Fundamental of Mathematics; Division by Zero Calculus and a New Axiom

Saburou Saitoh
Institute of Reproducing Kernels
Kawauchi-cho, 5-1648-16, Kiryu 376-0041, JAPAN
saburou.saitoh@gmail.com

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Abstract: Based on the preprint survey paper ([33]), we will discuss the theoritical point of the division by zero calculus. We will need a new axiom for our mathematics. The contents in this paper seem to be serious for our mathematics and for our world history with the materials in [33]. So, the author hopes that the related mathematicians, mathematical scientists and others check and consider the topics from various viewpoints.

Key Words: Division by zero, division by zero calculus, differential equation, analysis, infinity, discontinuous, point at infinity, Laurent expansion, conformal mapping, stereographic projection, Riemann sphere, horn torus, elementary geometry, zero and infinity, $1/0 = 0/0 = z/0 = \tan(\pi/2) = 0$, axiom, Riemann zeta function.

AMS Mathematics Subject Classifications: 00A05, 00A09, 42B20, 30E20.

1 Backgrounds

In order to state the contents in a self-contained manner, we first state the essence on the division by zero calculus. And in the sequel, we would like to discuss related problems from the theoritical viewpoint.

For the long history of division by zero, see [2, 26]. S. K. Sen and R. P. Agarwal [34] quite recently referred to our paper [12] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book in the following way:

“Thou shalt not divide by zero” remains valid eternally.

However, in [30] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

For the long tradition on the division by zero, people may not be accepted our new results against many clear evidences on our division by zero, however, a physicist stated as follows:

Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out (2017.10.14.08:55).

The common sense on the division by zero with the long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on differential coefficients we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In this paper, in a new and definite sense, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$. In particular, we will introduce several fundamental concepts in calculus, Euclidean geometry, analytic geometry, complex analysis and differential equations. We will see new properties on the Laurent expansion, singularity, derivative, extension of solutions of differential equations beyond analytical and isolated singularities, and reduction problems of differential equations. On Euclidean geometry and analytic geometry, we will find new fields by the concept of the division by zero. We will give many concrete properties in mathematical sciences from the viewpoint of the division by zero. We will know that the division by zero is our elementary and fundamental mathematics.

The contents in ([33]) are as follows.

1. Introduction.

2. Division by zero.
3. Division by zero calculus.
4. We can divide the numbers and analytic functions by zero.
5. General division and usual division.
6. Division by zero calculus.
7. Derivatives of functions.
8. Differential equations.
9. Euclidean spaces and division by zero calculus.
10. Analytic functions and division by zero calculus.
11. The Descartes circle theorem.
12. Horn torus models and division by zero calculus – a new world.

2 Division by zero

The division by zero with the mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [12] for example.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [12, 37]:

Proposition 2.1 *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that*

$$F(a, b)F(c, d) = F(ac, bd)$$

for all

$$a, b, c, d \in \mathbf{C}$$

and

$$F(a, b) = \frac{a}{b}, \quad a, b \in \mathbf{C}, b \neq 0.$$

Then, we obtain, for any $a \in \mathbf{C}$

$$F(a, 0) = 0.$$

In the long mysterious history of the division by zero, this proposition seems to be decisive.

Following Proposition 2.1, we should **define**

$$F(b,0) = \frac{b}{0} = 0,$$

and we should consider that for the mapping

$$W = f(z) = \frac{1}{z}, \tag{2.1}$$

the image $f(0)$ of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere. As the representation of the point at infinity on the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value at the origin by limiting like $+\infty$ and $-\infty$ or by the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotele. – However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([16, 18]) for example. Therefore, the division by zero will give great impact to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations at an undergraduate level and furthermore to our basic idea for the space and universe.

The simple field structure containing division by zero was established by M. Yamada ([15]) in a natural way. For a simple introduction, H. Okumura [17] discovered the very simple essence that:

To divide by zero is to multiply by zero.

For the operator properties of the generalized fractions, see [37].

