

Special Relativity → Quantum Mechanics

The SRQM Interpretation of Quantum Mechanics

A Tensor Study of Physical 4-Vectors

Using Special Relativity (SR) as a starting point, then noting a few empirical 4-Vector facts, one can derive the Principles that are normally considered to be Axioms of Quantum Mechanics (QM).

Since many of the QM Axioms are rather obscure, this seems a more logical and understandable paradigm than QM as a separate theory from SR, and sheds light on the origin and meaning of the QM Principles.

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

or: Why General Relativity (GR) is *NOT* wrong
or: Don't bet against Einstein ;) ;)
or: QM, the easy way...

*And yes,
I did the Math...*

Special Relativity → Quantum Mechanics

The SRQM Interpretation of Quantum Mechanics

A Tensor Study of Physical 4-Vectors

4-Vectors are a fantastic language/tool for describing the physics of both relativistic and quantum phenomena. They easily show many interesting properties and relations of our Universe, and do so in a simple and concise mathematical way. Due to their tensorial nature, these SR 4-Vectors are automatically coordinate-frame invariant, and can be used to generate *ALL* of the physical SR Lorentz Scalar tensors and higher-index-count SR tensors. Let me repeat: You can mathematically build *ALL* the Lorentz Scalars and larger SR tensors from SR 4-Vectors.

4-Vectors are likewise easily shown to be related to the standard 3-vectors that are used in Newtonian classical mechanics, Maxwellian classical electromagnetism, and standard quantum theory.

Why 4-Vectors as opposed to some of the more abstract mathematical approaches to QM? Because the components of 4-Vectors are physical properties that can actually be empirically measured. Experiment is the ultimate arbiter of which theories actually correspond to reality. If your quantum logics and string theories give no testable/measurable predictions, then they are basically useless for real physics.

In this treatise, I will demonstrate how 4-Vectors are used in the context of Special Relativity, and then show that their use in Relativistic Quantum Mechanics is really not fundamentally different. Quantum Principles then emerge in a natural and elegant way.

I also introduce the [SRQM Diagramming Method](#), an instructive graphical charting-method, which visually shows how the SRQM 4-Vectors, Lorentz 4-Scalars, and 4-Tensors are all related to each other. This symbolic representation clarifies a lot of physics and is a great tool for teaching and understanding.

SRQM

Some Physics Abbreviations
& NotationA Tensor Study
of Physical 4-VectorsSciRealm.org
John B. Wilson

GR = General Relativity
 SR = Special Relativity
 CM = Classical Mechanics
 EM = ElectroMagnetism
 QM = Quantum Mechanics
 RQM = Relativistic Quantum Mechanics
 NRQM = Non-Relativistic Quantum Mechanics
 QFT = Quantum Field Theory
 QED = Quantum ElectroDynamics
 RWE = Relativistic Wave Equation
 KG = Klein-Gordon (Relativistic Quantum) Eqn
 PDE = Partial Differential Equation
 H = The Hamiltonian = $\gamma(\mathbf{P}_T \cdot \mathbf{U})$
 L = The Lagrangian = $-(\mathbf{P}_T \cdot \mathbf{U})/\gamma$
 $\nabla = 3\text{-gradient} = (\partial_x, \partial_y, \partial_z) = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$
 $\partial = 4\text{-Gradient} = \partial^\mu = (\partial/c, -\nabla)$; $\partial_\mu = (\partial/c, \nabla)$
 S = The Action (4-TotalMomentum $\mathbf{P}_T = -\partial[S]$)
 $\Phi = \text{The Phase (4-TotalWaveVector } \mathbf{K}_T = -\partial[\Phi])$
 $\tau = \text{Proper Time (Invariant Rest Time)} = t_o$
 $\Sigma = \text{Sum of Range}$; $\Pi = \text{Product of Range}$
 $\Delta = \text{Difference}$; d = Differential ; $\partial = \text{Partial}$

$\beta = \text{Relativistic Beta} = \mathbf{v}/c = \{0..1\}\hat{n}$; $\mathbf{v} = 3\text{-velocity} = \{0..c\}\hat{n}$
 $\gamma = \text{Relativistic Gamma} = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\beta \cdot \beta} = \{1..\infty\}$
 $D = \text{Relativistic Doppler} = 1/[\gamma(1-|\beta|\cos[\theta])]$
 $\Lambda^\mu_\nu = \text{Lorentz (SpaceTime) Transform}$
 $I_{(3)} = 3\text{D Identity Matrix}$; $I_{(4)} = 4\text{D Identity Matrix} = \text{Diag}[1,1,1,1]$
 $\delta^{ij} = \delta^i_j = \delta_{ij} = I_{(3)} = \{1 \text{ if } i=j, \text{ else } 0\}$ 3D Kronecker delta
 $\delta^{\mu\nu} = \delta^\mu_\nu = \delta_{\mu\nu} = I_{(4)} = \{1 \text{ if } \mu=\nu, \text{ else } 0\}$ 4D Kronecker Delta
 $\eta^{\mu\nu} \rightarrow \eta_{\mu\nu} \rightarrow \text{Diag}[1, -I_{(3)}]_{\text{rect}}$ Minkowski "Flat SpaceTime" Metric
 $\eta^\mu_\nu = \delta^\mu_\nu = \text{Diag}[1, I_{(3)}] = I_{(4)} = g^\mu_\nu$ {also true in GR} (1,1)-Tensor Metric
 $\epsilon^{ijk} = 3\text{D Levi-Civita anti-symmetric permutation symbol}$
 $\epsilon^{\mu\nu\rho\sigma} = 4\text{D Levi-Civita Anti-symmetric Permutation Symbol}$
 {other upper:lower index combinations possible for Levi-Civita symbol}

Tensor-Index & 4-Vector Notation:

$A^i = \mathbf{a} = (a^1, a^2, a^3)$: 3-vector [Latin index {1..3}]
 $A^\mu = \mathbf{A} = (a^0, a^1, a^2, a^3)$: 4-Vector [Greek index {0..3}]
 $A^\mu B_\mu = A_\nu B^\nu = \mathbf{A} \cdot \mathbf{B}$: Einstein Sum : Dot Product : Inner Product
 $A^\mu B^\nu = \mathbf{A} \otimes \mathbf{B}$: Tensor Product : Outer Product
 $A^\mu B^\nu - A^\nu B^\mu = \mathbf{A} \wedge \mathbf{B}$: Wedge Product : Exterior Product
 $A^\mu B^\nu - A^\mu B^\nu = 0^{\mu\nu}$: (2,0)-Zero Tensor
 $A^\mu B^\nu - B^\nu A^\mu = [A^\mu, B^\nu] = [\mathbf{A}, \mathbf{B}]$: Commutation
 $A^\mu B^\nu - B^\mu A^\nu = ???$

Special Relativity → Quantum Mechanics

The SRQM Interpretation: Links

A Tensor Study
of Physical 4-Vectors

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<http://scirealm.org/SRQM.pdf>

See also:

<http://scirealm.org/SRQM.html> (alt discussion)

<http://scirealm.org/SRQM-RoadMap.html> (main SRQM website)

<http://scirealm.org/4Vectors.html> (4-Vector study)

<http://scirealm.org/SRQM-Tensors.html> (Tensor & 4-Vector Calculator)

<http://scirealm.org/SciCalculator.html> (Complex-capable RPN Calculator)

or Google “SRQM”

<http://scirealm.org/SRQM.pdf> (this document, most current ver at scirealm.org)

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

SRQM Study: Physical/Mathematical Tensors

Tensor Types: 4-Scalar, 4-Vector, 4-Tensor

Component Types: Temporal, Spatial, Mixed

A Tensor Study of Physical 4-Vectors

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John B. Wilson

Matrix Format

SRQM Diagram Format

Each 4D index = {0,1..3} = Tensor Rank 4

SR 4-Scalar S

S

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

SRQM Diagram Ellipse:
4-Scalars, 0 index
4*0 = 0 corners
4⁰ = (1) = 1 component

1 Temporal + 3 Spatial
= 4 SpaceTime Dimensions

(m,n)-Tensor has:
(m) # upper-indices
(n) # lower-indices

SR:Minkowski Metric
 $\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$
 Diag[1,-1,-1,-1] = Diag[1,-I₍₃₎] = Diag[1,-δ^{jk}]
 {in Cartesian form} "Particle Physics" Convention
 $\{\eta_{\mu\nu}\} = 1/\{\eta^{\mu\nu}\} : \eta_\mu^\nu = \delta_\mu^\nu$ **Tr[η^{μν}]=4**

SR 4-Vector V^μ

V⁰ V¹ V² V³

SR 4-Vector
(1,0)-Tensor V
V^μ = (V^μ) = (V⁰, v) = (V⁰, vⁱ)
→ (v^t, v^x, v^y, v^z)

SRQM Diagram Rectangle:
4-Vectors, 1 index
4*1 = 4 corners
4¹ = (1+3) = 4 components

SR 4-CoVector = "Dual" 4-Vector
(0,1)-Tensor aka. One-Form
C_μ = η_{μσ}C^σ = (c_μ) = (c₀, c_i) → (c_t, c_x, c_y, c_z)
= (c⁰, -c) = (c⁰, -cⁱ) → (c^t, -c^x, -c^y, -c^z)

SR:Lorentz Transform
 $\partial_\nu[R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu$
 $\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$
 $\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$
Det[Λ^{μ_ν}]=±1 **Λ_{μν}Λ^{μν}=4**

SR 4-Tensor T^{μν}

T ⁰⁰	T ⁰¹	T ⁰²	T ⁰³
T ¹⁰	T ¹¹	T ¹²	T ¹³
T ²⁰	T ²¹	T ²²	T ²³
T ³⁰	T ³¹	T ³²	T ³³

SR 4-Tensor
(2,0)-Tensor T
T^{μν} =
[T⁰⁰, T^{0k}]
[T^{j0}, T^{jk}]
→
[T^{tt}, T^{tx}, T^{ty}, T^{tz}]
[T^{xt}, T^{xx}, T^{xy}, T^{xz}]
[T^{yt}, T^{yx}, T^{yy}, T^{yz}]
[T^{zt}, T^{zx}, T^{zy}, T^{zz}]

SRQM Diagram Octagon:
4-Tensors, 2 index
4*2 = 8 corners
4² = (1+6+9) = 16 components
for 2-index tensors:
6 Anti-Symmetric (Skew)
+10 Symmetric
===== 16 General components

<p>SR Mixed 4-Tensor (1,1)-Tensor T_{μ^ν} = η_{μρ}T^{ρν} = [T_{0⁰}, T_{0^k}] [T_{j⁰}, T_{j^k}] = [+T⁰⁰, +T^{0k}] [-T^{j0}, -T^{jk}]</p>	<p>SR Mixed 4-Tensor (1,1)-Tensor T^{μ_ν} = η^{ρν}T_{μρ} = [T^{0₀}, T^{0_k}] [T^{j₀}, T^{j_k}] = [+T⁰⁰, -T^{0k}] [+T^{j0}, -T^{jk}]</p>	<p>SR Lowered 4-Tensor (0,2)-Tensor T_{μν} = η_{μρ}η_{νσ}T^{ρσ} = [T₀₀, T_{0k}] [T_{j0}, T_{jk}] = [+T⁰⁰, -T^{0k}] [-T^{j0}, +T^{jk}]}</p>
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Temporal region: blue
Spatial region: red
Mixed TimeSpace region: purple
The mnemonic being red and blue mixed make purple

<p>SR 4-Tensor (2,0)-Tensor T^{μν} (1,1)-Tensor T^{μ_ν} or T_{μ^ν} (0,2)-Tensor T_{μν}</p>	<p>SR 4-Vector (1,0)-Tensor V^μ = V = (v⁰, v) SR 4-CoVector (0,1)-Tensor V_μ = (v₀, -v)</p>	<p>SR 4-Scalar (0,0)-Tensor S Lorentz Scalar</p>
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Technically, all these objects are "SR 4-Tensors", but we usually reserve the name "4-Tensor" for objects with 2 or more indices, and use the (m,n)-Tensor notation to specify all the objects more precisely.

Trace[T^{μν}] = η_{μν}T^{μν} = T^{μ_μ} = T
V·V = V^{μ}η_{μν}V^ν = [(v⁰)² - v·v] = (v⁰)² = Lorentz Scalar}

Special Relativity → Quantum Mechanics

SRQM Diagramming Method

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The SRQM Diagramming Method shows the properties and relationships of various physical objects in a graphical way. This "flowchart" method aids understanding.

Representation: 4-Scalars by ellipses, 4-Vectors by rectangles, 4-Tensors by octagons. Physical/mathematical equations and descriptions inside each shape/object. Sometimes there will be additional clarifying descriptions around a shape/object.

Relationships: Lorentz Scalar Products or tensor compositions of different 4-Vectors are on simple lines between the related 4-Vectors. Lorentz Scalar Products of a single 4-Vector, or Invariants of Tensors, are next to that object and highlighted in a different color.

Flow: Objects that are some function of a Lorentz Scalar with another 4-Vector or 4-Tensor are on lines with arrows indicating the direction of flow. (ex. multiplication)

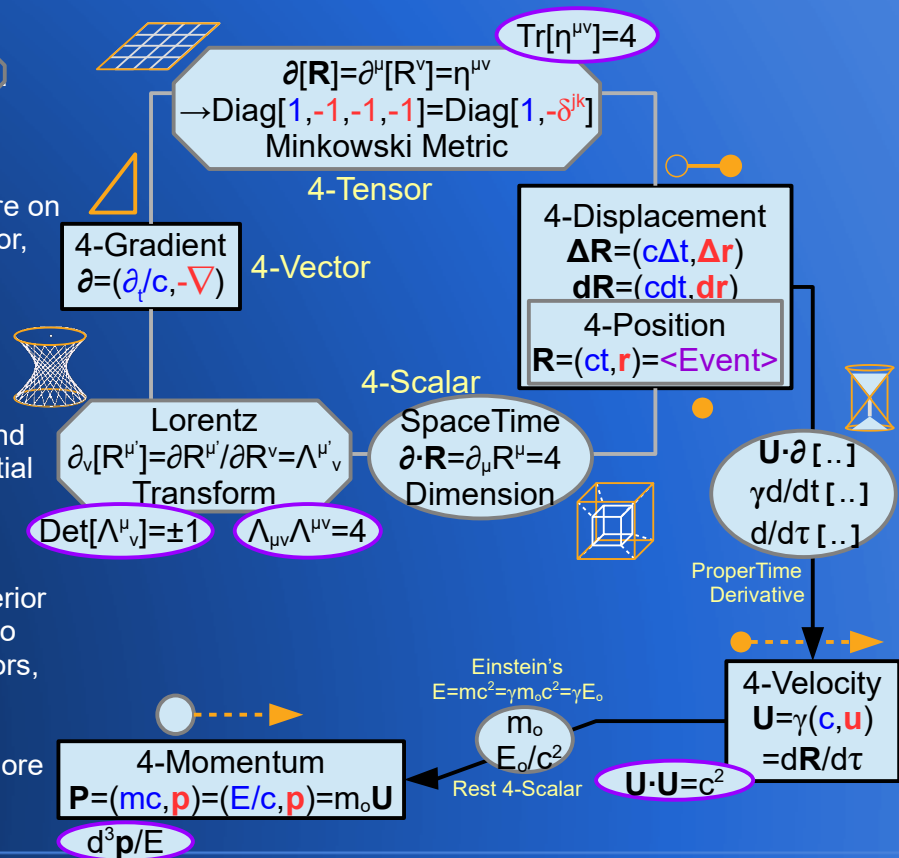
Properties: Some objects will also have a symbol representing its properties nearby, and sometimes there will be color highlighting within the object to emphasize temporal-spatial properties. I typically use blue=Temporal, red=Spatial, purple=mixed TimeSpace.

Alternate ways of writing 4-Vector expressions in physics:

(A · B) is a 4-Vector style, which uses vector-notation (ex. inner product "dot=" or exterior product "wedge="), and is typically more compact, always using **bold** UPPERCASE to represent the 4-Vector, ex. **(A · B)** = $(A^\mu \eta_{\mu\nu} B^\nu)$, and **bold** lowercase to represent 3-vectors, ex. **(a · b)** = $(a^j \delta_{jk} b^k)$. Most 3-vector rules have analogues in 4-Vector mathematics.

(A^μη_{μν}B^ν) is a Ricci Calculus style, which uses tensor-index-notation and is useful for more complicated expressions, especially to clarify those expressions involving tensors with more than one index, such as the Faraday EM Tensor $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial \wedge A)$

SRQM Diagramming Method



SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

Special Relativity → Quantum Mechanics

SRQM Tensor Invariants

A Tensor Study of Physical 4-Vectors

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One of the extremely important properties of Tensor Mathematics is the fact that there are numerous ways to generate Tensor Invariants. These Invariants lead to Physical Properties that are fundamental in our Universe. They are totally independent of the coordinate systems used to measure them. Thus, they represent symmetries that are inherent in the fabric of SpaceTime. See the Cayley-Hamilton Theorem, esp. for the AntiSymmetric Tensor Products.

Trace Tensor Invariant: $Tr[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = \Sigma[\text{EigenValues } \lambda_n]$ for $T^\mu{}_\nu$

Determinant Tensor Invariant: $Det[T^{\mu\nu}] = \Pi[\text{EigenValues } \lambda_n]$ for $T^\mu{}_\nu$

Inner Product Tensor Invariant: $IP[T^{\mu\nu}] = T^{\mu\nu} T_{\mu\nu}$

4-Divergence Tensor Invariant: $4\text{-Div}[T^\mu] = \partial_\mu T^\mu = \partial \cdot T = \partial T^\mu / \partial X^\mu : 4\text{-Div}[T^{\mu\nu}] = S^\nu = \partial_\mu T^{\mu\nu}$

Lorentz Scalar Product Tensor Invariant: $LSP[T^\mu, S^\nu] = T^\mu \eta_{\mu\nu} S^\nu = T^\mu S_\mu = T_\nu S^\nu = T \cdot S$

Phase Space Tensor Invariant: $PS[T^\mu] = (d^3\mathbf{t} / t^0) = (dt^1 dt^2 dt^3 / t^0)$ for $(T \cdot T) = \text{constant}$

The Ratio of 4-Vector Magnitudes (Ratio of Rest Value 4-Scalars): $T \cdot T / S \cdot S = (t^0 / s^0)^2$

Tensor Eigenvalues $\lambda_n = \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \}$

The various AntiSymmetric Tensor Products, etc.

$T^\alpha{}_\alpha = \text{Trace} = \Sigma[\text{EigenValues } \lambda_n]$ for (1,1)-Tensors

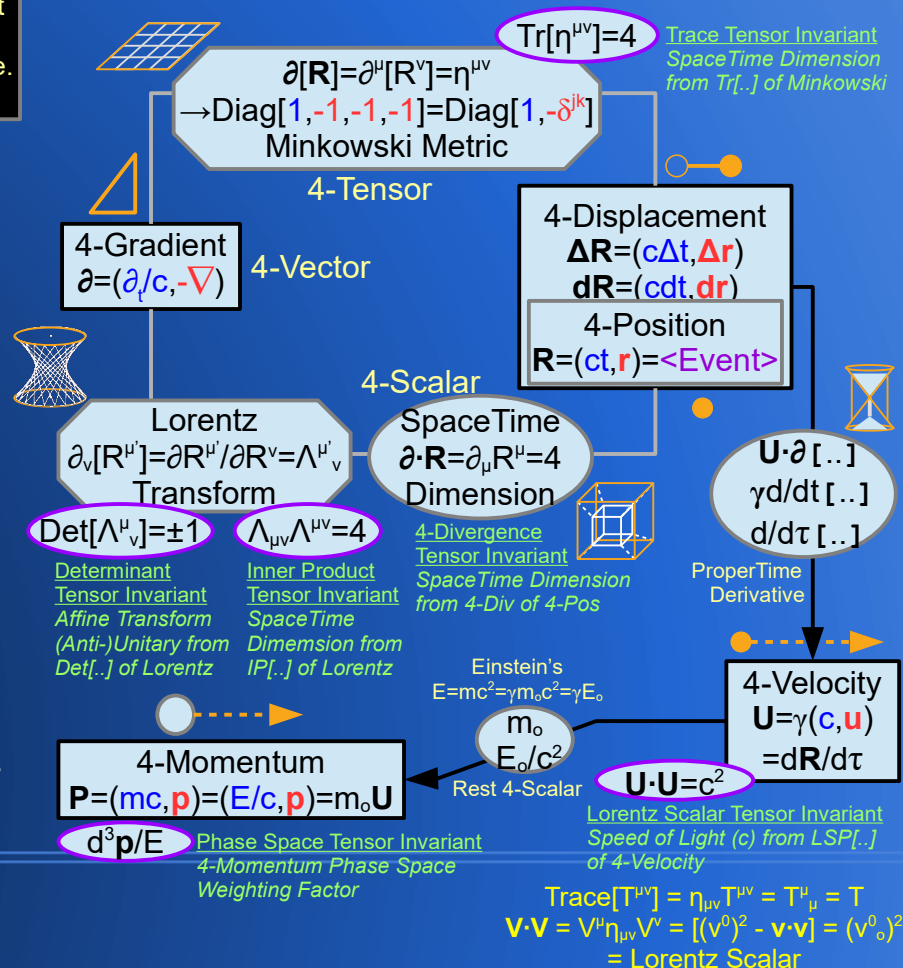
$T^\alpha{}_{[\alpha} T^\beta{}_{\beta]} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product}$

$T^\alpha{}_{[\alpha} T^\beta{}_\beta T^\gamma{}_\gamma] = \text{Asymm Tri-Product} \rightarrow ?\text{Name?}$

$T^\alpha{}_{[\alpha} T^\beta{}_\beta T^\gamma{}_\gamma T^\delta{}_\delta] = \text{Asymm Quad-Product} \rightarrow 4\text{D Determinant} = \Pi[\text{EigenValues } \lambda_n]$ for (1,1)-Tensors

These are not all always independent, some invariants are functions of other invariants.

