} \cdot \frac{(e\cdot\pi)^{3}}{1\cdot 10^{10}} + \cdots \qquad (3b)$$

Table 4

Table 5

The fine structure constant α is defined differently, usually, then in the preceding discussion based upon Equation 1 above. The usual definition is that α is set equal to v/c, where “v” is the velocity of the electron in the first Bohr orbit of H and “c” is the speed of light. Therefore, in the usual definition it is taken as the square of the quantity identified is Equation 1. In the NIST CODATA compilation [3] the value of α (= v/c) is 0.0072973525664(17). Table 3 and Table 5, below, indicate the accuracy of the squares of the approximations calculated using Equation 2 and Equation 3, to the latter definition of the fine structure constant, respectively. Note that out to 3rd order, in Table 5, the approximation based upon Equation 3 is not as good as in the Feynman definition of α, being good to just better than 99.99%. Results in Table 2, Table 3, Table 4 and Table 5 reflect the fact that the approximations calculated by Equation 2 and Equation 3 are accurate to only 6 significant figures, i.e. beyond 6 significant figures the numbers have uncertainty in them due to the level of approximation in Equation 2 and Equation 3, and for this reason the associated numbers are rounded at 6 significant figures in order to be consistent with the limitations of the approximation employed.

From the very good agreement of the “2017 series”, at beyond 99.99% agreement with the experimental value of the fine structure constant, and the even better agreement of the “2019 series”, at beyond 99.999% agreement with Feynman’s definition of α, it is left for the reader to thus speculate whether indeed α, at some time in the 13.8 billion year history of the Universe, may have in fact taken on the precise value given by the “2019 series”. In such a situation, one would find that the experimentally determined value of α is coincidently expressible in closed form as a simple series in terms of powers of $e \bullet \pi$, and this consequence might thus lead to other productive speculations. One could, for example, envision a Universe which was more mathematically precise in its physics and physical laws at some time in the past and this might have led to a more mathematically precise chemistry…and on the Earth-like planets perhaps a more favorable biology as well.

REFERENCES

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[3] P.J. Mohr, B.N. Taylor and D.B. Newell, "Fine Structure Constant", in CODATA Internationally recommended 2014 values of the fundamental physical constants, National Institute of Standards and Technology (NIST), Gaithersburg, MD, USA, 2015.

Table 1: Physical constants and mathematical constants associated with Sommerfeld number

Symbol	physical quantity	value
k_c	Coulomb's law constant	8.98755×10^9 N·m²/C²
q	electric charge	1.60217×10^{-19} C
$h/2\pi$	quantum of action	1.05457×10^{-34} J·s
c	speed of light	2.99792×10^8 m/s
π	circle constant	3.14159
e	Euler constant	2.71828
α	fine structure constant	0.0854245

Table 2: Approximations α' to the Feynman fine structure constant "√(v/c)"

Order	√(v/c)	√(v/c)'	√(v/c)'/√(v/c) x 100
0^{th}	0.0854245	------	100.000
1^{st}	0.0854245	0.0853973	99.9681
2^{nd}	0.0854245	0.0854216	99.9966
3^{rd}	0.0854245	0.0854217	99.9967

Table 3: Approximations α' to the Sommerfeld fine structure constant "v/c"

Order	(v/c)	(v/c)'	(v/c)'/(v/c) x 100
0^{th}	0.00729735	-------	100.000
1^{st}	0.00729735	0.00729269	99.9361
2^{nd}	0.00729735	0.00729684	99.9930
3^{rd}	0.00729735	0.00729686	99.9932

Table 4: Approximations α'' to the Feynman fine structure constant "√(v/c)"

Order	√(v/c)	√(v/c)''	√(v/c)''/√(v/c) x 100
0^{th}	0.0854245	------	100.000
1^{st}	0.0854245	0.0853973	99.9681
2^{nd}	0.0854245	0.0854235	99.9989
3^{rd}	0.0854245	0.0854236	99.9990

Table 5: Approximations α" to the Sommerfeld fine structure constant "v/c"

Order	(v/c)	(v/c)"	(v/c)"/(v/c) x 100
0th	0.00729735	-------	100.000
1st	0.00729735	0.00729269	99.9361
2nd	0.00729735	0.00729719	99.9978
3rd	0.00729735	0.00729719	99.9979