3 Division by zero calculus

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of unique determination of the results and for other reasons.

We will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{3.1}$$

we **define** the identity

$$f(a) = C_0. \tag{3.2}$$

Apart from the motivation, we define the division by zero calculus by (3.2). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered **as a fundamental assumption like an axiom.**

In addition, we will refer to an interesting viewpoint of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus.

We **define** the value of the function $f(z)$ at the singular point $z = a$ with the above Cauchy integral.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer

In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples to the author by the system that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x = 0) = 1,$$

and others following the questions of the author.

4 We can divide the numbers and analytic functions by zero

In the division by zero like $1/0, 0/0$ the important problem was on their definitions. We will give our interpretation.

Based on the division by zero calculus, the meaning (definition) of

$$\frac{1}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 1/z$. Similarly, the definition

$$\frac{0}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 0/z$.

In the division by zero, the essential problem was in the sense of the division by zero (**definition**) $z/0$. Many confusions and simple history of division by zero may be looked in [23].

In order to give the precise meaning of division by zero, we will give a simple and affirmative answer, for a famous rule that we are not permitted to divide the numbers and functions by zero. In our mathematics, **prohibition** is a famous word for the division by zero.

For any analytic function $f(z)$ around the origin $z = 0$ that is permitted to have any singularity at $z = 0$ (of course, any constant function is permitted), we can consider the value, by the division by zero calculus

$$\frac{f(z)}{z^n} \tag{4.1}$$

at the point $z = 0$, for any positive integer n. This will mean that from **the form** we can consider it as follows:

$$\frac{f(z)}{z^n} \mid_{z=0} . \tag{4.2}$$

For example,

$$\frac{e^x}{x^n} \mid_{x=0} = \frac{1}{n!}.$$

This is the definition of our division by zero (general fraction). In this sense, we can divide the numbers and analytic functions by zero. For $z \neq 0$, $\frac{f(z)}{z^n}$ means the usual division of the function $f(z)$ by z^n.

The content of this subsection was presented by [30].

Surprisingly enough, Brahmāgupta said 0/0=0 in 628, thirteen hundred years ago. However, our world history shows that his result is wrong and we have still in confusions on the division by zero.

However, his result and idea are right.

5 General division and usual division

Since the native division by zero $z/0$ in the sense that from $z/0 = X$ to $z = X \cdot 0 = 0$ is impossible for $z \neq 0$, we introduced its sense by the division by zero calculus. However, in our many formulas in mathematics and mathematical sciences we can see that they have the natural senses; that is for (4.2), we have:

$$\frac{f(z)}{z^n} \mid_{z=0} = \frac{f(0)}{0^n}.$$

However, this is, in general, not valid. Indeed, for the function $f(z) = \sin z$, we have

$$\frac{\sin z}{z} \mid_{z=0} = \frac{\sin 0}{0} = \frac{0}{0} = 0,$$

however, we have, by the division by zero calculus

$$\frac{\sin z}{z} \ |_{z=0} = 1.$$

For the functions $f(z) = 1/z$ and $g(z) = zf(z)$, we have $f(0) = 0$ and $g(0) = 1$ by the division by zero calculus, but we have another result in this way $g(0) = 0 \times f(0) = 0 \times 0 = 0$.

This does not imply any incompleteness of mathematics, that is why, for example, for the product $f(z)g(z)$ of two analytic functions $f(z)$ and $g(z)$ and for the value of $f(z)g(z)$ at a singular point $z = a$, we can consider its value in the both senses; that is,

$$f(z)g(z)|_{z=a}$$

and

$$f(z)|_{z=a} \cdot g(z)|_{z=a}.$$

Those values are, in general, different, in the division by zero calculus.

See [14, 16, 22, 25] for many examples.

For many applications, see the original survey paper ([33]) and the references in this paper.