SRQM Diagramming Method



SRQM Study: Physical/Mathematical Tensors

Tensor Types: 4-Scalar, 4-Vector, 4-Tensor

Examples – Venn Diagram

A Tensor Study of Physical 4-Vectors


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Physical 4-Tensors: Objects which have Invariant 4D SpaceTime properties

0 index-count Tensors:

SR 4-Scalar (0,0)-Tensors

- EM Charge ($Q = \int \rho d^3\mathbf{x}$)
- Lorentz Scalar S
- Speed-of-Light ($c = \sqrt{[\mathbf{U} \cdot \mathbf{U}]}$)
- #dimensionless
- RestMass (m_0)
- Planck's Const (h)
- $\delta^4[\mathbf{X} - \mathbf{X}_0]$
- ProperTime $\mathbf{U} \cdot \partial = d/d\tau = \gamma d/dt$ Derivative
- $V_0 = \int \gamma d^3\mathbf{x}$
- SpaceTime Dimension $\partial \cdot \mathbf{R} = \partial_\mu \mathbf{R}^\mu = 4$
- $\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$
- $\text{Det}[\Lambda^\mu_\nu] = \pm 1$
- $\text{Tr}[\eta^{\mu\nu}] = 4$
- $d^4\mathbf{X} = c dt \cdot dx \cdot dy \cdot dz$



1 index-count Tensors:

SR 4-Vector (1,0)-Tensors

$$V^\mu = \mathbf{V} = (v^\mu)$$

$$= (v^0, \mathbf{v}) = (v^0, v^1, v^2, v^3)$$

4-Position $\mathbf{R} = \mathbf{R}^\mu = (ct, \mathbf{r}) = \langle \text{Event} \rangle$
→ (ct, x, y, z)

4-Velocity $\mathbf{U} = \mathbf{U}^\mu = (c, \mathbf{u})$
= d \mathbf{R} /d τ

4-Momentum $\mathbf{P} = \mathbf{P}^\mu = (mc, \mathbf{p}) = m_0 \mathbf{U}$
= (E/c, \mathbf{p}) = (E_0/c^2) \mathbf{U}

SR 4-CoVector = "Dual" 4-Vector (0,1)-Tensors aka. One-Forms

$$C_\mu = \eta_{\mu\sigma} C^\sigma = (C_\mu) = (C_0, C_i) \rightarrow (C_i, C_x, C_y, C_z)$$

$$= (c^0, -\mathbf{c}) = (c^0, -c^1, -c^2, -c^3)$$

Gradient One-Form

$$\partial_\mu = (\partial/c, \nabla)$$

$$\rightarrow (\partial/c, \partial_x, \partial_y, \partial_z)$$

$$= (\partial/c\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)$$

2 index-count Tensors:

SR 4-Tensor (2,0)-Tensors

$$T^{\mu\nu} =$$

$$[T^{00}, T^{0k}]$$

$$[T^{j0}, T^{jk}]$$

Minkowski Metric $\eta^{\mu\nu} = \partial^\mu [R^\nu] = \partial[\mathbf{R}] = V^{\mu\nu} + H^{\mu\nu}$

Faraday EM 4-Tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge \mathbf{A}$

Perfect Fluid 4-Tensor $T^{\mu\nu} = (\rho_{eo}) V^{\mu\nu} + (-p_o) H^{\mu\nu}$

SR Mixed 4-Tensor (1,1)-Tensors

$$T^\mu_\nu = \eta_{\rho\nu} T^{\mu\rho}$$

$$=$$

$$[T^0_0, T^0_k]$$

$$[T^j_0, T^j_k]$$

Lorentz Transforms $\partial_\nu [R^\mu] = \Lambda^\mu_\nu$

Projection (Mixed) Tensors P^μ_ν

Temporal Projection $P^\mu_\nu \rightarrow V^\mu_\nu$

Spatial Projection $P^\mu_\nu \rightarrow H^\mu_\nu$

Lorentz Boost $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$

Lorentz ParityInverse $\Lambda^\mu_\nu \rightarrow P^\mu_\nu$

SR Lowered 4-Tensor (0,2)-Tensors

$$T_{\mu\nu} = \eta_{\mu\rho} \eta_{\nu\sigma} T^{\rho\sigma}$$

$$=$$

$$[T_{00}, T_{0k}]$$

$$[T_{j0}, T_{jk}]$$

Lowered Minkowski Metric $\partial_\mu [R_\nu] = \eta_{\mu\nu} = (\cdot \cdot)$ Metric

Projection Tensors $P_{\mu\nu}$

Temporal Proj. $P_{\mu\nu} \rightarrow V_{\mu\nu}$

Spatial Proj. $P_{\mu\nu} \rightarrow H_{\mu\nu}$

Higher index-count Tensors:

SR & GR 4-Tensors T^{\dots}

Riemann Curvature Tensor

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \rightarrow 0^{\rho}_{\sigma\mu\nu} \text{ for SR "Flat" Minkowski SpaceTime}$$

Weyl (Conformal) Curvature Tensor

$$C^{\rho}_{\sigma\mu\nu} = \text{Traceless part of Riemann } [R^{\rho}_{\sigma\mu\nu}]$$

SR 4-Tensor

(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or $T_{\mu}{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector

(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar

(0,0)-Tensor S
Lorentz Scalar

Ricci Decomposition of Riemann Tensor

$$R^{\rho}_{\sigma\mu\nu} = S^{\rho}_{\sigma\mu\nu} \text{ (scalar part)} + E^{\rho}_{\sigma\mu\nu} \text{ (semi-traceless part)} + C^{\rho}_{\sigma\mu\nu} \text{ (traceless part)}$$

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$$

SRQM 4-Vectors = (1,0)-Tensors

4-Tensors = (2+ index)-Tensors

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4-Vector = Type (1,0)-Tensor

- 4-Position $\mathbf{R} = R^\mu = (ct, \mathbf{r})$
- 4-Velocity $\mathbf{U} = U^\mu = \gamma(\mathbf{c}, \mathbf{u}) = (\gamma c, \gamma \mathbf{u})$
- 4-Unit Temporal $\mathbf{T} = T^\mu = \gamma(1, \boldsymbol{\beta}) = (\gamma, \gamma \boldsymbol{\beta})$
- 4-Momentum $\mathbf{P} = P^\mu = (E/c, \mathbf{p})$
- 4-Total Momentum $\mathbf{P}_T = P_T^\mu = (E_T/c = H/c, \mathbf{p}_T) = \sum_n [P_n]$
- 4-Force $\mathbf{F} = F^\mu = \gamma(\dot{E}/c, \mathbf{f})$
- 4-Wave Vector $\mathbf{K} = K^\mu = (\omega/c, \mathbf{k})$
- 4-Total Wave Vector $\mathbf{K}_T = K_T^\mu = (\omega_T/c, \mathbf{k}_T) = \sum_n [K_n]$
- 4-Current Density $\mathbf{J} = J^\mu = (\rho c, \mathbf{j})$
- 4-Vector Potential $\mathbf{A} = A^\mu = (\phi/c, \mathbf{a})$
- 4-Potential Momentum $\mathbf{Q} = Q^\mu = q\mathbf{A} = (V/c = \phi q/c, q\mathbf{a})$
- 4-Gradient $\partial_R = \partial = \partial^\mu = \partial/\partial R_\mu = (\partial_t/c, -\nabla)$
- 4-Number Flux $\mathbf{N} = N^\mu = n(\mathbf{c}, \mathbf{u}) = (nc, n\mathbf{u})$
- 4-Spin $\mathbf{S} = S^\mu = (s^0, \mathbf{s}) = (\mathbf{s} \cdot \boldsymbol{\beta}, \mathbf{s})$

SI Units

- [m]
- [m/s]
- [dimensionless]
- [kg · m/s]
- [kg · m/s]
- [N = kg · m/s²]
- [rad/m]
- [rad/m]
- [C/m² · s]
- [T · m = kg · m/C · s]
- [kg · m/s]
- [1/m]
- [#/m² · s]
- [J · s = N · m · s = kg · m²/s]

[Temporal : Spatial] components

- [Time (t) : Space (r)]
- [Temporal "Velocity" Factor (γ) : Spatial Velocity (γu)]
- [Temporal "Velocity" Factor (γ) : Spatial Velocity Factor (γβ)]
- [Energy (E) : 3-Momentum (p)]
- [Total Energy (E_T) = Hamiltonian (H) : 3-Total Momentum (p_T)
- [Power (γĖ) : Force (γf)]
- [Angular Frequency (ω) : 3-Angular Wave Number (k)]
- [Total Angular Frequency (ω_T) : 3-Total Angular Wave Number (k_T)
- [Charge Density (ρ) : 3-Current Density = 3-Charge Flux (j)]
- [Scalar Potential (φ) : 3-Vector Potential (a)], typically the EM versions (φ_{EM}) : (a_{EM})
- [Potential Energy (V) : 3-Potential Momentum (q = qa)]
- [Time Differential (∂_t) : Spatial 3-Gradient (∇ = ∂_{x̂})
- [Number Density (n) : Spatial 3-Number Flux (n = nu)]
- [Temporal Spin (s⁰) : Spatial 3-Spin (s)]

4-Tensor = Type (2,0)-Tensor

Faraday EM Tensor $F^{\mu\nu} = \begin{bmatrix} 0 & -e^i/c \\ +e^j/c & -\epsilon^{ij}_k b^k \end{bmatrix}$

[T = kg/C · s]

[Temporal-Temporal : Temporal-Spatial : Spatial-Spatial] components

[0 : 3-Electric-Field (e = eⁱ) : 3-Magnetic-Field (b = b^k)] $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

4-Angular Momentum $M^{\mu\nu} = \begin{bmatrix} 0 & -cn^i \\ +cn^j & -\epsilon^{ij}_k l^k \end{bmatrix}$
Tensor

[J · s = N · m · s = kg · m²/s]

[0 : 3-Mass-Moment (n = nⁱ) : 3-Angular-Momentum (l = l^k)]

$M^{\mu\nu} = \mathbf{X} \wedge \mathbf{P} = X^\mu P^\nu - X^\nu P^\mu$

Minkowski Metric $\eta^{\mu\nu} = \text{Diag}[1, -\delta^{jk}] = V^{\mu\nu} + H^{\mu\nu}$

[dimensionless]

[1 : 0 : -I₍₃₎] = [1 : 0 : -δ^{jk}]

$\eta^{\mu\nu} = \partial^\mu [R^\nu] = V^{\mu\nu} + H^{\mu\nu}$

Temporal Projection Tensor $V^{\mu\nu} = \text{Diag}[1, 0]$

[dimensionless]

[1 : 0 : 0]

$V^{\mu\nu} = T^\mu T^\nu$

Spatial Projection Tensor $H^{\mu\nu} = \text{Diag}[0, -\delta^{jk}]$

[dimensionless]

[0 : 0 : -I₍₃₎]

$H^{\mu\nu} = \eta^{\mu\nu} - T^\mu T^\nu$

Perfect-Fluid Stress-Energy Tensor $T^{\mu\nu} \rightarrow \text{Diag}[\rho_e, \mathbf{p}, \mathbf{p}, \mathbf{p}]$

[J/m³ = N/m² = kg · m/s²] [ρ_e : 0 : pI₍₃₎] = [ρ_e : 0 : pδ^{jk}]

$T^{\mu\nu} = (\rho_{ee} + p_o) T^\mu T^\nu - (p_o) \partial^\mu [R^\nu]$

$T^{\mu\nu} = (\rho_{ee}) V^{\mu\nu} + (-p_o) H^{\mu\nu}$

SR 4-Tensor

- (2,0)-Tensor $T^{\mu\nu}$
- (1,1)-Tensor T^μ_ν or T_μ^ν
- (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector

- (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
- SR 4-CoVector
- (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar

- (0,0)-Tensor S
- Lorentz Scalar

4-Tensors can be constructed from the Tensor Products of 4-Vectors. Technically, 4-Tensors refer to all SR objects (4-Scalars, 4-Vectors, etc), but we typically reserve the name 4-Tensor for SR Tensors of 2 or more indices

SRQM 4-Scalars = (0,0)-Tensors = Lorentz Scalars → Physical Constants

A Tensor Study of Physical 4-Vectors

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4-Scalar = Type (0,0)-Tensor	SI Units:	4-Scalar = Type (0,0)-Tensor
RestTime:ProperTime (t_0) = (τ)	[s]	$(\tau) = [\mathbf{R}\cdot\mathbf{U}]/[\mathbf{U}\cdot\mathbf{U}] = [\mathbf{R}\cdot\mathbf{R}]/[\mathbf{R}\cdot\mathbf{U}]$ **Time as measured in the at-rest frame**
RestTime:ProperTime Differential (dt_0) = ($d\tau$)	[s]	$(d\tau) = [d\mathbf{R}\cdot\mathbf{U}]/[\mathbf{U}\cdot\mathbf{U}]$ **Differential Time as measured in the at-rest frame**
Speed of Light (c)	[m/s]	$(c) = \text{Sqrt}[\mathbf{U}\cdot\mathbf{U}] = [\mathbf{T}\cdot\mathbf{U}]$ with 4-UnitTemporal $\mathbf{T} = \gamma(1, \boldsymbol{\beta})$
RestMass (m_0)	[kg]	$(m_0) = [\mathbf{P}\cdot\mathbf{U}]/[\mathbf{U}\cdot\mathbf{U}] = [\mathbf{P}\cdot\mathbf{R}]/[\mathbf{U}\cdot\mathbf{R}]$ ($m_0 \rightarrow m_e$) as Electron RestMass
RestEnergy (E_0)	[J = kg·m ² /s ²]	$(E_0) = [\mathbf{P}\cdot\mathbf{U}]$
RestAngFrequency (ω_0)	[rad/s]	$(\omega_0) = [\mathbf{K}\cdot\mathbf{U}]$
RestChargeDensity (ρ_0)	[C/m ³]	$(\rho_0) = [\mathbf{J}\cdot\mathbf{U}]/[\mathbf{U}\cdot\mathbf{U}] = (q)[\mathbf{N}\cdot\mathbf{U}]/[\mathbf{U}\cdot\mathbf{U}] = (q)(n_0)$
RestScalarPotential (ϕ_0)	[V = J/C = kg·m ² /C·s ²]	$(\phi_0) = [\mathbf{A}\cdot\mathbf{U}]$, ($\phi_0 \rightarrow \phi_{EM0}$) as the EM version RestScalarPotential
ProperTimeDerivative ($d/d\tau$)	[1/s]	$(d/d\tau) = [\mathbf{U}\cdot\partial] = \gamma(d/dt)$ **Note that the 4-Gradient is to right of 4-Velocity**
RestNumberDensity (n_0)	[#/m ³]	$(n_0) = [\mathbf{N}\cdot\mathbf{U}]/[\mathbf{U}\cdot\mathbf{U}]$
SR Phase (Φ_{phase})	[rad] _{angle}	$(\Phi_{\text{phase}}) = -[\mathbf{K}\cdot\mathbf{R}] = (\mathbf{k}\cdot\mathbf{r} - \omega t)$: $(\Phi_{\text{phase}}) = -[\mathbf{K}_T\cdot\mathbf{R}] = (\mathbf{k}_T\cdot\mathbf{r} - \omega_T t)$
SR Action (S_{action})	[J·s] _{action}	$(S_{\text{action, free}}) = -[\mathbf{P}\cdot\mathbf{R}] = (\mathbf{p}\cdot\mathbf{r} - Et)$: $(S_{\text{action}}) = -[\mathbf{P}_T\cdot\mathbf{R}] = (\mathbf{p}_T\cdot\mathbf{r} - E_T t)$ **Units [Action] = [Momentum]·[Length] = [Energy]·[Time]**
Planck Constant (h)	[J·s = N·m·s = kg·m ² /s]	$(\hbar) = [\mathbf{P}\cdot\mathbf{U}]/[\mathbf{K}\cdot\mathbf{U}] = [\mathbf{P}\cdot\mathbf{R}]/[\mathbf{K}\cdot\mathbf{R}]$
Planck Reduced:Dirac Constant ($\hbar = h/2\pi$)	[J·s = N·m·s = kg·m ² /s]	$(4) = [\partial\cdot\mathbf{R}] = \text{Tr}[\eta^{\alpha\beta}]$ **4-Divergence[4-Position] = Trace[MinkowskiMetric] = SR Dimension**
SpaceTime Dimension (4)	[dimensionless]	$\partial\cdot\mathbf{F}^{\alpha\beta} = (\mu_0)\mathbf{J} = (1/\epsilon_0 c^2)\mathbf{J}$ Maxwell EM Eqn. $\mu_0\epsilon_0 = 1/c^2$
Electric Constant (ϵ_0)	[F/m = C ² ·s ² /kg·m ³]	$\partial\cdot\mathbf{F}^{\alpha\beta} = (\mu_0)\mathbf{J} = (1/\epsilon_0 c^2)\mathbf{J}$ Maxwell EM Eqn. $\mu_0\epsilon_0 = 1/c^2$
Magnetic Constant (μ_0)	[H/m = kg·m/C ²]	$\mathbf{U}\cdot\mathbf{F}^{\alpha\beta} = (1/q)\mathbf{F}$ Lorentz Force Eqn. ($q \rightarrow -e$) as Electron Charge
EM Charge (q)	[C]	$(Q) = \int \rho d^3\mathbf{x} = \int \rho_0 \gamma d^3\mathbf{x}$ Integration of volume charge density
EM Charge (Q) *alt method*	[C]	$(N) = \int n d^3\mathbf{x} = \int n_0 \gamma d^3\mathbf{x}$ Integration of volume number density
Particle # (N)	[#]	$(V_0) = \int \gamma d^3\mathbf{x}$ Integration of volume elements (Riemannian Volume Form)
Rest Volume (V_0)	[m ³]	$(\rho_{eo}) = V_{\alpha\beta} T^{\alpha\beta} = \text{Temporal "Vertical" Projection of Perfect Fluid Stress-Energy Tensor}$
RestEnergyDensity (ρ_{eo})	[J/m ³ = N/m ² = kg·m/s ²]	$(\rho_o) = (-1/3)H_{\alpha\beta} T^{\alpha\beta} = \text{Spatial "Horizontal" Projection of Perfect Fluid Stress-Energy Tensor}$
RestPressure (ρ_o)	[J/m ³ = N/m ² = kg·m/s ²]	
Faraday InnerProduct Invariant $2(\mathbf{b}\cdot\mathbf{b}\cdot\mathbf{e}\cdot\mathbf{e}/c^2)$	[T ² = kg ² /C ² ·s ²]	$2(\mathbf{b}\cdot\mathbf{b}\cdot\mathbf{e}\cdot\mathbf{e}/c^2) = F^{\alpha\beta} F_{\alpha\beta}$
Faraday Determinant Invariant $(\mathbf{e}\cdot\mathbf{b}/c)^2$	[T ⁴ = kg ⁴ /C ⁴ ·s ⁴]	$(\mathbf{e}\cdot\mathbf{b}/c)^2 = \text{Det}[F^{\alpha\beta}]$

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(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
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SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Lorentz Scalars can be constructed from the Lorentz Scalar Product of 4-Vectors

SRQM Study: Physical 4-Vectors

Some SR 4-Vectors and Symbols

A Tensor Study of Physical 4-Vectors

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4-Gradient
 $\partial = \partial^\mu = (\partial/c, -\nabla)$
 $\rightarrow (\partial/c, -\partial_x, -\partial_y, -\partial_z)$
 $= (\partial/c\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$

Gradient 4-Vector
 $\partial^\mu = (\partial/c, -\nabla)$
 $\partial_\mu = (\partial/c, \nabla)$
 Gradient One-Form

4-Displacement
 $\Delta R = \Delta R^\mu = (c\Delta t, \Delta \mathbf{r}) = \mathbf{R}_2 - \mathbf{R}_1$ {finite}
 $dR = dR^\mu = (cdt, d\mathbf{r})$ {infinitesimal}

4-Position
 $R = R^\mu = (ct, \mathbf{r}) = \langle \text{Event} \rangle$
 $\rightarrow (ct, x, y, z)$
 alt. notation $\mathbf{X} = X^\mu$

Lorentz Invariant, but not Poincaré Invariant

4-Velocity
 $U = U^\mu = \gamma(\mathbf{c}, \mathbf{u})$
 $= dR/d\tau = cT$

4-Unit Temporal
 $T = T^\mu = \gamma(1, \boldsymbol{\beta})$
 $= \gamma(1, \mathbf{u}/c) = U/c$

