A note on Ramanujan's integral

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ABSTRACT. We give some remarks on Ramanujan's integral:

$\int_0^\infty f(x)\,dx = \frac{2}{3}\sqrt{\pi}$.

keywords: Definite integral , Ramanujan , number Pi.

I. Introduction .

In Ramanujan's Notebooks Part II,(Chapter 13, p.187) appears:

(1) $\int_0^\infty \frac{e^{-x^2}-1+x^2}{x^4}\,dx = \frac{2}{3}\sqrt{\pi}$

In this note we give some formulas related with (1).

II. Formulas .

(2) $\frac{2}{3}\sqrt{\pi} = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!\,(2n-3)} + \frac{1}{2}\Gamma\left(-\frac{3}{2}, 1\right)$

(3) $\Gamma\left(-\frac{3}{2}, 1\right) = 2\int_1^\infty e^{-x^2} x^{-4}\,dx$

(4) $\Gamma\left(-\frac{3}{2}, 1\right) = \cfrac{e^{-1}}{1+\cfrac{5/2}{1+\cfrac{1}{1+\cfrac{7/2}{1+\cfrac{2}{1+\cfrac{9/2}{1+\dots}}}}}}$

(5) $\frac{2}{3}\sqrt{\pi} = \int_0^{1/4}\phi(x)\,dx - \frac{\alpha}{4} + \sum_{n=2}^{\infty} \frac{(-1)^n \alpha^{2n-3}}{n!\,(2n-3)}$

where

(6) $\alpha = \sqrt{2+2\,e^{-1-e^{-1-e^{-1-\dots}}}} = 1.5990400512564241\ldots$

(7) $\phi = f^{-1}$, $f(x) = \frac{e^{-x^2}-1+x^2}{x^4}$

(8) $\phi(x) = \sqrt{g(x, g(x, g(x, \ldots)))}$, $g(x, z) = \frac{1+\sqrt{1-4x+4x\,e^{-z}}}{2x}$, $0 \le x \le 1/4$

III. Related Integrals .

$$\frac{2}{3}\sqrt{\pi} = \int_0^{\infty}\left\{1 - x^2\left(1 - e^{-1/x^2}\right)\right\} dx$$

$$\frac{2}{3}\sqrt{\pi} = \frac{2}{3} + \int_0^1 x^2\, e^{-1/x^2}\, dx + \int_0^1 x^{-4}\left(e^{-x^2} - 1 + x^2\right) dx$$

$$\frac{2}{3}\sqrt{\pi} = \frac{2}{3} + \int_1^{\infty} x^{-4}\, e^{-x^2}\, dx + \int_1^{\infty}\left\{1 - x^2\left(1 - e^{-1/x^2}\right)\right\} dx$$

$$\sqrt{\pi} = \int_0^{\infty} x^{-2}\left(1 - e^{-x^2}\right) dx$$

$$2\sqrt{\pi} = \int_0^{\infty} x^{-3/2}\left(1 - e^{-x}\right) dx$$

$$\int_{\beta}^{\infty} x^{-2}\sqrt{x - x\, e^{-x + x\, e^{-x + x\, e^{-x + \dots}}}}\, dx = \sqrt{\pi} + 1 - e^{-1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n+1)!}$$

$$\beta = \frac{e}{e-1}$$

Remarks:

- $\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$; $e = \sum_{n=0}^{\infty} \frac{1}{n!}$
- $\Gamma(a, x)$ is the incomplete gamma function.

IV. References .

[1] Berndt, B.C.: Ramanujan’s Notebooks, Part II, Springer-Verlag,New York, 1989.

[2] Boros, G., and Moll, V.H.: Irresistible Integrals, Cambridge University Press, 2004.

[3] Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals, Series and Products, 7th ed.,Edited by Alan Jeffrey and Daniel Zwillinger,Academic Press, 2007.

[4] Ramanujan, S.: Notebooks. Vols. 1, 2, Tata Institute of Fundamental Research,Bombay,1957.