6 What is mathematics?

We have still curious situations and opinions on the division by zero; in particular, two great challengers Jakub Czajko [7] and Ilija Barukčić [4] on the division by zero in connection with physics stated recently that we do not have the definition of the division $0/0$, however $0/0 = 1$. They seem to think that a truth is based on physical objects and is not on our mathematics.

In particular, J. Czajko [9] stated in Section 9 as follows:

> Mathematics is mainly about forms and operations, and thus is truthless, but its objects must not only be consistent but also realistic, i.e. procedurally operational and structurally constructible. Yet presence of realistic operations and existence of constructible structures for the operations to be performed on the structures should be confirmed by experimental results. Mathematical truths cannot be established

> by abstract mathematical means alone. Yet the unconventional division by zero can reveal where the mathematical truth is about to vanquish due to unsubstantiated existential postulates or arbitrarily decreed operations. Mathematics must not be forced into submission by decrees, for enforcing nonsenses can backfire by producing faulty/contradictory conclusions, the acceptance of which can lead to failures.

In such a case, we will not be able to continue discussions on the division by zero more, because for mathematicians, they will not be able to follow their logics more. However, then we would like to ask for the question that what are the values and contributions of your articles and discussions. We will expect some contributions, of course.

This question will reflect to our mathematicians contrary. We stated for the estimation of mathematics in [27] as follows: Mathematics is a collection of relations and, good results are fundamental, beautiful, and give good impacts to human beings.

With this estimation, we stated that the Euler formula

$$e^{\pi i} = -1$$

is the best result in mathematics in details in:

No.81, May 2012 (pdf 432kb) www.jams.or.jp/kaiho/kaiho-81.pdf

In order to show the importance of our division by zero and division by zero calculus we are requested to show their importance with many examples; over 900 items from an elementary mathematics. However, with the results stated in the references, we think the importance of our division by zero may be definitely stated already and clearly.

7 New axiom; Fundamental of mathematics

Of course, mathematics is developing on some axioms; basic assumptions. For the division by zero calculus, we will need the basic assumption for the definition. We gave many and many motivations and applications, already. Therefore, we have to discuss it whether it is suitable as a new axiom. Without the acceptance as a new axiom, our all the results obtained by the division by zero are looked as the results under the axiom. The results obtained by the division by zero that may be stated in our established mathematics have to be checked in the framework of our established mathematics. Note that here many and many results may be accepted and so for the sake of many and many results, our division by zero calculus may be accepted. However, I would like to point out this point as an important point for our mathematics.

For this point, the e-mails of J.M.R. Caballero are very interested in:

> Dear Saitoh,
>
> In Isabelle/HOL, we can define and redefine every function in different ways. So, logarithm of zero depend upon our definition. The best definition is the one which simplify the proofs the most. According to the experts, z/0 = 0 is the best definition for division by zero.
>
> $$\tan(\pi/2) = 0$$
>
> $$\log 0 =$$
>
> is undefined (but we can redefine it as 0)
>
> $$e^0 = 1$$
>
> (but we can redefine it as 0)
>
> $$0^0 = 1$$
>
> (but we can redefine it as 0).
>
> In the attached file you will find some versions of logarithms and exponentials satisfying different properties. This file can be opened with the software Isabelle/HOL from this webpage: https://isabelle.in.tum.de/
>
> Kind Regards,
>
> José M.
>
> (2017.2.17.11:09).
>
> At 2019.3.4.18:04 for my short question, we received:
>
> It is as it was programmed by the HOL team.
>
> Jose M.

On Mar 4, 2019, Saburou Saitoh wrote:

Dear José M.

I have the short question.

For your outputs for the division by zero calculus, for the input, is it some direct or do you need some program???