4-Acceleration
 $A = A^\mu = \gamma(c\boldsymbol{\gamma}', \boldsymbol{\gamma}'\mathbf{u} + \boldsymbol{\gamma}\mathbf{a})$
 $= dU/d\tau = d^2R/d\tau^2$

Minkowski Metric
 $\partial[R] = \partial^\mu[R^\nu] = \eta^{\mu\nu}$

Lorentz Transform
 $\partial_\nu[R^\mu] = \Lambda^{\mu\nu}$

SpaceTime Dimension
 $\partial R = \partial_\mu R^\mu = 4$

4-Momentum
 $P = P^\mu = (mc, \mathbf{p}) = (mc, m\mathbf{u}) = m_0U$
 $= (E/c, \mathbf{p}) = (E_0/c^2)U$

4-WaveVector
 $K = K^\mu = (\omega/c, \mathbf{k}) = (\omega_0/c^2)U$
 $= (\omega/c, \omega\hat{n}/v_{\text{phase}}) = (1/cT, \hat{n}/\lambda)$

4-(EM)VectorPotential
 $A = A^\mu = (\phi/c, \mathbf{a}) = (\phi_0/c^2)U$
 $A_{EM} = A_{EM}^\mu = (\phi_{EM}/c, \mathbf{a}_{EM})$

4-(Vector)PotentialMomentum
 $Q = Q^\mu = (q\phi/c, q\mathbf{a}) = (V/c, \mathbf{q})$
 $= qA = (q\phi_0/c^2)U = (V_0/c^2)U$

4-ChargeFlux : 4-CurrentDensity
 $J = J^\mu = (\rho c, \mathbf{j}) = \rho(\mathbf{c}, \mathbf{u}) = \rho_0U$
 $= qn_0U = qN$

4-(Dust)NumberFlux
 $N = N^\mu = (nc, \mathbf{n}) = n(\mathbf{c}, \mathbf{u}) = n_0U$

4-ThermalVector
4-InverseTemperatureMomentum
 $\Theta = \Theta^\mu = (\theta^0, \boldsymbol{\theta}) = (c/k_B T, \mathbf{u}/k_B T) = (\theta_0/c)U$
 $= (1/k_B T)(\mathbf{c}, \mathbf{u}) = (1/k_B \gamma T)U = (1/k_B T_0)U$

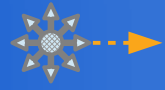
4-Force
 $F = F^\mu = \gamma(\dot{E}/c, \mathbf{f})$
 $= dP/d\tau$

4-MassFlux
4-MomentumDensity
 $G = G^\mu = (\rho_m c, \mathbf{g}) = \rho_m(\mathbf{c}, \mathbf{u})$
 $= m_0N = n_0m_0U$

4-HeatEnergyFlux
 $Q = Q^\mu = (\rho_E c, \mathbf{q}) = \rho_E(\mathbf{c}, \mathbf{u})$
 $= E_0N = n_0E_0U = c^2G$

4-PureEntropyFlux
 $S_{\text{ent_pure}} = S_{\text{ent_pure}}^\mu$
 $= (S_{\text{ent_pure}}^0, \mathbf{S}_{\text{ent_pure}})$
 $= S_{\text{ent}} N = n_0 S_{\text{ent}} U$

4-HeatEntropyFlux
 $S_{\text{ent_heat}} = (S_{\text{ent_heat}}^0, \mathbf{S}_{\text{ent_heat}})$
 $= S_{\text{ent}} N + Q/T_0 = S_{\text{ent}} N + E_0N/T_0$
 $= n_0(S_{\text{ent}} + E_0/T_0)U$



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
 (0,2)-Tensor $T_{\mu\nu}$

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 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

4-Vector $V = V^\mu = (v^\mu) = (v^0, \mathbf{v}) = (v^0, \mathbf{v})$
SR 4-Vector $V = V^\mu = (\text{scalar} * c^{\pm 1}, \mathbf{3-vector})$

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Study: Physical 4-Tensors

Some SR 4-Tensors and Symbols

A Tensor Study of Physical 4-Vectors

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Lorentz Identity Transform
 $\Lambda^{\mu}_{\nu} \rightarrow \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu} = I_{(4)}$

t	x	y	z
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

$\begin{matrix} 1 & 0_j \\ 0^i & \delta_j \end{matrix}$

← Discrete Continuous →
SR: Lorentz Transforms

Lorentz x-Boost Transform
 $\Lambda^{\mu}_{\nu} \rightarrow B^{\mu}_{\nu}$

t	x	y	z
γ	$-\beta\gamma$	0	0
$-\beta\gamma$	γ	0	0
0	0	1	0
0	0	0	1

t	x	y	z
$\cosh[\theta]$	$-\sinh[\theta]$	0	0
$-\sinh[\theta]$	$\cosh[\theta]$	0	0
0	0	1	0
0	0	0	1

General Time-Space Boost
 $\begin{matrix} \gamma & -\gamma\beta_j \\ -\gamma\beta^i & (\gamma-1)\beta^i\beta_j/(\beta\cdot\beta)+\delta^i_j \end{matrix}$

Symmetric Mixed 4-Tensor

Lorentz Time-Reverse Transform
 $\Lambda^{\mu}_{\nu} \rightarrow T^{\mu}_{\nu}$

t	x	y	z
-1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

$\begin{matrix} -1 & 0_j \\ 0^i & \delta_j \end{matrix}$

SR: Minkowski Metric
 $\partial[R] = \partial^{\mu}R^{\nu} = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu}$
 $\rightarrow \text{Diag}[1, -1_{(3)}] = \text{Diag}[1, -\delta^{jk}]$

t	x	y	z
1	0	0	0
0	-1	0	0
0	0	-1	0
0	0	0	-1

$\begin{matrix} 1 & 0^j \\ 0^i & -\delta^{ij} \end{matrix}$

"Particle Physics" Convention Symmetric

Lorentz z-Rotation Transform
 $\Lambda^{\mu}_{\nu} \rightarrow R^{\mu}_{\nu}$

t	x	y	z
1	0	0	0
0	$\cos[\theta]$	$-\sin[\theta]$	0
0	$\sin[\theta]$	$\cos[\theta]$	0
0	0	0	1

General Space-Space Rotation
 $\begin{matrix} 1 & 0_j \\ 0^i & (\delta^i_j - n^i n_j) \cos(\theta) - (\epsilon^i_{jk} n^k) \sin(\theta) + n^i n_j \end{matrix}$

Non-symmetric Mixed 4-Tensor

Lorentz Space-Parity Transform
 $\Lambda^{\mu}_{\nu} \rightarrow P^{\mu}_{\nu}$

t	x	y	z
1	0	0	0
0	-1	0	0
0	0	-1	0
0	0	0	-1

$\begin{matrix} 1 & 0_j \\ 0^i & -\delta^i_j \end{matrix}$

SpaceTime Dimension
 $\partial \cdot R = \partial_{\mu} R^{\mu} = \text{Tr}[\eta^{\mu\nu}] = 4$

Lorentz Transform $\partial_{\nu}[R^{\mu}] = \Lambda^{\mu}_{\nu}$
 $\begin{matrix} [\Lambda^0_0, \Lambda^0_j] & \text{temporal-spatial-mixed} \\ [\Lambda^i_0, \Lambda^i_j] & \text{components} \end{matrix}$

Lorentz ComboPT Transform
 $\Lambda^{\mu}_{\nu} \rightarrow (PT)^{\mu}_{\nu} = -I_{(4)}$

t	x	y	z
-1	0	0	0
0	-1	0	0
0	0	-1	0
0	0	0	-1

$\begin{matrix} -1 & 0_j \\ 0^i & -\delta^i_j \end{matrix}$

Perfect Fluid
 $T^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}$

t	x	y	z
ρ_e	0	0	0
0	p	0	0
0	0	p	0
0	0	0	p

$\begin{matrix} \rho_e & 0^j \\ 0^i & p\delta^{ij} \end{matrix}$

4-Tensor Symmetric

Faraday EM
 $F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha} = \partial \wedge A$

t	x	y	z
0	$-e^x/c$	$-e^y/c$	$-e^z/c$
$+e^x/c$	0	$-b^z$	$+b^y$
$+e^y/c$	$+b^z$	0	$-b^x$
$+e^z/c$	$-b^y$	$+b^x$	0

0	$-e/c$
$+e^i/c$	$-\epsilon^{ij}_k b^k$

0	$-e/c$
$+e^T/c$	$-\nabla \wedge a$

4-Tensor Anti-symmetric

4-AngularMomentum
 $M^{\alpha\beta} = X^{\alpha}P^{\beta} - X^{\beta}P^{\alpha} = X \wedge P$

t	x	y	z
0	$-cn^x$	$-cn^y$	$-cn^z$
$+cn^x$	0	$+l^z$	$-l^y$
$+cn^y$	$-l^z$	0	$+l^x$
$+cn^z$	$+l^y$	$-l^x$	0

0	$-cn^j$
$+cn^i$	$\epsilon^{ij}_k l^k$

0	$-cn$
$+cn^T$	$x \wedge p$

4-Tensor Anti-symmetric

SR 4-Tensor
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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Note that all the Lorentz Transforms and the Minkowski Metric are dimensionless

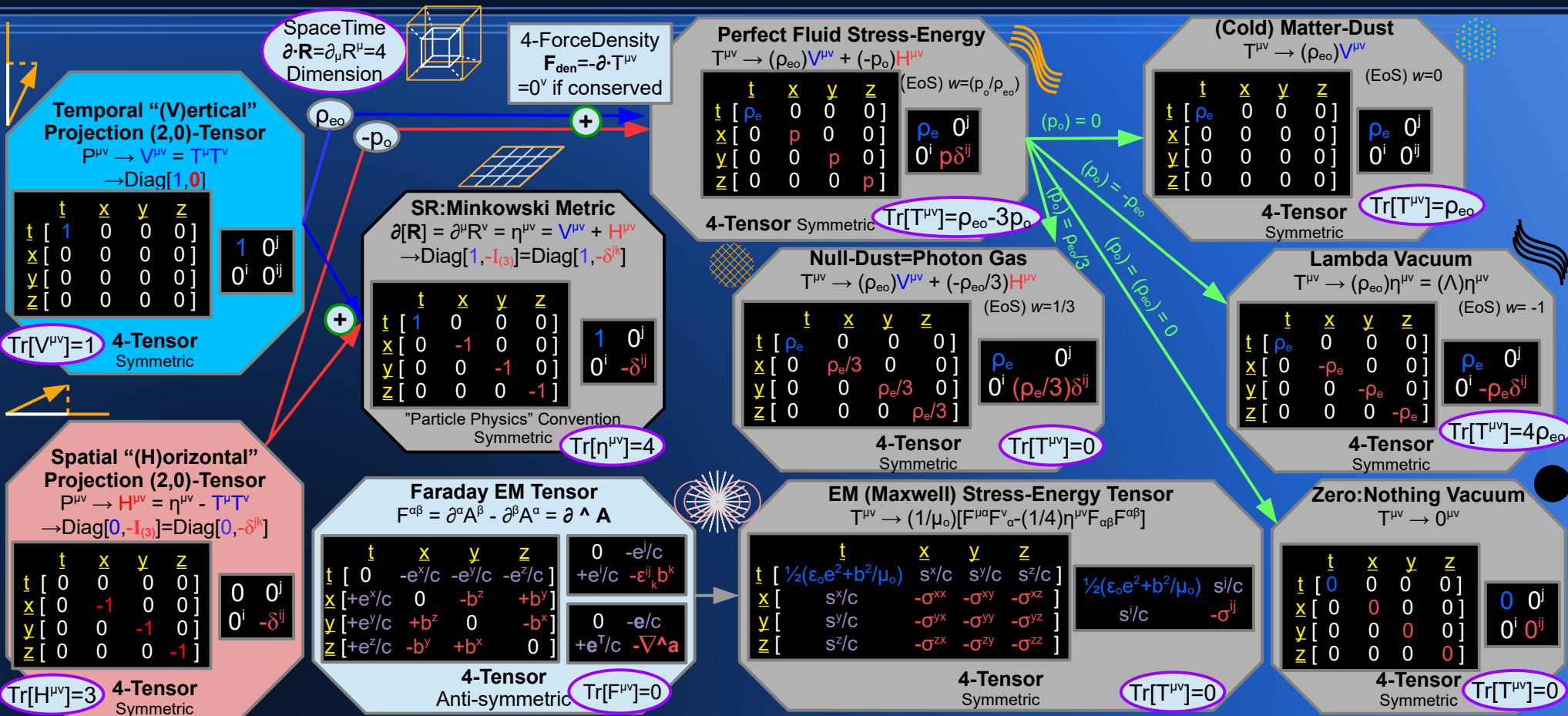
Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Study: Physical 4-Tensors

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SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (\mathbf{v}_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Note that the Projection Tensors and the Minkowski Metric are dimensionless

Equation of State (EoS) $w = (p_o/\rho_{eo})$

Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$

SRQM Study: Physical 4-Tensors

Projection 4-Tensors

A Tensor Study of Physical 4-Vectors

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Temporal "(V)ertical" Projection (2,0)-Tensor
 $P^{\mu\nu} \rightarrow V^{\mu\nu} = T^{\mu}T^{\nu}$
 $\rightarrow \text{Diag}[1, 0]$

$\text{Tr}[V^{\mu\nu}] = 1$

t	x	y	z
1	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

$\begin{matrix} 1 & 0^j \\ 0^i & 0_{ij} \end{matrix}$

4-Tensor Symmetric

Temporal "(V)ertical" Projection (1,1)-Tensor
 $P^{\mu}_{\nu} \rightarrow V^{\mu}_{\nu} = T^{\mu}T_{\nu}$
 $\rightarrow \text{Diag}[1, 0]$

$\text{Tr}[V^{\mu}_{\nu}] = 1$

t	x	y	z
1	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

$\begin{matrix} 1 & 0_j \\ 0^i & 0_{ij} \end{matrix}$

4-Tensor Symmetric

Temporal "(V)ertical" Projection (0,2)-Tensor
 $P_{\mu\nu} \rightarrow V_{\mu\nu} = T_{\mu}T_{\nu}$
 $\rightarrow \text{Diag}[1, 0]$

$\text{Tr}[V_{\mu\nu}] = 1$

t	x	y	z
1	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

$\begin{matrix} 1 & 0_j \\ 0_i & 0_{ij} \end{matrix}$

4-Tensor Symmetric

Spatial "(H)orizontal" Projection (2,0)-Tensor
 $P^{\mu\nu} \rightarrow H^{\mu\nu} = \eta^{\mu\nu} - T^{\mu}T^{\nu}$
 $\rightarrow \text{Diag}[0, -\mathbf{I}_{(3)}] = \text{Diag}[0, -\delta^k_k]$

$\text{Tr}[H^{\mu\nu}] = 3$

t	x	y	z
0	0	0	0
0	-1	0	0
0	0	-1	0
0	0	0	-1

$\begin{matrix} 0 & 0^j \\ 0^i & -\delta^k_k \end{matrix}$

4-Tensor Symmetric

Spatial "(H)orizontal" Projection (1,1)-Tensor
 $P^{\mu}_{\nu} \rightarrow H^{\mu}_{\nu} = \eta^{\mu}_{\nu} - T^{\mu}T_{\nu}$
 $\rightarrow \text{Diag}[0, \mathbf{I}_{(3)}] = \text{Diag}[0, \delta^k_k]$

$\text{Tr}[H^{\mu}_{\nu}] = 3$

t	x	y	z
0	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

$\begin{matrix} 0 & 0_j \\ 0^i & \delta^k_k \end{matrix}$

4-Tensor Symmetric

Spatial "(H)orizontal" Projection (0,2)-Tensor
 $P_{\mu\nu} \rightarrow H_{\mu\nu} = \eta_{\mu\nu} - T_{\mu}T_{\nu}$
 $\rightarrow \text{Diag}[0, -\mathbf{I}_{(3)}] = \text{Diag}[0, -\delta^k_k]$

$\text{Tr}[H_{\mu\nu}] = 3$

t	x	y	z
0	0	0	0
0	-1	0	0
0	0	-1	0
0	0	0	-1

$\begin{matrix} 0 & 0_j \\ 0_i & -\delta^k_k \end{matrix}$

4-Tensor Symmetric



Perfect Fluid
 $T^{\mu\nu} \rightarrow (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}$

t	x	y	z
ρ_e	0	0	0
0	p	0	0
0	0	p	0
0	0	0	p

$\begin{matrix} \rho_e & 0^j \\ 0^i & p\delta^{ij} \end{matrix}$

4-Tensor Symmetric
 $\text{Tr}[T^{\mu\nu}] = \rho_{eo} - 3p_o$

$P^{\mu}_{\nu} = P^{\mu\alpha}\eta_{\alpha\nu}$
 $P_{\mu\nu} = P^{\alpha\beta}\eta_{\alpha\mu}\eta_{\beta\nu}$

The projection tensors can work on 4-Vectors to give a new 4-Vector, or on 4-Tensors to give either a 4-Scalar component or a new 4-Tensor.

4-UnitTemporal $T^{\mu} = \gamma(1, \beta)$
 4-Generic $A^{\nu} = (a^0, \mathbf{a}) = (a^0, a^1, a^2, a^3)$

$V^{\mu}A^{\nu} = (1 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3, 0 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3, 0 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3, 0 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3) = (a^0, 0, 0, 0) = (a^0, \mathbf{0})$: Temporal Projection

$H^{\mu}_{\nu}A^{\nu} = (0 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3, 0 \cdot a^0 + 1 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3, 0 \cdot a^0 + 0 \cdot a^1 + 1 \cdot a^2 + 0 \cdot a^3, 0 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 1 \cdot a^3) = (0, a^1, a^2, a^3) = (0, \mathbf{a})$: Spatial Projection

$V_{\mu\nu}T^{\mu\nu} = V_{\mu\nu}[(\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}] = (\rho_{eo})V_{\mu\nu}V^{\mu\nu} = (\rho_{eo})$: $(\rho_{eo}) = V_{\mu\nu}T^{\mu\nu}$
 $H_{\mu\nu}T^{\mu\nu} = H_{\mu\nu}[(\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}] = (-p_o)H_{\mu\nu}H^{\mu\nu} = (-3p_o)$: $(p_o) = (-1/3)H_{\mu\nu}T^{\mu\nu}$

$V^{\mu}_{\alpha}T^{\alpha\nu} = V^{\mu}_{\alpha}[(\rho_{eo})V^{\alpha\nu} + (-p_o)H^{\alpha\nu}] = (\rho_{eo})V^{\mu}_{\alpha}V^{\alpha\nu} + 0 = (\rho_{eo})V^{\mu\nu} = \text{Diag}[\rho_{eo}, 0, 0, 0]$
 $H^{\mu}_{\alpha}T^{\alpha\nu} = H^{\mu}_{\alpha}[(\rho_{eo})V^{\alpha\nu} + (-p_o)H^{\alpha\nu}] = 0 + (-p_o)H^{\mu}_{\alpha}H^{\alpha\nu} = (-p_o)H^{\mu\nu} = \text{Diag}[0, p, p, p]$

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Note that the Projection Tensors are dimensionless

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

Special Relativity → Quantum Mechanics RoadMap of SR→QM

A Tensor Study of Physical 4-Vectors

4-Gradient=**Alteration** of SR <Events>

SR SpaceTime Dimension=4
SR SpaceTime Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves

START HERE: 4-Position=**Location** of SR <Events> in SpaceTime

$$\partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla = -(m_0 c / \hbar)^2 = -(\omega_0 / c)^2 = (\partial_\tau / c)^2$$

SR d'Alembertian & Klein-Gordon Relativistic Quantum Wave Relation
Schrödinger QWE is $\{ |v| < c \}$ limit of KG
[SR → QM]

4-WaveVector=**Substantiation** of SR Wave <Events>
oscillations proportional to mass:energy & 3-momentum

$$K \cdot K = (\omega/c)^2 - k \cdot k = (m_0 c / \hbar)^2 = (\omega_0 / c)^2 = (1/cT_0)^2$$

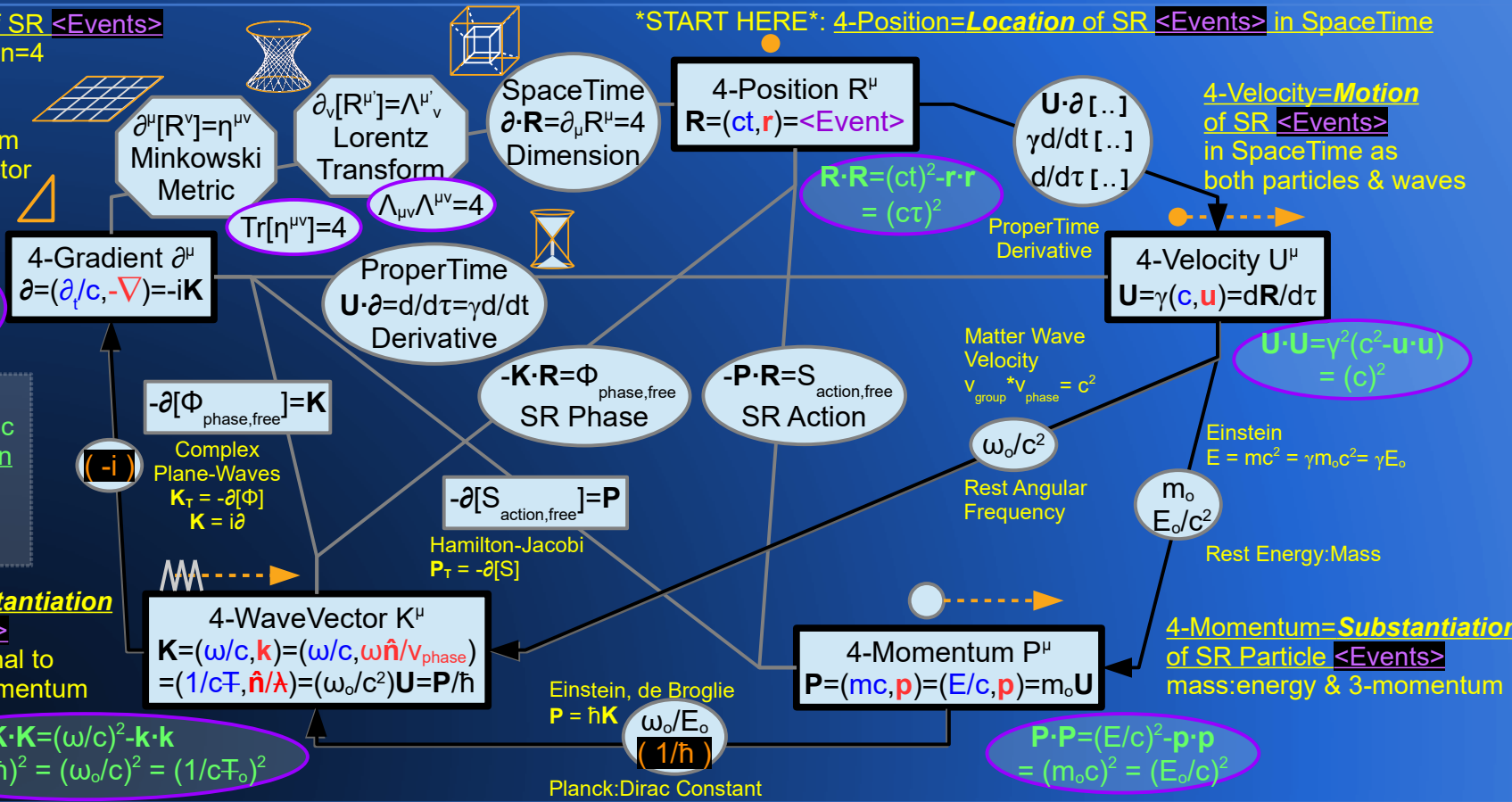
SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$



SRQM:

SR → QM Interpretation Simplified

A Tensor Study
of Physical 4-VectorsSciRealm.org
John B. Wilson
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http://scirealm.org/SRQM.pdfSRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR,
although technically SR is itself the low-curvature limiting-case of GR

{c,τ,m₀,ħ,i}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors:

Related by these SR Lorentz Invariants

4-Position	$\mathbf{R} = (ct, \mathbf{r})$	= <Event>	$(\mathbf{R} \cdot \mathbf{R}) = (c\tau)^2$
4-Velocity	$\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$	$= (\mathbf{U} \cdot \partial)\mathbf{R} = (d/d\tau)\mathbf{R} = d\mathbf{R}/d\tau$	$(\mathbf{U} \cdot \mathbf{U}) = (c)^2$
4-Momentum	$\mathbf{P} = (E/c, \mathbf{p})$	$= m_0\mathbf{U}$	$(\mathbf{P} \cdot \mathbf{P}) = (m_0c)^2$
4-WaveVector	$\mathbf{K} = (\omega/c, \mathbf{k})$	$= \mathbf{P}/\hbar$	$(\mathbf{K} \cdot \mathbf{K}) = (m_0c/\hbar)^2$
4-Gradient	$\partial = (\partial_t/c, -\nabla)$	$= -i\mathbf{K}$	$(\partial \cdot \partial) = -(m_0c/\hbar)^2 = KG \text{ Eqn} \rightarrow RQM \rightarrow QM$

 $|v| \ll c$

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn,
and thence to QM via the low-velocity limit { $|v| \ll c$ }, giving the Schrödinger Eqn.

The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR → QM by John B. Wilson (SciRealm@aol.com)

SRQM 4-Vector Topics Covered

SR & QM via 4-Vector Diagrams

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

Mostly SR Stuff

4-Vector Basics
 Paradigm Assumptions, Where is Quantum Gravity?
 Minkowski SpaceTime, <Events>, WorldLines, Minkowski Metric
 4-Scalars, 4-Vectors, 4-Tensors & Tensor Invariants, Cayley-Hamilton
 SR 4-Vector Connections
 SR Lorentz Transforms, CPT Symmetry, Trace Identification, Antimatter
 Fundamental Physical Constants = Lorentz Scalar Invariants
 Projection Tensors: Temporal (V) & Spatial (H)
 Stress-Energy Tensors, Perfect Fluids, Special Cases (Dust,Radiation,etc)
 Invariant Intervals, Measurement, Causality, Relativity
 SpaceTime Kinematics & Dynamics, ProperTime Derivative
 Einstein's $E = mc^2 = \gamma m_0 c^2 = \gamma E_0$, Rest Mass, Rest Energy, Invariants
 SpaceTime Orthogonality: Time-like velocity, Space-like acceleration
 Relativity of Simultaneity, Time Dilation, Length Contraction
 SR Motion * Lorentz Scalar = Interesting Physical 4-Vector
 SR Conservation Laws & Local Continuity Equations, Symmetries
 Relativistic Doppler Effect, Relativistic Aberration Effect
 SR Wave-Particle Relation, Invariant d'Alembertian Wave Eqn, SR Waves
 SpaceTime is 4D: $\partial \cdot \mathbf{R} = \partial_\mu R^\mu = 4$, $\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$, $\text{Tr}[\eta^{\mu\nu}] = 4$, $\mathbf{A} = A^\mu = (a^0, a^1, a^2, a^3)$
 Minimal Coupling = Interaction with a (Vector)Potential
 Conservation of 4-TotalMomentum
 SR Hamiltonian:Lagrangian Connection
 Lagrangian, Lagrangian Density
 Hamilton-Jacobi Equation (differential), Relativistic Action (integral)
 Euler-Lagrange Equations
 Relativistic Equations of Motion, Lorentz Force Equation
 c^2 Invariant Relations, The Speed-of-Light (c)
 Thermodynamic 4-Vectors, Unruh-Hawking Radiation, Particle Distributions

Mostly QM & SRQM Stuff

Relativistic Quantum Wave Equations
 Klein-Gordon Equation/ Fundamental Quantum Relation
 RoadMap from SR to QM: SR→QM, SRQM 4-Vector Connections
 QM Schrödinger Relation
 QM Axioms? - No, (QM Principles derived from SR) = SRQM
 Relativistic Wave Equations: based on mass & spin & velocity
 Klein-Gordon, Dirac, Proca, Maxwell, Weyl, Pauli, Schrödinger, etc.
 Classical Limits $|v| \ll c$
 Photon Polarization
 Linear PDE's→
 {Principle of Superposition, Hilbert Space, <Bra|,|Ket> Notation}
 Canonical QM Commutation Relations ← derived from SR
 Heisenberg Uncertainty Principle (due to non-zero commutation)
 Pauli Exclusion Principle (Fermion), Bose Aggregation Principle (Boson)
 Complex 4-Vectors
 CPT Theorem, Lorentz Invariance, Poincaré Invariance, Isometry
 Hermetian Generators, Unitarity, Anti-Unitarity
 QM → Classical Correspondence Principle, similar to SR → Classical
 Quantum Probability
 The Compton Effect = Photon:Electron Interaction (neglecting Spin Effects)
 Photon Diffraction, Crystal-Electron Diffraction, The Kapitza-Dirac Effect
 The \hbar Relation, Einstein-de Broglie, Planck:Dirac
 The Aharonov-Bohm Effect, The Josephson Junction Effect
 Noether's Theorem, Continuous Symmetries, Conservation Laws
 Dimensionless Quantities
 Quantum Relativity: GR is ***NOT*** wrong, *Never bet against Einstein* :)
 Quantum Mechanics is Derivable from Special Relativity, SR→QM, SRQM

SRQM = The [SR→QM] Interpretation of Quantum Mechanics
 = Special Relativity → Quantum Mechanics

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 1)

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

There are some paradigm assumptions that need to be cleared up:

Relativistic Physics ****IS NOT**** the generalization of Classical Physics.

Classical Physics ****IS**** the low-velocity limiting-case approximation of Relativistic Physics $\{ |v| \ll c \}$.

This includes (Newtonian) Classical Mechanics and Classical QM, (meaning the non-relativistic Schrödinger QM Equation).

Classical EM is for the most part already compatible with Special Relativity.

However, Classical EM doesn't include intrinsic spin, even though spin is a result of SR Poincaré Invariance, not QM.

So far, in all of my research, if there was a way to get a result classically,
then there was usually a much simpler way to get the result using 4-Vectors and SRQM relativistic thinking.

Likewise, a lot of QM results make much more sense when approached from SRQM.

Einstein Energy:Mass Eqn: $\mathbf{P} = m_0\mathbf{U} \rightarrow \{ E = mc^2 = \gamma m_0 c^2 = \gamma E_0 : \mathbf{p} = m\mathbf{u} = \gamma m_0 \mathbf{u} \}$	Einstein-de Broglie Relation: $\mathbf{P} = \hbar\mathbf{K} \rightarrow \{ E = \hbar\omega : \mathbf{p} = \hbar\mathbf{k} \}$
Hamiltonian: $H = \gamma(\mathbf{P}_T \cdot \mathbf{U})$ { Relativistic } $\rightarrow (T + V) = (E_{\text{kinetic}} + E_{\text{potential}})$ { Classical-limit only, $ u \ll c$ }	Complex Plane-Wave Relation: $\mathbf{K} = i\partial \rightarrow \{ \omega = i\partial_t : \mathbf{k} = -i\nabla \}$
Lagrangian: $L = -(\mathbf{P}_T \cdot \mathbf{U})/\gamma$ { Relativistic } $\rightarrow (T - V) = (E_{\text{kinetic}} - E_{\text{potential}})$ { Classical-limit only, $ u \ll c$ }	Schrödinger Relations: $\mathbf{P} = i\hbar\partial \rightarrow \{ E = i\hbar\partial_t : \mathbf{p} = -i\hbar\nabla \}$
SR Wave Eqn (differential format): $\mathbf{K}_T = -\partial[\Phi_{\text{phase}}] = \mathbf{P}_T/\hbar \rightarrow \{ \omega_T = -\partial[\Phi] : \mathbf{k}_T = \nabla[\Phi] \}$	Canonical QM Commutation Relations inc. QM Time-Energy: $[P^\mu, X^\nu] = i\hbar\eta^{\mu\nu} \rightarrow \{ [x^0, p^0] = [t, E] = -i\hbar : [x^j, p^k] = i\hbar\delta^{jk} \}$
Hamilton-Jacobi Eqn (differential format): $\mathbf{P}_T = -\partial[S_{\text{action}}] = \hbar\mathbf{K}_T \rightarrow \{ E_T = -\partial_t[S] : \mathbf{p}_T = \nabla[S] \}$	Minimal Coupling: $\mathbf{P} = \mathbf{P}_T - q\mathbf{A} \rightarrow \{ E = E_T - q\phi : \mathbf{p} = \mathbf{p}_T - q\mathbf{a} \}$
Action Equation (integral format): $\Delta S_{\text{action}} = -\int_{\text{path}} \mathbf{P}_T \cdot d\mathbf{X} = -\int_{\text{path}} (\mathbf{P}_T \cdot \mathbf{U}) d\tau = \int_{\text{path}} L dt$	Josephson Junction Relation (differential format): $\mathbf{A} = -(\hbar/q)\partial[\Delta\Phi_{\text{pot}}]$
SR/QM Wave Equation (integral format): $\Delta\Phi_{\text{phase}} = -\int_{\text{path}} \mathbf{K}_T \cdot d\mathbf{X} = -\int_{\text{path}} (\mathbf{K}_T \cdot \mathbf{U}) d\tau = \Delta S_{\text{action}}/\hbar$	Aharonov-Bohm Relation (integral format): $\Delta\Phi_{\text{pot}} = -(q/\hbar)\int_{\text{path}} \mathbf{A} \cdot d\mathbf{X}$
Euler-Lagrange Equation: $(\mathbf{U} = (d/d\tau)\mathbf{R}) \rightarrow (\partial_R = (d/d\tau)\partial_U)$	Compton Scattering: $\Delta\lambda = (\lambda' - \lambda) = (\hbar/m_0c)(1 - \cos[\theta])$
Hamilton's Equations: $(d/d\tau)[\mathbf{X}] = (\partial/\partial\mathbf{P}_T)[H_0]$ & $(d/d\tau)[\mathbf{P}_T] = (\partial/\partial\mathbf{X})[H_0]$	Klein-Gordon Relativistic Quantum Wave Eqn: $\partial \cdot \partial = -(m_0c/\hbar)^2$
d'Alembertian Wave Equation: $\partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla$, with solutions $\sim \sum_n e^{\pm(Kn \cdot X)}$	

4-Vector formulations are all extremely easy to derive in SRQM and are all relativistically covariant.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 2)

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

There are some paradigm assumptions that need to be cleared up:

SR 4D Physical 4-Vectors **ARE NOT** generalizations of Classical/Quantum 3D Physical 3-vectors. While a “mathematical” Euclidean (n+1)D-vector is the generalization of a Euclidean (n)D-vector, the “physical” analogy ends there.

Minkowskian SR 4-Vectors **ARE** the primitive elements of 4D Minkowski SR SpaceTime. Classical/Quantum Physical 3-vectors are just the spatial components of SR Physical 4-Vectors. There is also a fundamentally-related Classical/Quantum Physical scalar related to each 3-vector, which is just the temporal component scalar of a given SR Physical 4-Vector.

ex. 4-Position $\mathbf{R} = (r^\mu) = (r^0, \mathbf{r}) = (ct, \mathbf{r}) \rightarrow (ct, x, y, z)$: 4-Momentum $\mathbf{P} = (p^\mu) = (p^0, \mathbf{p}) = (E/c, \mathbf{p}) \rightarrow (E/c, p^x, p^y, p^z)$

These Classical/Quantum {scalar}+{3-vector} are the dual {temporal}+{spatial} components of a single SR 4-Vector = (temporal scalar * $c^{\pm 1}$, spatial 3-vector) with SR lightspeed factor ($c^{\pm 1}$) to give correct overall dimensional units.

While different observers may see different "values" of the Classical/Quantum components (v^0, v^1, v^2, v^3) from their point-of-view in SpaceTime, each will see the same actual SR 4-Vector \mathbf{V} and its magnitude $|\mathbf{V}|$ at a given <Event> in SpaceTime.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 3)

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

There are some paradigm assumptions that need to be cleared up:

We will ****NOT**** be employing the commonly-(mis)used Newtonian classical limits $\{c \rightarrow \infty\}$ and $\{\hbar \rightarrow 0\}$.

Neither of these is a valid physical assumption, for the following reasons:

[1]

Both (c) and (\hbar) are unchanging Physical Constants and Lorentz Invariants.

Taking a limit where these change is non-physical. They are **CONSTANT**.

Many, many experiments verify that these constants have not changed over the lifetime of the universe.

This is one reason for the 2019 Redefinition of SI Base Units on Fundamental Constants $(c, \hbar, e, k_B, N_A, K_{CD}, \Delta V_{CS})$.

[2]

Let $E = pc$. If $c \rightarrow \infty$, then $E \rightarrow \infty$. Then Classical EM light rays/waves have infinite energy.

Let $E = \hbar\omega$. If $\hbar \rightarrow 0$, then $E \rightarrow 0$. Then Classical EM light rays/waves have zero energy.

Obviously neither of these is true in the Newtonian limit.

In Classical EM and Classical Mechanics, (c) remains a large but finite constant.

Likewise, (\hbar) remains very small but never becomes zero.

The correct way to take the limits is via:

The low-velocity non-relativistic limit $\{ |\mathbf{v}| \ll c \}$, which is a physically-occurring situation.

The Hamilton-Jacobi non-quantum limit $\{ \hbar |\nabla \cdot \mathbf{p}| \ll (\mathbf{p} \cdot \mathbf{p}) \}$, which is a physically-occurring situation.

SRQM: A treatise of SR → QM by John B. Wilson (SciRealm@aol.com)

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 4)

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

There are some paradigm assumptions that need to be cleared up:

We will *NOT* be implementing the common {→lazy and extremely misguided} convention of setting physical constants to the value of (dimensionless) unity, often called “Natural Units”, to hide them from equations; nor using mass (m) instead of (m_0) as the RestMass. Likewise for other components vs Lorentz Scalars with naughts, like energy (E) vs (E_0) as the RestEnergy.

One sees this very often in the literature. The usual excuse cited is “For the sake of brevity”.

Well, the “sake of brevity” forsakes “clarity”

The *ONLY* situation in which setting constants to unity is practical or advisable is in numerical simulation. When teaching physics, or trying to understand physics, it helps when equations are dimensionally correct. In other words, the technique of dimensional analysis is a powerful tool that should not be disdained. i.e. Brevity aids speed of computation, Clarity aids understanding.

The situation of using “naught = 0” for rest-values, such as (m_0) for RestMass and (E_0) for RestEnergy: Is intrinsic to SR, is a very good idea, absolutely adds clarity, identifies Lorentz Scalar Invariants, and will be explained in more detail later. Essentially, the relativistic gamma (γ) pairs with a (Lorentz scalar:rest value 0) to make a relativistic component: $m = \gamma m_0$, $E = \gamma E_0$. Note the multiple equivalent ways that one can write 4-Vectors using these rules:

$$\begin{aligned} \text{4-Momentum } \mathbf{P} = P^\mu = (p^\mu) &= (p^0, \mathbf{p}^i) = (mc, \mathbf{p}) = m_0 \mathbf{U} = m_0 \gamma(\mathbf{c}, \mathbf{u}) = \gamma m_0 (\mathbf{c}, \mathbf{u}) = m(\mathbf{c}, \mathbf{u}) = (mc, m\mathbf{u}) = (mc, \mathbf{p}) \\ &= (E/c, \mathbf{p}) = (E_0/c^2) \mathbf{U} = (E_0/c^2) \gamma(\mathbf{c}, \mathbf{u}) = \gamma (E_0/c^2) (\mathbf{c}, \mathbf{u}) = (E/c^2) (\mathbf{c}, \mathbf{u}) = (E/c, E\mathbf{u}/c^2) = (E/c, \mathbf{p}) \end{aligned}$$

It is damn hard enough just to get the minus-signs right in GR/SR, as there are different metric-conventions available.

This notation makes clear what is relativistic vs. invariant, temporal vs. spatial

BTW, I prefer the “Particle Physics” Metric-Convention (+,-,-,-). {Makes rest values positive, fewer minus signs to deal with}

Show the physical constants and naughts in the work. They deserve the respect and you will benefit.

You can always set constants to unity later, when you are doing your numerical simulations.

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 5)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

There are some paradigm assumptions that need to be cleared up:

Many physics books say that the Electric field \mathbf{E} and the Magnetic field \mathbf{B} are the “real” physical objects, and that the EM scalar-potential ϕ and the EM 3-vector-potential \mathbf{A} are just “calculational/mathematical” artifacts.

Neither of these statements is relativistically correct.

All of these physical EM properties: $\{\mathbf{E}, \mathbf{B}, \phi, \mathbf{A}\}$ are actually just the components of SR tensors, and as such, their magnitudes will vary in different observers’ reference-frames.

The truly SR invariant physical objects are:

The 4-Gradient ∂ , the 4-VectorPotential \mathbf{A} , and their combination via exterior (wedge= \wedge) product into the Faraday EM Tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge \mathbf{A}$

Given this SR knowledge, we demote the physical property symbols, (the tensor components) to their lower-case equivalents $\{\mathbf{e}, \mathbf{b}, \phi, \mathbf{a}\}$.

Temporal-spatial components of 4-Tensor $F^{\alpha\beta}$: **electric 3-vector field \mathbf{e}** .
Spatial-spatial components of 4-Tensor $F^{\alpha\beta}$: **magnetic 3-vector field \mathbf{b}** .
Temporal component of 4-Vector \mathbf{A} : **EM scalar-potential ϕ** .
Spatial components of 4-Vector \mathbf{A} : **EM 3-vector-potential \mathbf{a}** .

Note that the speed-of-light (c) plays a prominent role in the component definitions. Also, QM requires the 4-VectorPotential \mathbf{A} as explanation of the Aharonov-Bohm Effect. Again, all the higher-index-count SR tensors can be built from fundamental 4-Vectors.