With best regards, Sincerely yours,

Saburou Saitoh 2019.3.4.18:00

Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019; 9:45-10:00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [10],

he kindly sent the message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Surprisingly enough, he sent his e-mail at 2019.3.30.18:42 as follows:

Nevertheless, you can use that $x/0 = 0$, following the rules from Isabelle/HOL and you will obtain no contradiction. Indeed, you can check this fact just downloading Isabelle/HOL: https://isabelle.in.tum.de/

and copying the following code

theory DivByZeroSatoih imports Complex Main begin

theorem T: ‹x/0 + 2000 = 2000› for x :: complex by simp

Further:

Dear Saitoh, The system Isabelle/HOL uses that $x/0 = 0$ for all x. HOL stands for Higher Order Logic. So, the higher order logic agrees with your definition of division by zero.

https://en.wikipedia.org/wiki/HOL (proof assistant)

Kind Regards, Jose M.

It seems that he is checking our axiom.

8 Axiom for the division by zero

Of course, for any axiom, we wish to some simple good representation, because we have several equivalent representations, in general. We consider the axiom of the division by zero by its definition. How will be the following representation?:

Axiom of the division by zero: *For any negative integer n and for the function, for any fixed a*

$$f_n(z) = (z - a)^n$$

we have

$$f_n(a) = 0.$$

Of course, if the equation $f_n(z) = (z - a)^n = 0$ has a solution, then the solution has to be a. Indeed, we considered to solve an equation by extending our concept in our mathematics.

9 Remarks

As a history of mathematics, European (containing the USA) people did not like to consider ZERO and VOID for long years, meanwhile people of India will have a long history and deep culture for them. These feelings seem to be still valid with the problem of division by zero.

10 Conclusions

The contents in this paper seem to be serious for our mathematics and for our world history with the materials in [33]. So, the author expects that the related mathematicians, mathematical scientists and others check and consider the topics from various viewpoints.

As a typical new result, we would like to refer to the famous problem for the values of the Riemann zeta function at positive odd integer points and its complete solution from ([24]):

Recall the expansion

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{k=1}^{\infty}\left(\frac{1}{z+k} - \frac{1}{k}\right) \tag{10.1}$$

([5], page 53). We obtain, taking $n-1; (n > 2)$ order derivative, by the division by zero calculus

$$\zeta(n) = \frac{(-1)^n}{(n-1)!}\psi^{(n-1)}(z)|_{z=0}. \tag{10.2}$$

Recall the expansion

$$\psi(z+1) = -\gamma + \sum_{k=2}^{\infty}(-1)^k\zeta(k)z^{k-1} \quad (|z| < 1) \tag{10.3}$$

([1], page 259, 6.3.14). Then we obtain

$$\frac{\psi(z+1)}{z^{n-1}} = \frac{-\gamma}{z^{n-1}} + \sum_{k=2}^{\infty}(-1)^k\zeta(k)\frac{z^{k-1}}{z^{n-1}} \quad (|z| < 1). \tag{10.4}$$

Hence, by the division by zero calculus, we obtain, for $n > 2$

$$\zeta(n) = (-1)^n\frac{\psi(z+1)}{z^{n-1}}|_{z=0}. \tag{10.5}$$

Then, by using (10.3), we obtain for $n = 2$, by MATHEMATICA

$$\zeta(3) = 1 - \frac{\psi^{(2)}(2)}{2} \sim 1.20206.$$

Note that with MATHEMATICA, we can derive the Laurent expansion for many analytic functions and so we can obtain the division by zero calculus for many analytic functions.

In general, we have

Theorem:

$$\zeta(n) = 1 - \frac{\psi^{(n-1)}(2)}{n-1}. \tag{10.6}$$

These values may be calculated easily as follows:

$$\zeta(5) = 1 - \frac{1}{24}\psi^{(4)}(2) \sim 1.03693,$$

$$\zeta(6) = \frac{\pi^6}{945} \sim 1.01734$$

$$\zeta(7) = 1 - \frac{1}{720}\psi^{(6)}(2) \sim 1.00835,$$

$$\zeta(8) = \frac{\pi^8}{9450} \sim 1.00408,$$

$$\zeta(9) = 1 - \frac{1}{40320}\psi^{(8)}(2) \sim 1.00201.$$

Note that the value of the function $\psi(z)$ may be calculated easily by MATHEMATICA.