4-Gradient
 $\partial = \partial^\mu = (\partial/c, -\nabla)$
 $\rightarrow (\partial/c, -\partial_x, -\partial_y, -\partial_z)$



4-(EM)VectorPotential
 $\mathbf{A} = A^\mu = (\phi/c, \mathbf{a})$
 $\mathbf{A}_{EM} = A_{EM}^\mu = (\phi_{EM}/c, \mathbf{a}_{EM})$

Faraday EM Tensor
 $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge \mathbf{A}$
 \rightarrow
 $\begin{bmatrix} F^{tt} & F^{tx} & F^{ty} & F^{tz} \\ F^{xt} & F^{xx} & F^{xy} & F^{xz} \\ F^{yt} & F^{yx} & F^{yy} & F^{yz} \\ F^{zt} & F^{zx} & F^{zy} & F^{zz} \end{bmatrix}$
 $=$
 $\begin{bmatrix} 0 & -e^x/c & -e^y/c & -e^z/c \\ +e^x/c & 0 & -b^z & +b^y \\ +e^y/c & +b^z & 0 & -b^x \\ +e^z/c & -b^y & +b^x & 0 \end{bmatrix}$
 $=$
 $\begin{bmatrix} 0 & -e/c \\ +e/c, & -\epsilon^{ij} b^k \end{bmatrix}$
 $=$
 $\begin{bmatrix} 0 & -e/c \\ +e^T/c, & -\nabla^A \mathbf{a} \end{bmatrix}$

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 6)

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

There are some paradigm assumptions that need to be cleared up:

A number of QM philosophies make the assertion that particle “properties” do not “exist” until measured. The assertion is based on the Heisenberg Uncertainty Principle, and more specifically on quantum non-zero commutation, in which a measurement on one property of a particle alters a non-commuting property of the same particle.

That is an incorrect analysis. Properties define particles: what they do, how they interact with other particles. Particles and their properties “exist” independently of human intervention or observation. The correct way to analyze this is to understand what a measurement is: the arrangement of some number of fundamental particles in a particular manner as to allow an observer to get information about one or more of the subject particle’s properties. Typically this involves “counting” spacetime events and using SR invariant intervals as a basis of measurement.

Some properties are indeed non-commuting. This simply means that it is not possible to arrange a set of particles in such a way as to measure (ie. obtain “complete” information about) both of the subject particle’s non-commuting properties at the same spacetime event. The measurement arrangement events can be done at best sequentially, and the temporal order of these events makes a difference in observed results. EPR-Bell, however, allows one to “infer” properties on a subject particle by making a measurement on a different {space-like separated but entangled} particle. This does **not** imply FTL signalling. It just updates local partial-information about particles that interacted/entangled then separated.

So, a better way to think about it is this: The “measurement” of a property does not “exist” until a physical setup event is arranged. Non-commuting properties require different physical arrangements in order for the properties to be measured, and the temporally-first measurement alters the particle’s properties in a minimum sort of way, which affects the latter measurement. All observers agree on the order of temporally-separated spacetime events. However, individual observers may have different sets of partial information about the same particle(s).

This makes way more sense than the subjective belief that a particle’s property doesn’t exist until it is observed, which is about as unscientific and laughable a statement as I can imagine.

Relativity is the system of measurement that QM has been looking for

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 7)

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

There are some paradigm assumptions that need to be cleared up:

Correct Notation is critical for understanding physics

Unfortunately, there are a number of “sloppy” notations in relativistic and quantum physics.

Incorrect: Using T^{ii} as a Trace of tensor T^{ij} , or $T^{\mu\mu}$ as a Trace of tensor $T^{\mu\nu}$

T^{ii} is just the diagonal part of 3-tensor T^{ij} , the components: $T^{ii} = \text{Diag}[T^{11}, T^{22}, T^{33}]$

T_i^i is the Trace of 3-tensor T^{ij} : $T_i^i = T_1^1 + T_2^2 + T_3^3 = 3\text{-trace}[T^{ij}] = T^{11} + T^{22} + T^{33}$ in the Euclidean Metric $E^{ij} = \delta^{ij}$

$T^{\mu\mu}$ is just the diagonal part of 4-Tensor $T^{\mu\nu}$, the components: $T^{\mu\mu} = \text{Diag}[T^{00}, T^{11}, T^{22}, T^{33}]$

T_{μ}^{μ} is the Trace of 4-Tensor $T^{\mu\nu}$: $T_{\mu}^{\mu} = T_0^0 + T_1^1 + T_2^2 + T_3^3 = 4\text{-Trace}[T^{\mu\nu}] = T^{00} - T^{11} - T^{22} - T^{33}$ in the Minkowskian Metric $\eta^{\mu\nu}$

Incorrect: Hiding factors of (c) in relativistic equations, ex. $E = m$

The use of “natural units” leads to a lot of ambiguity, and one loses the ability to do dimensional analysis.

Wrong: $E=m$: Energy is *not* identical to mass.

Correct: $E=mc^2$: Energy is related to mass via the speed-of-light, ie. mass is a type of concentrated energy.

Incorrect: Using m instead of m_0 for rest mass, Using E instead of E_0 for rest energy

Correct: $E = mc^2 = \gamma m_0 c^2 = \gamma E_0$

E & m are relativistic internal components of 4-Momentum $\mathbf{P}=(mc, \mathbf{p})=(E/c, \mathbf{p})$ which vary in different reference-frames.

E_0 & m_0 are Lorentz Scalar Invariants, the rest values, which are the same, even in different reference-frames: $\mathbf{P}=m_0\mathbf{U}=(E_0/c^2)\mathbf{U}$

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Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 8)

A Tensor Study
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There are some paradigm assumptions that need to be cleared up:

Incorrect: Using the same symbol for a tensor-index and a component

The biggest offender in many books for this one is quantum commutation.

Unclear because (i) means two different things in one equation.

Better: ($i = \sqrt{-1}$) is the imaginary unit ; { j,k } are tensor-indices

In general, any equation which uses complex-number math should reserve (i) for the imaginary, not as a tensor-index.

Wrong: $[x^i, p^j] = i\hbar\delta^{ij}$
 Right: $[x^j, p^k] = i\hbar\delta^{jk}$
 Better: $[P^\mu, X^\nu] = i\hbar\eta^{\mu\nu}$

Incorrect: Using the 4-Gradient notation incorrectly

The 4-Gradient is a 4-Vector, a (1,0)-Tensor, which uses an upper index, and has a negative spatial component in SR.

The Gradient One-Form, its natural tensor form, a (0,1)-Tensor, uses a lower index in SR.

4-Gradient: $\partial = \partial^\mu = (\partial_t/c, -\nabla)$ Gradient One-Form: $\partial_\mu = (\partial_t/c, \nabla)$

Incorrect: Mixing styles in 4-Vector naming conventions

There is pretty much universal agreement on the **4-Momentum $\mathbf{P} = P^\mu = (E/c, \mathbf{p}) = (mc, \mathbf{p})$**

Do not in the same document use **4-Potential $\mathbf{A} = (\phi, \mathbf{A})$** : This is wrong on many levels.

The correct form is **4-Vector Potential $\mathbf{A} = A^\mu = (\phi/c, \mathbf{a})$** , with (ϕ) as the scalar-potential & (\mathbf{a}) as the 3-vector-potential

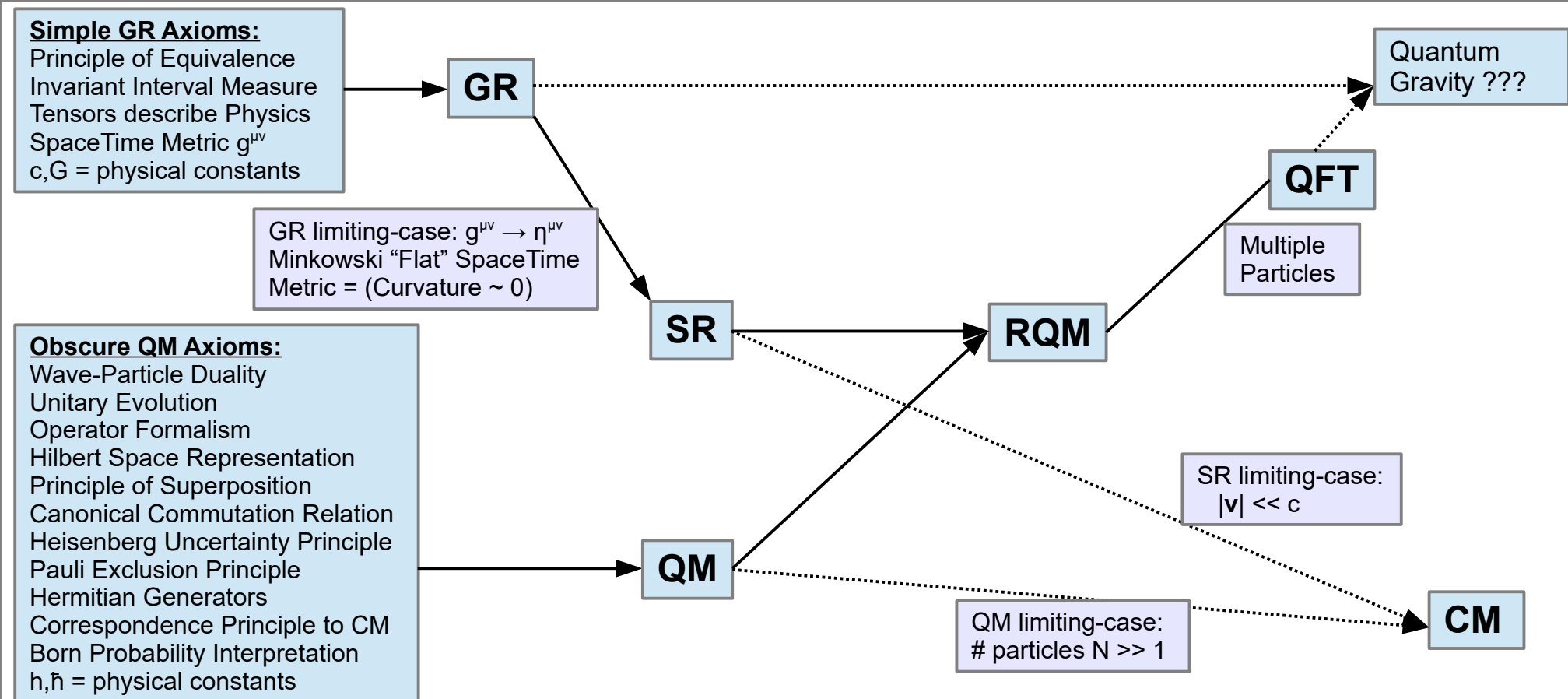
For all 4-Vectors, one should use a consistent notation:

The Upper-Case SpaceTime 4-Vector Names match the lower-case spatial 3-vector names

There is a (c) factor in the temporal component to give overall matching dimensional units for the entire 4-Vector
 4-Vector components are typically lower-case with a few historical exceptions, mainly energy E , energy-density e or p

Old Paradigm: QM (as I was taught)

SR and QM as separate theories

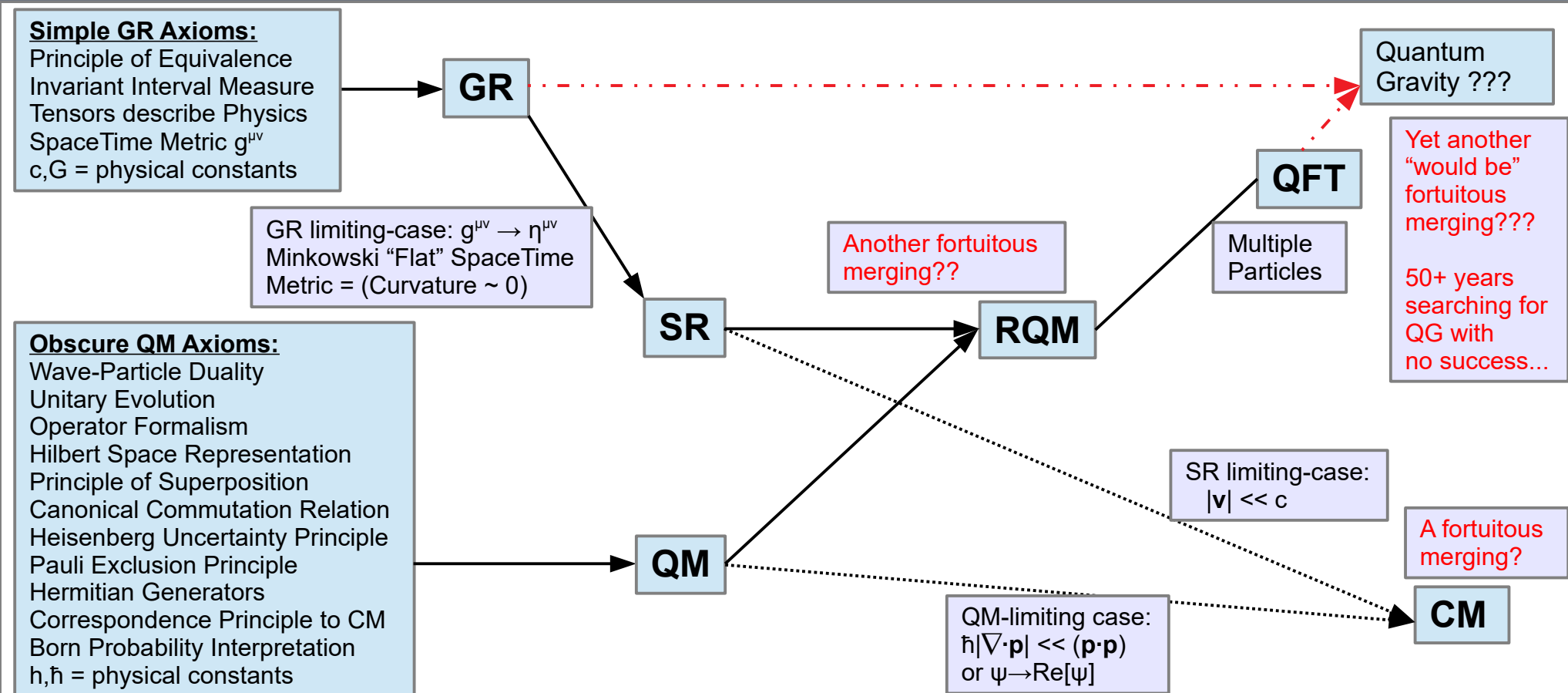


This was the QM paradigm that I was taught while in Grad School; everyone trying for Quantum Gravity

Old Paradigm: QM (years later)

SR and QM still as separate theories

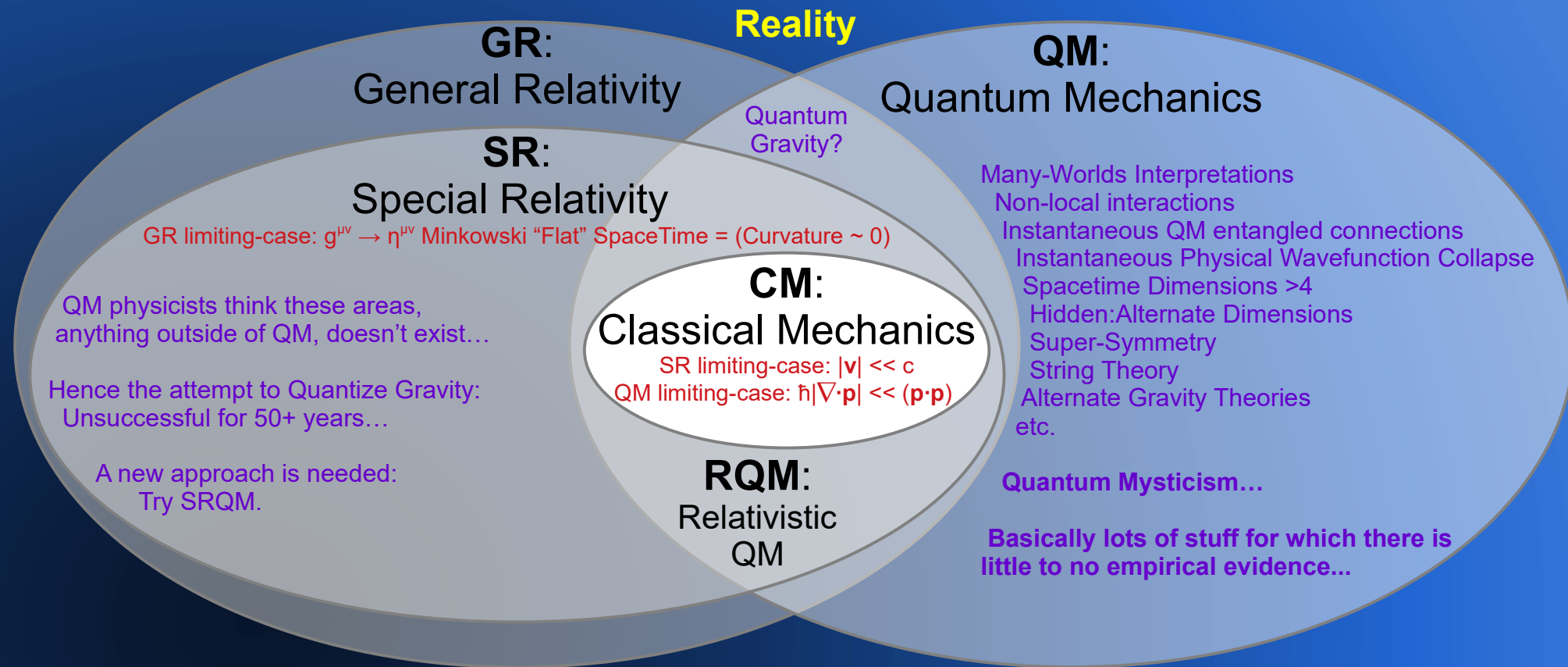
QM limiting-case better defined, still no QG



It is known that QM + SR "join nicely" together to form RQM, but problems with RQM + GR...

Physical Theories as Venn Diagram

Which regions are real?



Many QM physicists believe that the regions outside of QM don't exist...
SRQM Interpretation would say that the regions outside of GR probably don't exist...

Physical Limit-Cases as Venn Diagram

Which limit-regions use which physics?

A Tensor Study
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Reality

Quantum Gravity? Actual GR?

QM limit-case: $\hbar|\nabla\cdot\mathbf{p}| \ll (\mathbf{p}\cdot\mathbf{p})$
or $\psi \rightarrow \text{Re}[\psi]$
Change by a few quanta
has negligible effect
on overall state

GR limit-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
Minkowski "Flat" SpaceTime
= (Curvature ~ 0)

SRQM

Special Relativity → Relativistic QM

Classical SR

Classical (non-QM)
Special Relativity

RQM

Relativistic QM

Classical GR
Classical (non-QM)
General Relativity

CM

Classical
Mechanics

QM

Non-relativistic
Quantum
Mechanics

Large gravity
fields typically lead
to relativistic speeds $|v| \sim c$

SR limit-case: $|v| \ll c$

Instead of taking the Physical Theories as set, examine Physical Reality and then apply various limiting-conditions.

What do we then call the various regions?

As we move inwards from any region on the diagram, we are adding more stringent conditions which give physical limiting-cases of "larger" theories.

If one is in Classical GR, one can get Classical SR by moving toward the Minkowski SpaceTime limit.

If one is in RQM, one can get Classical SR by moving toward the Hamilton-Jacobi non-QM limit, or to standard QM by moving toward the SR low-velocity limit.

Looking at it this way, I can define SRQM to be equivalent to Minkowski SpaceTime, which contains RQM, and leads to Classical SR, or QM, or CM by taking additional limits.

My assertion:

There is no "Quantized Gravity"
Actual GR contains SRQM and Classical GR.
Perhaps "Gravitizing QM"...

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

Special Relativity → Quantum Mechanics

Background: Proven Physics

A Tensor Study
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Both General Relativity (GR) and Special Relativity (SR) have passed very stringent tests of multiple varieties. Likewise, Relativistic Quantum Mechanics (RQM) and Quantum Mechanics (QM) have passed all tests within their realms of validity: {generally micro-scale systems, but a few special macro-scale systems ex. Bose-Einstein condensates, superfluids, etc.}.

To date, however, there is no experimental indication that quantum effects "alter" the fundamentals of either SR or GR. Likewise, there are no known violations, QM or otherwise, of Local Lorentz Invariance (LLI) nor of Local Position/Poincaré Invariance (LPI). In fact, in all known experiments where both SR/GR and QM are present, QM respects the principles of SR/GR, whereas SR/GR modify the results of QM. All tested quantum-level particles, atoms, isotopes, super-positions, spin-states, etc. obey GR's Universality of FreeFall & Equivalence Principle and SR's $\{ E = mc^2 \}$ and speed-of-light (c) communication limit. Quantum-level atomic clocks are used to measure gravitational red:blue-shift effects. i.e. GR gravitational frequency-shift (time-dilation) alters atomic=quantum-level timing.

Some might argue that QM modifies the results of SR, such as via non-commuting measurements. However, that is an alteration of CM expectations, not SR expectations. In fact, there is a basic non-zero commutation relation fully within SR: $[\partial^\mu, X^\nu] = \eta^{\mu\nu}$ which will be derived from purely SR Principles in this treatise. The actual commutation part is not about (\hbar) or (i). Those are just multipliers.

On the other hand, GR Gravity *does* induce changes in quantum interference patterns and hence modifies QM: See the COW gravity-induced neutron QM interference experiments and the LIGO gravitational-wave detections via QM interferometry. Likewise, SR induces fine-structure splitting of spectral lines of atoms, "quantum" spin, spin magnetic moments, spin-statistics (fermions & bosons), antimatter, QED, Lamb shift, etc. - essentially requiring QM to be RQM to be valid.

Some QM scientists say that quantum entanglement is "non-local", but you still can't send any real messages/signals/information/particles faster than SR's speed-of-light (c). The only "non-local" aspect is the alteration of probabilities based on knowledge gained via measurement. A local measurement can alter the "partial information" known about a distant (entangled) system. There is no FTL communication to or alteration of the distant particle. Getting a Stern-Gerlach "up" here doesn't cause the distant entangled particle to suddenly start moving "down". One only knows "now" that it "will" go down if the distant experimenter actually performs the measurement.

QM respects the principles of SR/GR, whereas SR/GR modify the results of QM

Special Relativity → Quantum Mechanics

Background: GR Principles

A Tensor Study
of Physical 4-Vectors

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Principles/Axioms and Mathematical Consequences of GR:

Equivalence Principle: Inertial Motion = Geodesic Motion, Universality/Equivalency of Free-Fall, $Mass_{inertial} = Mass_{gravitational}$

Relativity Principle: SpaceTime M has a Lorentzian/pseudo-Riemannian Metric $g^{\mu\nu}$, SR:Minkowski Space rules apply locally ($\eta^{\mu\nu}$)

General Covariance Principle: Tensors describe Physics, Laws of Physics are independent of Coordinate System

Invariance Principle: Invariant Interval Measure comes from Tensor Invariance Properties, 4D SpaceTime from Invariant $Trace[g^{\mu\nu}] = 4$

Causality Principle: Minkowski Diagram/Light-Cone give {Time-Like, Light-Light(null), Space-Like} Measures and Causality Conditions

Einstein:Riemann's Ideas about Matter & Curvature:

Riemann(g) has 20 independent components → too many

Ricci(g) has 10 independent components = enough to describe gravitational field

{c,G} are Fundamental Physical Constants

To date, there are no violations of any of these GR Principles.

GR limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
Minkowski "Flat" SpaceTime
Metric = (Curvature ~ 0)

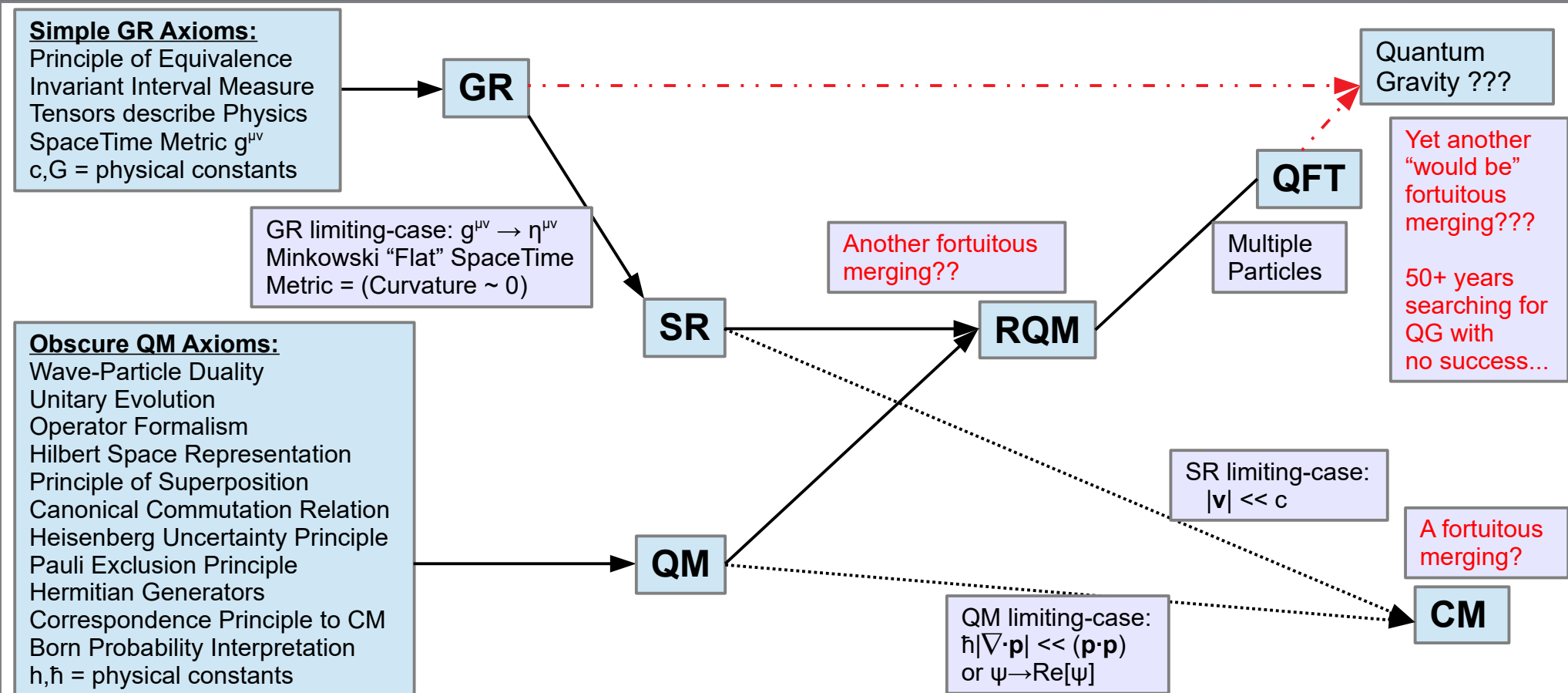
It is vitally important to keep the mathematics fixed to known physics. There are too many instances of trying to apply theoretical math to the physics (ex. String Theory – no physical evidence to date). It doesn't work that way. Nature is the arbiter of what math works with physics. Tensor mathematics applies well to SR and GR, which have been extremely well-tested now in a variety of physical situations.

All known experiments to date comply with all of these Principles, including QM

Old Paradigm: QM (for comparison)

SR and QM still as separate theories

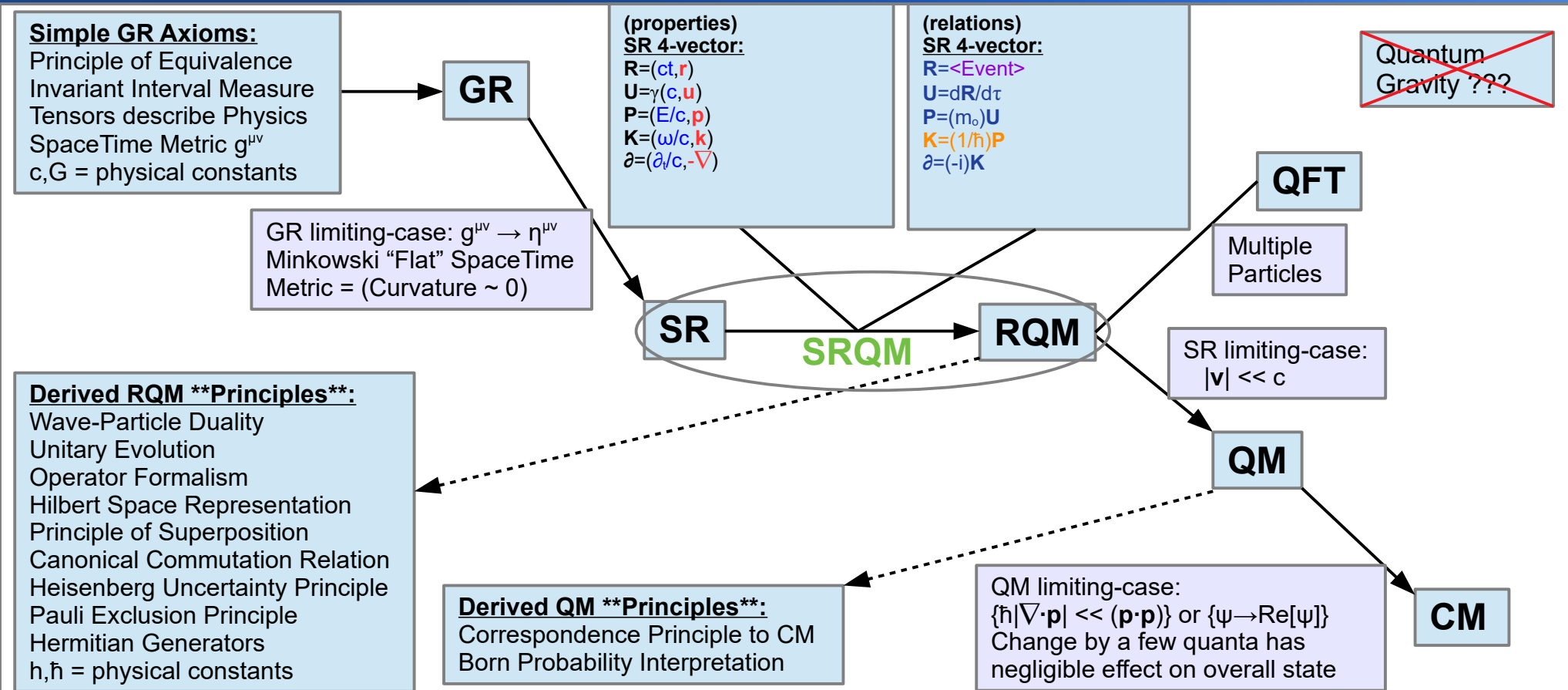
QM limiting-case better defined, still no QG



It is known that QM + SR "join nicely" together to form RQM, but problems with RQM + GR...

New Paradigm: SRQM or [SR→QM]

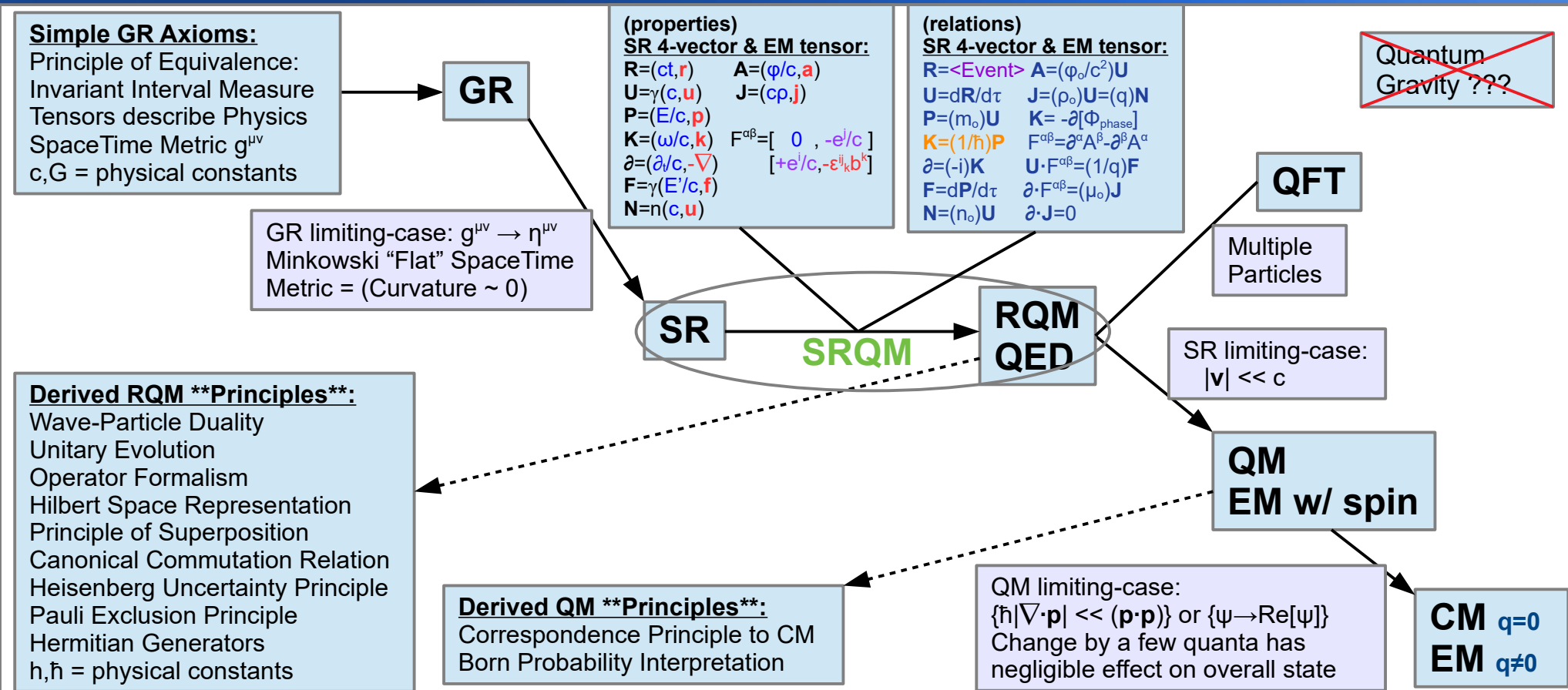
QM derived from SR + a few empirical facts



This new paradigm explains why RQM "miraculously fits" SR, but not necessarily GR

New Paradigm: SRQM w/ EM

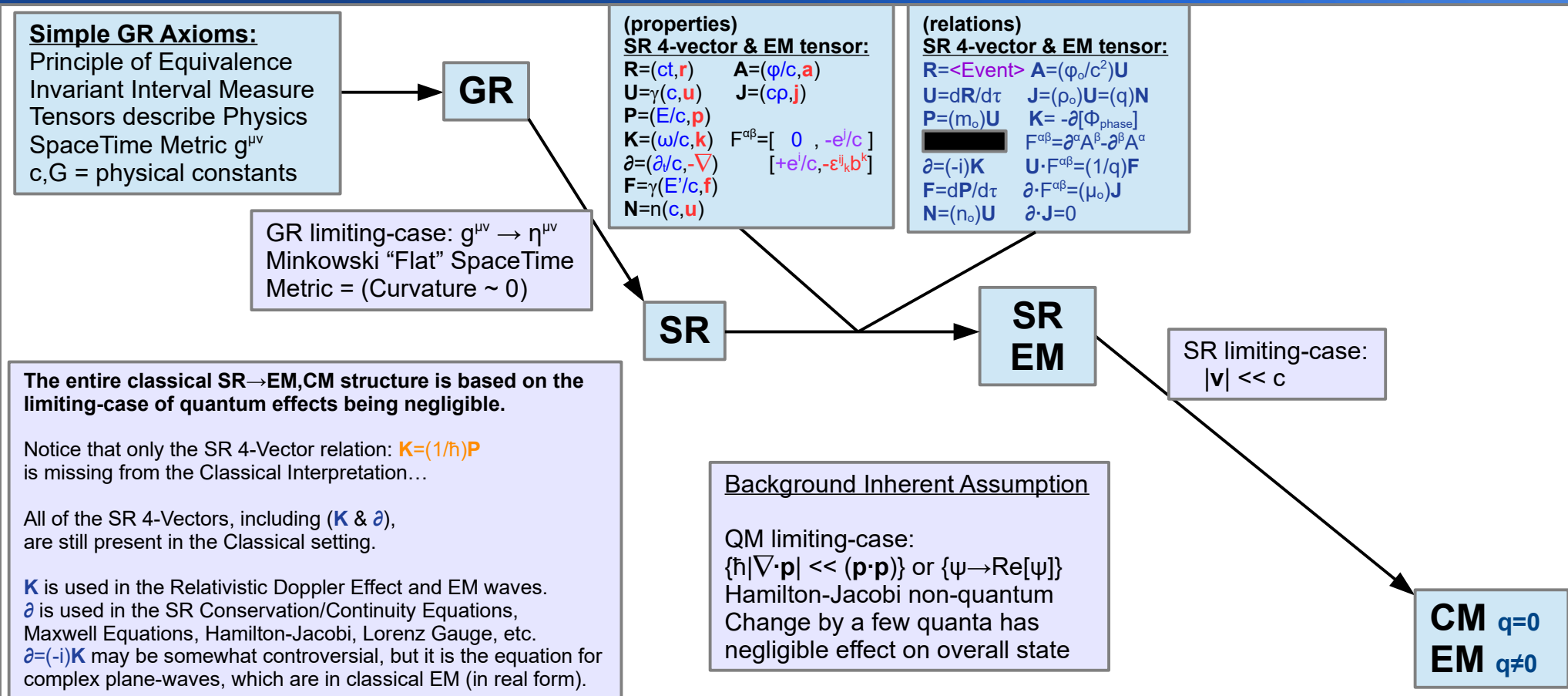
QM, EM, CM derived from SR + a few empirical facts



This new paradigm explains why RQM "miraculously fits" SR, but not necessarily GR

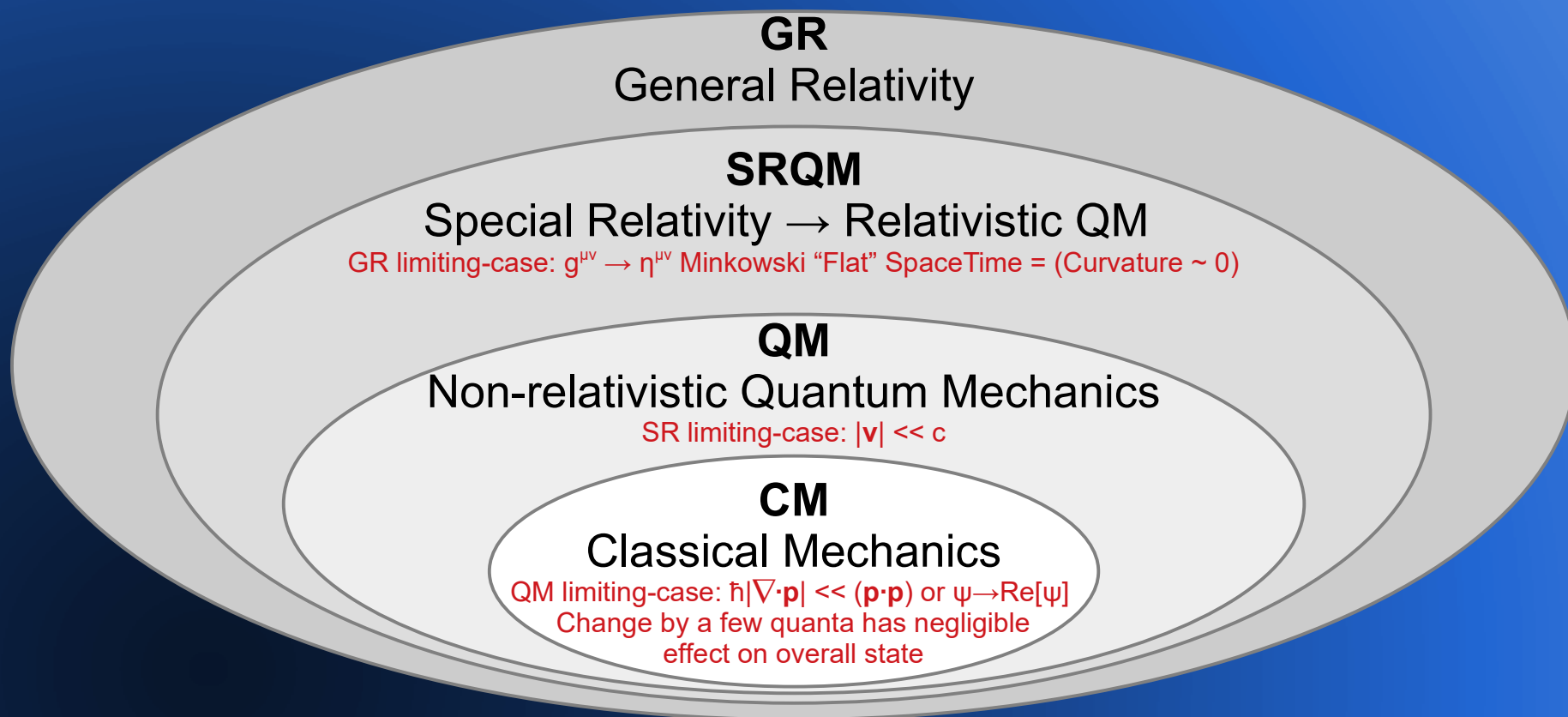
Classical SR w/ EM Paradigm (for comparison)

CM & EM derived from SR + a few empirical facts



This {Classical=non-QM} SR→EM,CM paradigm has been working successfully for decades

New Paradigm: SRQM View as Venn Diagram

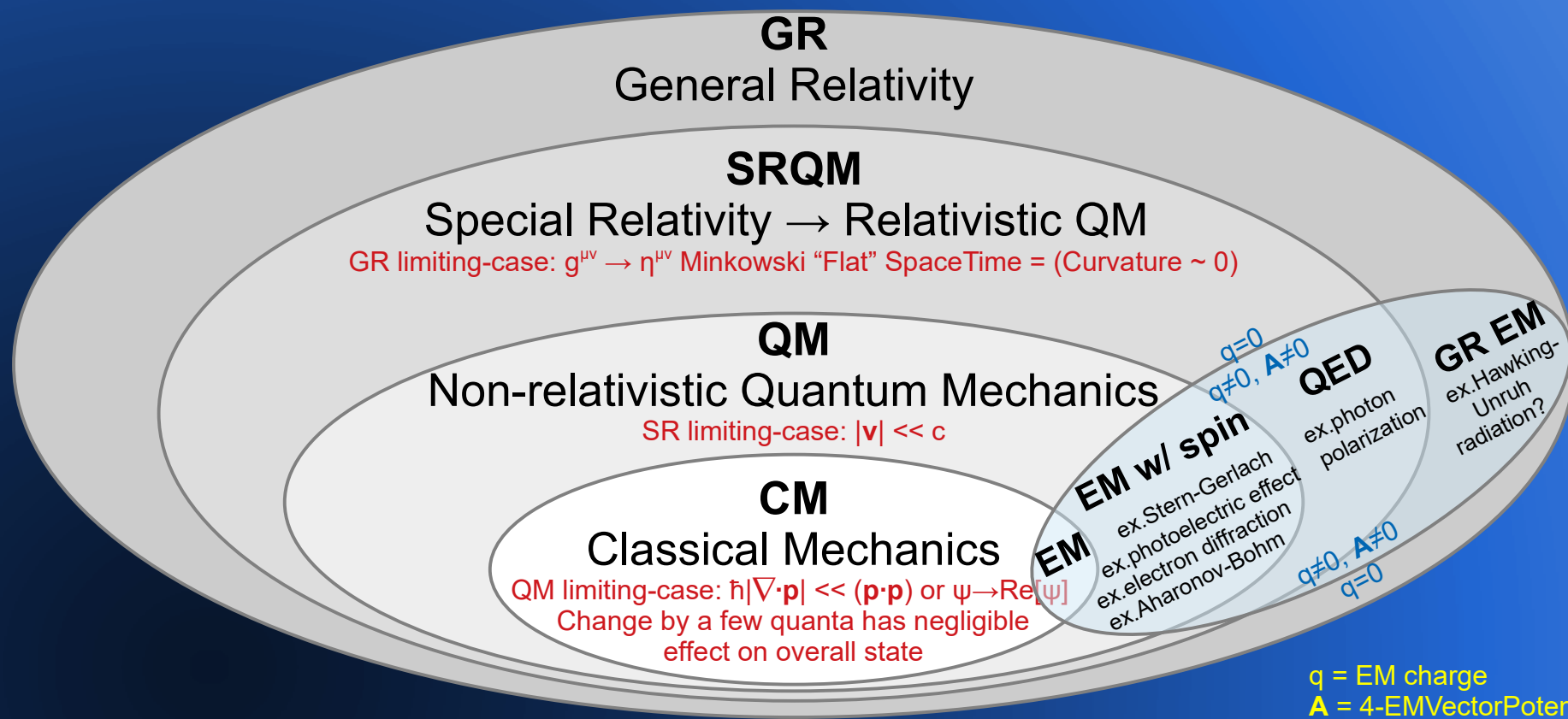


The SRQM view: Each level (range of validity) is a subset of the larger level.