In particular, note the well-known formulas:

$$\zeta(2n) = (-1)^{n+1}\,\frac{B_{2n}(2\pi)^{2n}}{2(2n)!},$$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

The following formula was given by S. Ramanujan (1887 - 1920)

$$\zeta(2n+1) = 2^{2n}\,\pi^{2n+1}\sum_{k=0}^{n+1}(-1)^{k+1}\,\frac{B_{2k}}{(2k)!}\,\frac{B_{2n+2-2k}}{(2n+2-2k)!}$$
$$-2\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\pi k}-1}.$$

For the known results, our results are the similar.

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References

[1] M. Abramowitz and I. Stengun, HANDBOOK OF MATHEMATICAL FUNCTIONS WITH FORMULAS, GRAPHS, AND MATHEMATICAL TABLES, Dover Publishings, Inc. (1972).

[2] C. B. Boyer, An early reference to division by zero, The Journal of the American Mathematical Monthly, **50** (1943), (8), 487- 491. Retrieved March 6, 2018, from the JSTOR database.

[3] J. P. Barukčić and I. Barukčić, Anti Aristotle - The Division Of Zero By Zero, ViXra.org (Friday, June 5, 2015) Germany. All rights reserved. Friday, June 5, 2015 20:44:59.

[4] I. Barukčić, Dialectical Logic - Negation Of Classical Logic, http://vixra.org/abs/1801.0256.

[5] K. Chandrasekharan, Lectures on The Riemann Zeta–Function By K. Chandrasekharan, Tata Institute of Fundamental Research, Bombay 1953.

[6] J. Czajko, On Cantorian spacetime over number systems with division by zero, Chaos, Solitons and Fractals, **21**(2004), 261-271. doi:10.1016/j.chaos.2003.12.046.

[7] J. Czajko, Equalized mass can explain the dark energy or missing mass problem as higher density of matter in stars amplifies their attraction, Available online at www.worldscientificnews.com WSN **80** (2017), 207-238. EISSN 2392-2192.

[8] J. Czajko, Algebraic division by zero implemented as quasigeometric multiplication by infinity in real and complex multispatial hyperspaces, World Scientific News **92**(2) (2018), 171-197.

[9] J. Czajko, Wave-particle duality of a 6D wavicle in two paired 4D dual reciprocal quasispaces of a heterogeneous 8D quasispatial structure, World Scientific News **127**(1) (2019), 1–55.

[10] W. W. Däumler, H. Okumura, V. V. Puha and S. Saitoh, Horn Torus Models for the Riemann Sphere and Division by Zero, viXra:1902.0223 submitted on 2019-02-12 18:39:18.

[11] R. Kaplan, THE NOTHING THAT IS A Natural History of Zero, OXFORD UNIVERSITY PRESS (1999).

[12] M. Kuroda, H. Michiwaki, S. Saitoh and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, Int. J. Appl. Math. **27** (2014), no 2, pp. 191-198, DOI: 10.12732/ijam.v27i2.9.

[13] T. Matsuura and S. Saitoh, Matrices and division by zero $z/0 = 0$, Advances in Linear Algebra & Matrix Theory, **6**(2016), 51-58 Published Online June 2016 in SciRes. http://www.scirp.org/journal/alamt http://dx.doi.org/10.4236/alamt.2016.62007.

[14] T. Matsuura, H. Michiwaki and S. Saitoh, $\log 0 = \log \infty = 0$ and applications, Differential and Difference Equations with Applications, Springer Proceedings in Mathematics & Statistics, **230** (2018), 293-305.

[15] H. Michiwaki, S. Saitoh and M.Yamada, Reality of the division by zero $z/0 = 0$, IJAPM International J. of Applied Physics and Math. **6**(2015), 1–8. http://www.ijapm.org/show-63-504-1.html

[16] H. Michiwaki, H. Okumura and S. Saitoh, Division by Zero $z/0 = 0$ in Euclidean Spaces, International Journal of Mathematics and Computation, **2**8(2017); Issue 1, 1-16.