New Paradigm: SRQM View w/ EM as Venn Diagram

A Tensor Study of Physical 4-Vectors

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John B. Wilson



The SRQM view: Each level (range of validity) is a subset of the larger level

SR language beautifully expressed with Physical 4-Vectors

A Tensor Study of Physical 4-Vectors

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Newton's laws of classical physics are greatly simplified by the use of physical 3-vector notation, which converts 3 separate space components, which may be different in various coordinate systems, into a single invariant object, a vector:
The basis values of these components can differ, yet still refer to the same overall 3-vector object.

Classical 3D objects styled this way to emphasize that they are actually just the separated components of SR 4-Vectors. The triangle/wedge (3 sides) represents splitting the components into a scalar and 3-vector.

3-vector = 3D (1,0)-tensor
 $\mathbf{a} = \mathbf{a}^i = (a^i) = (a^1, a^2, a^3)$

- (a^x, a^y, a^z) Cartesian/Rectangular 3D basis
- (a^r, a^θ, a^z) Polar/Cylindrical 3D basis
- (a^r, a^θ, a^ϕ) Spherical 3D basis

$\mathbf{a} \cdot \mathbf{a} = a^i \delta_{jk} a^k = |\mathbf{a}|^2$

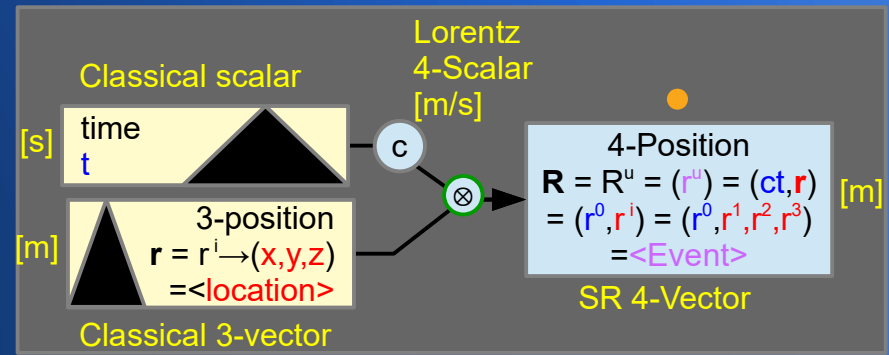
$\mathbf{A} \cdot \mathbf{A} = A^\mu \eta_{\mu\nu} A^\nu = (a^0)^2 - \mathbf{a} \cdot \mathbf{a} = (a^0_o)^2$

The scalar products of either type are basis-independent

4-Vector = 4D (1,0)-Tensor
 $\mathbf{A} = \mathbf{A}^\mu = (a^\mu) = (a^0, \mathbf{a}^i) = (a^0, \mathbf{a}) = (a^0, a^1, a^2, a^3)$

- (a^t, a^x, a^y, a^z) Cartesian/Rectangular 4D basis
- $(a^t, a^r, a^\theta, a^z)$ Polar/Cylindrical 4D basis
- $(a^t, a^r, a^\theta, a^\phi)$ Spherical 4D basis

SR is able to expand the concept of mathematical vectors into the Physical 4-Vector, which combines both (time) and (space) components into a single TimeSpace object: These 4-Vectors are elements of Minkowski 4D SR SpaceTime. Typically there is a speed-of-light factor (c) in the temporal component to make the dimensional units match. eg. $\mathbf{R} = (ct, \mathbf{r})$: overall dimensional units of [length] = SI Unit [m] This also allows the 4-Vector name to match up with the 3-vector name.




In this presentation:

I use the (+,-,-,-) metric signature, giving $\mathbf{A} \cdot \mathbf{A} = A^\mu \eta_{\mu\nu} A^\nu = [(a^0)^2 - \mathbf{a} \cdot \mathbf{a}] = (a^0_o)^2$
4-Vectors will use Upper-Case Letters, ex. \mathbf{A} ; 3-vectors will use lower-case letters, ex. \mathbf{a} ; I always put the (c) in the temporal component.
Vectors of both types will be in bold font; components and scalars in normal font and usually lower-case. 4-Vector name will match 3-vector name.
Tensor form will usually be normal font with a tensor index, ex. A^μ or a^i , with Greek TimeSpace index (0,1..3); Latin SpaceOnly index (1..3)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Classical (scalar 3-vector)
Galilean Invariant  Not Lorentz Invariant

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$

SR 4-Vectors & Lorentz Scalars

Frame-Invariant Equations

SRQM Diagramming Method

A Tensor Study of Physical 4-Vectors

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John B. Wilson

4-Vectors are type (1,0)-Tensors, Lorentz {4-}Scalars are type (0,0)-Tensors, 4-CoVectors are type (0,1)-Tensors, (m,n)-Tensors have (m) # upper-indices and (n) # lower-indices. $V^\mu, S, C_\mu, T^{\alpha\beta\gamma\dots\{m\text{ indices}\}}_{\mu\nu\dots\{n\text{ indices}\}}$

Any equation which employs only Tensors, such as those with only 4-Vectors and Lorentz 4-Scalars, (ex. $\mathbf{P} = m_0\mathbf{U}$) is automatically **Frame-Invariant**, or coordinate-frame-independent. One's frame-of-reference plays no role in the form of the overall equations. This is also known as being "Manifestly-Invariant". This is exactly what Einstein meant by his postulate: "The laws of physics should have the same form for all inertial observers". Use of the RestFrame-naught (${}_0$) helps show this.

4-Vector = 4D (1,0)-Tensor
 $\mathbf{A} = A^\mu = (a^\mu) = (a^0, \mathbf{a}^i) = (a^0, \mathbf{a}) = (a^0, a^1, a^2, a^3) \rightarrow (a^t, \mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^z)$

$\mathbf{A} \cdot \mathbf{A} = A^\mu \eta_{\mu\nu} A^\nu = (a^0)^2 - \mathbf{a} \cdot \mathbf{a} = (a^0_0)^2$

Unlike the 3D case (+), the 4D magnitude can be (+/-)

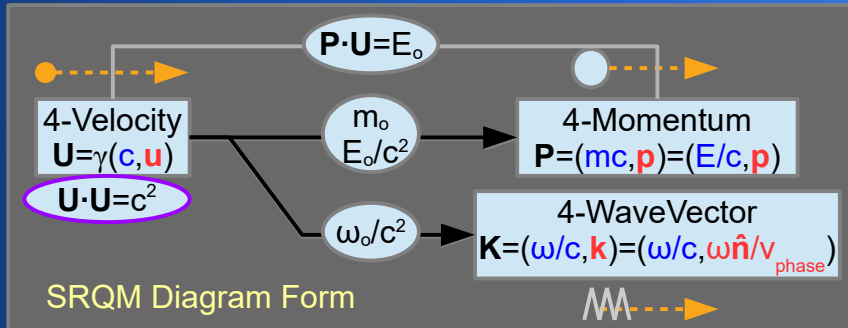
The components (a^0, a^1, a^2, a^3) of the 4-Vector \mathbf{A} can vary depending on the observer and their choice of coordinate system, but the 4-Vector $\mathbf{A} = A^\mu$ itself is invariant. Equations using only 4-Tensors, 4-Vectors, and Lorentz 4-Scalars are true for all inertial observers. The **SRQM Diagramming Method** makes this easy to see in a visual format, and will be used throughout this treatise. The following examples are frame-invariant equations:

$\mathbf{U} \cdot \mathbf{U} = (c)^2$
 $\mathbf{U} = \gamma(c, \mathbf{u})$
 $\mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p}) = m_0\mathbf{U} = (E_0/c^2)\mathbf{U}$
 $\mathbf{K} = (\omega/c, \mathbf{k}) = (\omega_0/c^2)\mathbf{U}$
 $\mathbf{P} \cdot \mathbf{U} = E_0$

Equation Form

The SRQM Diagram Form has all of the info of the Equation Form, but shows overall relationships and symmetries among the 4-Vectors much more clearly.

Blue: Temporal components
 Red: Spatial components
 Purple: Mixed TimeSpace components



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or T_{μ}^ν
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
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 (0,0)-Tensor S
 Lorentz Scalar

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 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SR 4-Vectors are primitive elements of Minkowski SpaceTime (4D)

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We want to be clear, however, that SR 4-Vectors are **NOT** generalizations of Classical or Quantum 3-vectors.

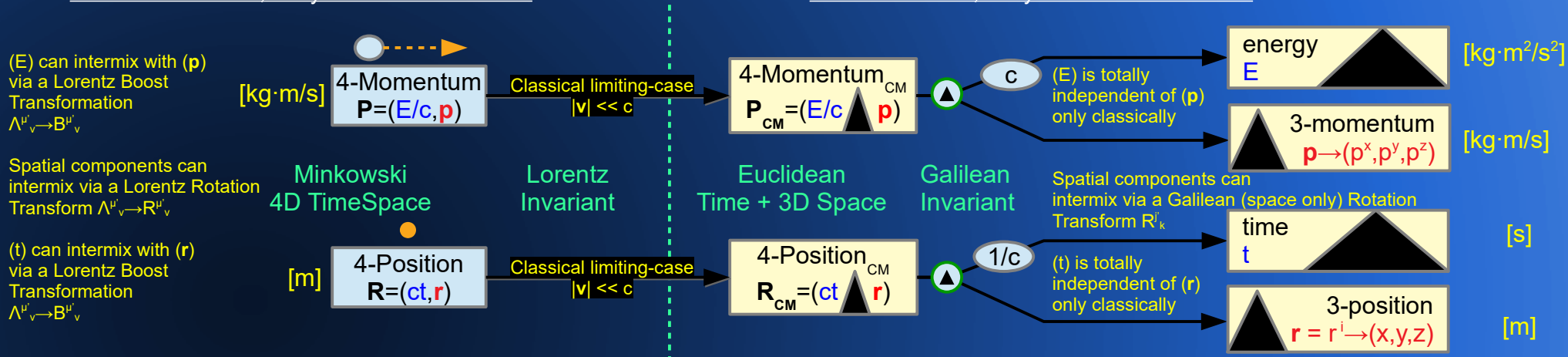
SR 4-Vectors are the primitive elements of Minkowski SpaceTime (4D) which incorporate both: a {temporal scalar element} and a {spatial 3-vector element} as components.
4-Vector $\mathbf{A} = (a^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3) = (a^0, \mathbf{a}) \rightarrow (a^t, \mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^z)$ with scalar (a^t) & 3-vector $\mathbf{a} \rightarrow (a^x, a^y, a^z)$

It is the Classical or Quantum 3-vector (\mathbf{a}) which is a limiting-case approximation of the spatial part of SR 4-Vector (\mathbf{A}) for $\{ |v| \ll c \}$.

i.e. The Energy (E) and 3-momentum (\mathbf{p}) as “separate” entities occurs only in the low-velocity limit $\{ |v| \ll c \}$ of the Lorentz Boost Transform. They are actually part of a single 4D entity: the 4-Momentum $\mathbf{P} = (E/c, \mathbf{p})$; with the components: temporal (E), spatial (\mathbf{p}), dependent on a frame-of-reference, while the overall 4-Vector \mathbf{P} is invariant. Likewise with (t) and (\mathbf{r}) in the 4-Position \mathbf{R} .

SR is Minkowskian; obeys Lorentz Invariance.

CM is Euclidean; obeys Galilean Invariance.



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

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Relations among SR 4-Vectors are Manifestly Invariant

A Tensor Study of Physical 4-Vectors

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John B. Wilson

Relations among 4-Vectors and Lorentz 4-Scalars are Manifestly Invariant, meaning that they are true in all inertial reference frames.

Consider a particle at a SpaceTime **<Event>** that has properties described by 4-Vectors **A** and **B**:

One possible relationship is that the two 4-Vectors are related by a Lorentz 4-Scalar (S): ex. **B = (S) A**.

How can one determine this? Answer: Make an experiment that empirically measures the tensor invariant **[B·A / A·A]**.

If **B = (S) A**

B·A = (S) A·A

(S) = [B·A / A·A] Note that this basically a vector projection.



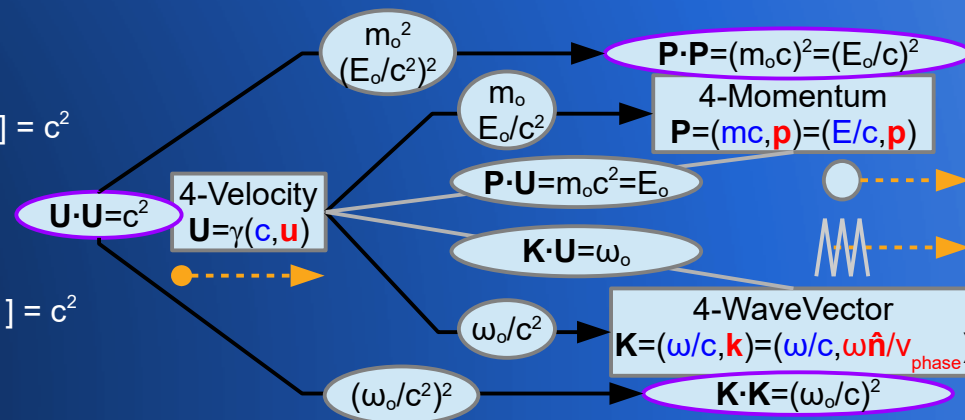
Run the experiment many times. If you always get the same result for (S), then it is likely that the relationship is true, and invariant.

Example: Measure $(S_P) = [\mathbf{P} \cdot \mathbf{U} / \mathbf{U} \cdot \mathbf{U}]$ for a given particle type.

Repeated measurement always give $(S_P) = m_0$

This makes sense because we know $[\mathbf{P} \cdot \mathbf{U}] = \gamma(E - \mathbf{p} \cdot \mathbf{u}) = E_0$ and $[\mathbf{U} \cdot \mathbf{U}] = c^2$

Thus, 4-Momentum $\mathbf{P} = (E_0/c^2)\mathbf{U} = (m_0)\mathbf{U} = (m_0)*4\text{-Velocity } \mathbf{U}$



Example: Measure $(S_K) = [\mathbf{K} \cdot \mathbf{U} / \mathbf{U} \cdot \mathbf{U}]$ for a given particle type.

Repeated measurement always give $(S_K) = (\omega_0/c^2)$

This makes sense because we know $[\mathbf{K} \cdot \mathbf{U}] = \gamma(\omega - \mathbf{k} \cdot \mathbf{u}) = \omega_0$ and $[\mathbf{U} \cdot \mathbf{U}] = c^2$

Thus, 4-WaveVector $\mathbf{K} = (\omega_0/c^2)\mathbf{U} = (\omega_0/c^2)*4\text{-Velocity } \mathbf{U}$

Since **P** and **K** are both related to **U**, this would also mean that the

4-Momentum **P** is related to the 4-WaveVector **K** in a particular manner for each given particle type... a hint for later...

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

Some SR Mathematical Tools

Definitions and Approximations

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

$\beta = v/c$: dimensionless Velocity Beta Factor { $\beta=(0..1)$; rest at ($\beta=0$); speed-of-light (c) at ($\beta=1$) }
 $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\beta\cdot\beta}$: dimensionless Lorentz Relativistic Gamma Factor { $\gamma=(1..\infty)$; rest at ($\gamma=1$); speed-of-light (c) at ($\gamma=\infty$) }

$(1+x)^n \sim (1 + nx + O[x^2])$ for { $|x| \ll 1$ } Approximation used for SR→Classical limiting-cases

Lorentz Transformation $\Lambda^\mu_\nu = \partial X^\mu / \partial X^\nu = \partial_\nu [X^\mu]$: a relativistic frame-shift, such as a rotation or velocity boost
 It transforms a 4-Vector in the following way: $X^\mu = \Lambda^\mu_\nu X^\nu$: with Einstein summation over the paired indices
 A typical Lorentz Boost Transformation $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$ for a linear-velocity frame-shift (x,t)-Boost in the \hat{x} -direction:

Lorentz x-Boost Transform $\Lambda^\mu_\nu \rightarrow B^\mu_\nu =$

	\hat{t}	\hat{x}	\hat{y}	\hat{z}
\hat{t}	γ	$-\beta\gamma$	0	0
\hat{x}	$-\beta\gamma$	γ	0	0
\hat{y}	0	0	1	0
\hat{z}	0	0	0	1

SR:Minkowski Metric
 $\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} = \mathbf{V}^{\mu\nu} + \mathbf{H}^{\mu\nu} \rightarrow \text{Diag}[1, -\mathbf{I}_{(3)}] = \text{Diag}[1, -\delta^{jk}]$

\hat{t}	\hat{x}	\hat{y}	\hat{z}
1	0	0	0
0	-1	0	0
0	0	-1	0
0	0	0	-1

$\begin{matrix} 1 & 0^j \\ 0^i & -\delta^{ij} \end{matrix}$

"Particle Physics" Convention Symmetric

SR:Minkowski Metric
 $\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} = \mathbf{V}^{\mu\nu} + \mathbf{H}^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -\mathbf{I}_{(3)}] = \text{Diag}[1, -\delta^{jk}]$
 {in Cartesian form} "Particle Physics" Convention
 $\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu^\nu = \delta_\mu^\nu$ **Tr $[\eta^{\mu\nu}] = 4$**

SR:Lorentz Transform
 $\partial_\nu [R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu$
 $\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$
 $\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$
Det $[\Lambda^\mu_\nu] = \pm 1$ **$\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$**

Original $A^\nu = (a^t, a^x, a^y, a^z)$
 Boosted $A^\mu = (a^t, a^x, a^y, a^z)' = \Lambda^\mu_\nu A^\nu \rightarrow B^\mu_\nu A^\nu = (\gamma a^t - \gamma\beta a^x, -\gamma\beta a^t + \gamma a^x, a^y, a^z)$ {for \hat{x} -boost Lorentz Transform}

$\mathbf{A}' \cdot \mathbf{B}' = (\Lambda^\mu_\nu A^\nu) \cdot (\Lambda^\sigma_\rho B^\rho) = \mathbf{A} \cdot \mathbf{B} = A^\mu \eta_{\mu\nu} B^\nu = A^\mu B_\mu = A_\nu B^\nu = \sum_{\nu=0..3} [a^\nu b_\nu] = \sum_{u=0..3} [a^u b_u] = (a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3)$
 $= (a^0 b^0 - \mathbf{a} \cdot \mathbf{b}) = (a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3)$
 using the **Einstein summation convention** where upper:lower paired-indices are summed over

$\partial[X] = \partial^\mu [X^\nu] = (\partial/c, -\nabla)(ct, \mathbf{x}) = \text{Diag}[\partial/c[ct], -\nabla[\mathbf{x}]] = \text{Diag}[1, -\mathbf{I}_{(3)}] = \text{Diag}[1, -1, -1, -1] = \eta^{\mu\nu}$ Minkowski "Flat" SpaceTime Metric



SpaceTime Dimension
 $\partial \cdot R = \partial_\mu R^\mu = 4$



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
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SRQM Diagram: The Basis of Classical SR Physics Special Relativity via 4-Vectors

A Tensor Study of Physical 4-Vectors

Focus on a few of the main SR Physical 4-Vectors.