[17] H. Okumura, To Divide by Zero is to Multiply by Zero, viXra:1811.0132., (2018).

[18] H. Okumura, S. Saitoh and T. Matsuura, Relations of 0 and ∞, Journal of Technology and Social Science (JTSS), **1**(2017), 70-77.

[19] H. Okumura and S. Saitoh, The Descartes circles theorem and division by zero calculus, https://arxiv.org/abs/1711.04961 (2017.11.14).

[20] H. Okumura and S. Saitoh, Harmonic Mean and Division by Zero, Dedicated to Professor Josip Pečarić on the occasion of his 70th birthday, Forum Geometricorum, **18** (2018), 155—159.

[21] H. Okumura and S. Saitoh, Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by H. Okumura and M. Watanabe, Forum Geometricorum, **18**(2018), 97-100.

[22] H. Okumura and S. Saitoh, Applications of the division by zero calculus to Wasan geometry, GLOBAL JOURNAL OF ADVANCED RESEARCH ON CLASSICAL AND MODERN GEOMETRIES" (GJARCMG), **7**(2018), 2, 44–49.

[23] H. Okumura and S. Saitoh, Wasan Geometry and Division by Zero Calculus, Sangaku Journal of Mathematics (SJM), **2** (2018), 57–73.

[24] H. Okumura and S. Saitoh, Values of the Riemann Zeta Function by Means of Division by Zero Calculus viXra:1907.0437 submitted on 2019-07-23 20:48:54.

[25] S. Pinelas and S. Saitoh, Division by zero calculus and differential equations. Differential and Difference Equations with Applications. Springer Proceedings in Mathematics & Statistics, **230** (2018), 399-418.

[26] H. G. Romig, Discussions: Early History of Division by Zero, American Mathematical Monthly, **3**1, No. 8. (Oct., 1924), 387-389.

[27] S. Saitoh, Nonlinear transforms and analyticity of functions, T. M. Rassias, Editor, Nonlinear Mathematical Analysis and Applications, Hadronic Press,Palm Harbor, FL34682-1577,USA:ISBN1-57485-044-X, (1998), 223–234.

[28] S. Saitoh, A reproducing kernel theory with some general applications, Qian,T./Rodino,L.(eds.): Mathematical Analysis, Probability and Applications - Plenary Lectures: Isaac 2015, Macau, China, Springer Proceedings in Mathematics and Statistics, **177**(2016), 151-182.

[29] S. Saitoh, Mysterious Properties of the Point at Infinity, arXiv:1712.09467 [math.GM](2017.12.17).

[30] S. Saitoh, We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense, viXra:1902.0058 submitted on 2019-02-03 22:47:53.

[31] S. Saitoh, Zero and Infinity; Their Interrelation by Means of Division by Zero, viXra:1902.0240 submitted on 2019-02-13 22:57:25.

[32] S. Saitoh, Division by Zero Calculus in Trigonometric Functions, viXra:1903.0566 submitted on 2019-03-31 15:59:03.

[33] S. Saitoh, What Was Division by Zero?; Division by Zero Calculus and New World, viXra:1904.0408 submitted on 2019-04-22 00:32:30.

[34] S.K.S. Sen and R. P. Agarwal, ZERO A Landmark Discovery, the Dreadful Volid, and the Unitimate Mind, ELSEVIER (2016).

[35] F. Soddy, The Kiss Precise, Nature **137**(1936), 1021. doi:10.1038/1371021a0.

[36] E. Sondheimer and A. Rogerson, NUMBERS AND INFINITY A Historical Account of Mathematical Concepts, Dover (2006) unabridged republication of the published by Cambridge University Press, Cambridge (1981).

[37] S.-E. Takahasi, M. Tsukada and Y. Kobayashi, Classification of continuous fractional binary operations on the real and complex fields, Tokyo Journal of Mathematics, **38**(2015), no. 2, 369-380.