4-Position
 $\mathbf{R} = R^\mu = (ct, \mathbf{r}) = \langle \text{Event} \rangle$

● $\langle \text{Event} \rangle$ Location

4-Velocity
 $\mathbf{U} = U^\mu = \gamma(\mathbf{c}, \mathbf{u})$

● → $\langle \text{Event} \rangle$ Motion

4-Gradient
 $\partial = \partial^\mu = (\partial/c, -\nabla)$

△ $\langle \text{Event} \rangle$ Alteration

4-Displacement
 $\Delta \mathbf{R} = (c\Delta t, \Delta \mathbf{r})$
 $d\mathbf{R} = (cdt, d\mathbf{r})$
4-Position
 $\mathbf{R} = (ct, \mathbf{r})$

4-Gradient
 $\partial = (\partial/c, -\nabla)$
 $= (\partial/c, -\partial_x, -\partial_y, -\partial_z)$
 $= (\partial/c\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$

4-Velocity
 $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$
 $= d\mathbf{R}/d\tau$

SRQM Diagram

These 4-Vectors give some of the main classical results of Special Relativity, including SR concepts like:

The Minkowski Metric, SpaceTime Dimension = 4, Lorentz Transformations $\langle \text{Events} \rangle$, Invariant Interval Measure, Causality (=Temporal Ordering)

The Invariant Speed-of-Light (c)

Invariant ProperTime (clock at rest), Invariant ProperLength (ruler at rest)

Time Dilation (clock moving), Length Contraction (ruler moving)

Relativity of Simultaneity, Minkowski Diagrams, Light Cone

Use of the Lorentz Scalar Product to make Lorentz Invariants

Invariant SR Wave Equations, via the d'Alembertian

Continuity Equations

etc.

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(2,0)-Tensor $T^{\mu\nu}$
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A Tensor Study of Physical 4-Vectors

SRQM Diagram: The Basis of Classical SR Physics Special Relativity via 4-Vectors

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The Basis of most all Classical SR Physics is in the SR Minkowski Metric of "Flat" SpaceTime, which can be generated from the 4-Position **R** and 4-Gradient ∂ , and determines the measurement between <Event>'s.

This Metric provides the relations between the 4-Vectors of SR: 4-Position **R**, 4-Gradient ∂ , 4-Velocity **U**.

The Tensor Invariants of these 4-Vectors give the: Invariant Interval Measures & Causality, from **R**·**R** Invariant d'Alembertian Wave Equation, from ∂ · ∂ Invariant Magnitude LightSpeed (c), from **U**·**U**

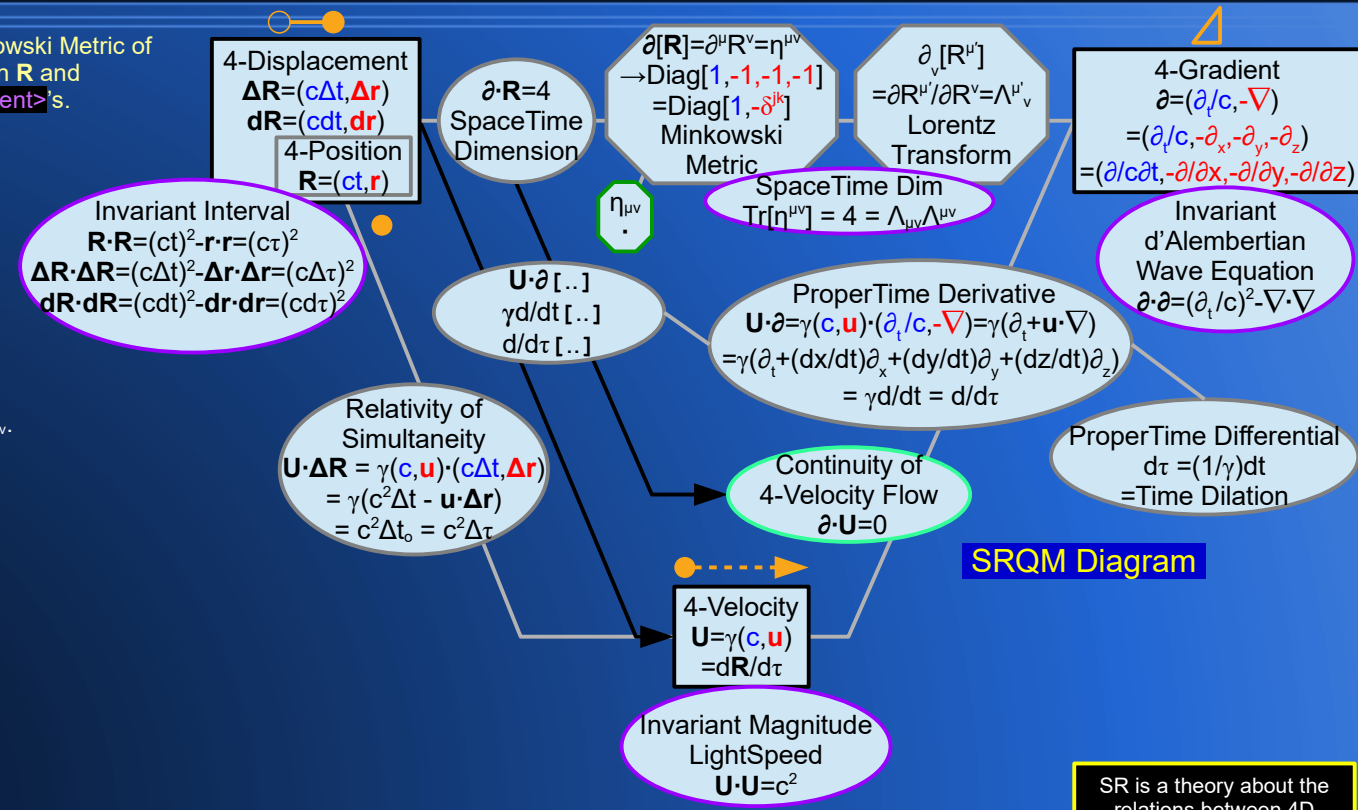
The relation between 4-Gradient ∂ and 4-Position **R** gives the Dimension of SpaceTime (4), the Minkowski Metric $\eta^{\mu\nu}$, and the Lorentz Transformations Λ^{μ}_{ν} .

The relation between 4-Gradient ∂ and 4-Velocity **U** gives the ProperTime Derivative $d/d\tau$. Rearranging gives the ProperTime Differential $d\tau$, which leads to Time Dilation & Length Contraction.

The ProperTime Derivative $d/d\tau$: acting on 4-Position **R** gives 4-Velocity **U** acting on the SpaceTime Dimension Lorentz Scalar gives the Continuity of 4-Velocity Flow.

The relation between 4-Displacement $\Delta\mathbf{R}$ and 4-Velocity **U** gives Relativity of Simultaneity.

One of the most important properties is the Tensor Invariant Lorentz Scalar Product (dot = \cdot), provided by the lowered- index form of the Minkowski Metric $\eta_{\mu\nu}$.



From here, each object will be examined in turn...

SR is a theory about the relations between 4D SpaceTime <Event>'s, ie. how they are measured

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν} (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_{\mu} = (\mathbf{v}_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
---------------------------------------------------------------------------------------------------------------------------------	---------------------------------------------------------------------------------------------------------------------------------------------------------------	--------------------------------------------------------

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

The Basis of Classical SR Physics 4-Position, 4-Displacement, 4-Differential

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SR → QM

A Tensor Study
of Physical 4-Vectors

4-Displacement $\Delta R = (c\Delta t, \Delta \mathbf{r}) = \mathbf{U}\Delta\tau = \mathbf{R}_2 - \mathbf{R}_1 = (ct_2 - ct_1, \mathbf{r}_2 - \mathbf{r}_1)$: {finite}
 4-Differential $dR = (cdt, d\mathbf{r}) = \mathbf{U}d\tau$: {infinitesimal}
 4-Position
 $\mathbf{R} = (ct, \mathbf{r}) = \langle \text{Event} \rangle$

The 4-Position is essentially one of the fundamental 4-Vectors of SR. It is the SpaceTime location of an **<Event>**, the basic element of Minkowski SpaceTime: a time (t) & a place (r) → (when, where) = (ct, r).

The 4-Position relates time to space via the fundamental physical constant (c): the speed-of-light = "(c)elerity ; (c)eleritas", which is used to give consistent dimensional units across all SR 4-Vectors.

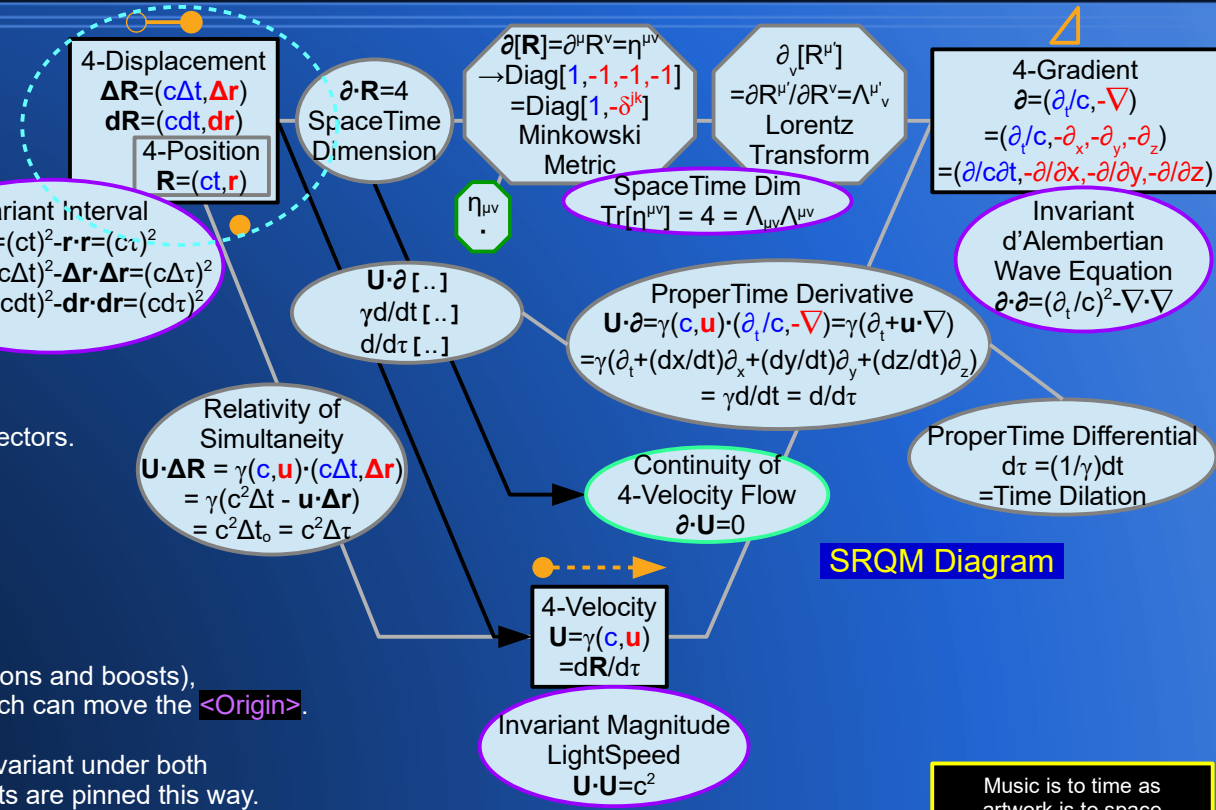
The 4-Position is a specific type of 4-Displacement, for which one of the endpoints is the origin, or 4-Zero.

$\mathbf{R}_2 \rightarrow \mathbf{R}, \mathbf{R}_1 \rightarrow \mathbf{Z}$
 $\Delta \mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1 \rightarrow \mathbf{R} - \mathbf{Z} = \mathbf{R}$
 4-Zero
 $\mathbf{Z} = (0, \mathbf{0}) = (0, 0, 0, 0) = \langle \text{Origin} \rangle$

As such, the 4-Position and 4-Zero are Lorentz Invariant (point rotations and boosts), but not Poincaré Invariant (Lorentz + time & space translations), which can move the **<Origin>**.

The general 4-Displacement and 4-Differential(Displacement) are invariant under both Lorentz and Poincaré transformations, since neither of their endpoints are pinned this way.

The 4-Differential(Displacement) is just the infinitesimal version of the finite 4-Displacement, and is used in the calculus of SR. $\mathbf{U} = dR/d\tau$: $dR = \mathbf{U}d\tau$



SRQM Diagram

Music is to time as
artwork is to space

4-Creativity
 $\odot = (\text{Music}, \text{Artwork})$

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
 SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

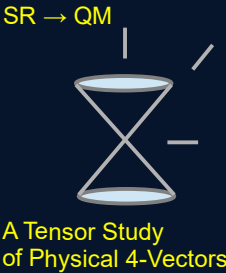
4-Position $\mathbf{R} = (ct, \mathbf{r}) = \langle \text{Event} \rangle$
 $\mathbf{R} = \int dR = \int \mathbf{U}d\tau = \int \gamma(\mathbf{c}, \mathbf{u})d\tau = \int (\mathbf{c}, \mathbf{u})\gamma d\tau = \int (\mathbf{c}, \mathbf{u})dt = (ct, \mathbf{r})$
 $\mathbf{R} = \sum \Delta \mathbf{R} = \sum \mathbf{U}\Delta\tau = \sum \gamma(\mathbf{c}, \mathbf{u})\Delta\tau = \sum (\mathbf{c}, \mathbf{u})\gamma\Delta\tau = \sum (\mathbf{c}, \mathbf{u})\Delta t = (ct, \mathbf{r})$

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

The Basis of Classical SR Physics Invariant Intervals, Causality, TimeSpace

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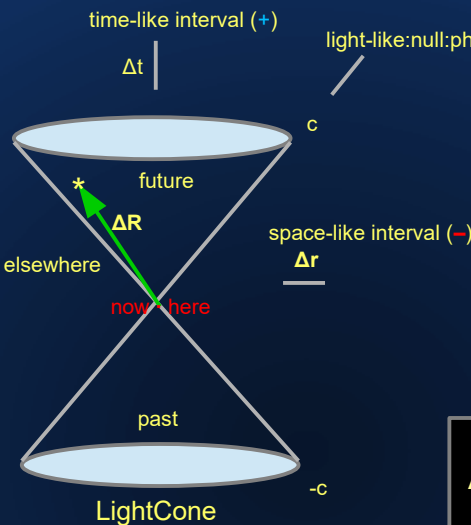
4-Displacement $\Delta R = (c\Delta t, \Delta r) = U\Delta\tau = R_2 - R_1 = (ct_2 - ct_1, r_2 - r_1)$: {finite}
 4-Differential $dR = (cdt, dr) = U d\tau$: {infinitesimal}
 4-Position
 $R = (ct, r) = \langle \text{Event} \rangle$

The Invariant Interval is the Lorentz Scalar Product of the {4-Position, 4-Displacement, 4-Differential} with itself, giving a magnitude-squared, which may be (+/-)

$$R \cdot R = (ct)^2 - r \cdot r = (ct_0)^2 = (c\tau)^2$$

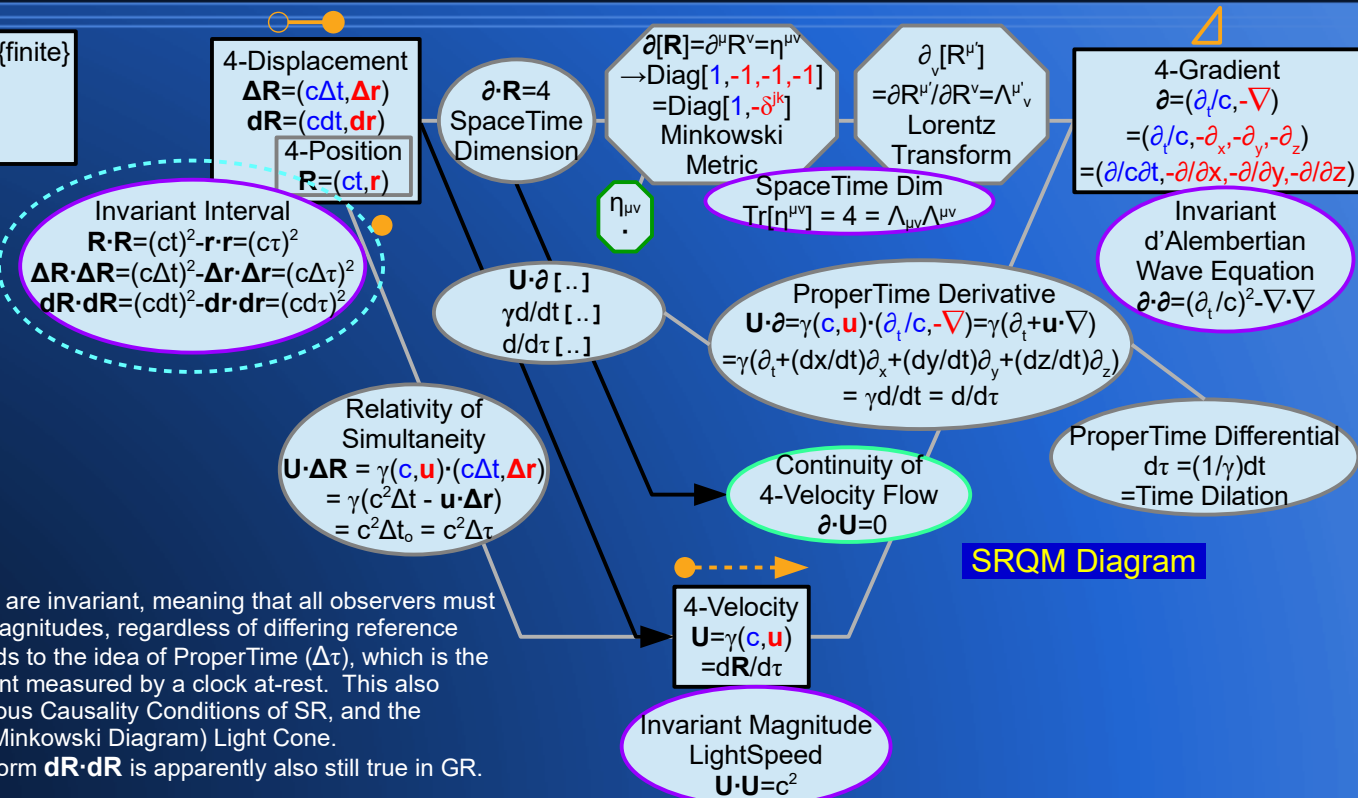
$$\Delta R \cdot \Delta R = (c\Delta t)^2 - \Delta r \cdot \Delta r = (c\Delta t_0)^2 = (c\Delta\tau)^2$$

$$dR \cdot dR = (cdt)^2 - dr \cdot dr = (cdt_0)^2 = (cd\tau)^2$$



The 4D intervals are invariant, meaning that all observers must agree on their magnitudes, regardless of differing reference frames. This leads to the idea of ProperTime ($\Delta\tau$), which is the time-displacement measured by a clock at-rest. This also leads to the various Causality Conditions of SR, and the concept of the (Minkowski Diagram) Light Cone. The differential form $dR \cdot dR$ is apparently also still true in GR.

$\Delta R \cdot \Delta R = [(c\Delta t)^2 - \Delta r \cdot \Delta r] =$	$(c\Delta\tau)^2$ Time-like:Temporal	(+) {causal = temporally-ordered}
	(0) Light-like:Null:Photonic	(0) {causal, maximum signal speed ($ \Delta r/\Delta t =c$)}
	$-(\Delta r_0)^2$ Space-like: Spatial	(-) {non-causal, spatially-extended}



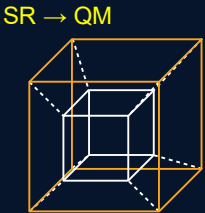
- SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$
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SRQM Diagram:

The Basis of Classical SR Physics SpaceTime Dimension = 4D

SciRealm.org
John B. Wilson



A Tensor Study of Physical 4-Vectors

4-Gradient
 $\partial=(\partial/c, -\nabla)$

4-Position
 $R=(ct, \mathbf{r})=\langle \text{Event} \rangle$

$\partial \cdot R = 4$: The 4-Divergence SpaceTime Dimension Relation

$\partial \cdot R$
 $= (\partial/c, -\nabla) \cdot (ct, \mathbf{r})$
 $= [(\partial/c) \cdot (ct) - (-\nabla) \cdot (\mathbf{r})]$
 $= (\partial_t[t] + \nabla \cdot \mathbf{r})$
 $= (\partial_t[t] + \partial_x[x] + \partial_y[y] + \partial_z[z])$
 $= (\partial[t]/\partial t + \partial[x]/\partial x + \partial[y]/\partial y + \partial[z]/\partial z)$
 $= (1+1+1+1)$
 $= 4$

SpaceTime Dimension
 $\partial \cdot R = \partial^\mu \eta_{\mu\nu} R^\nu = \partial_\nu R^\nu = 4$

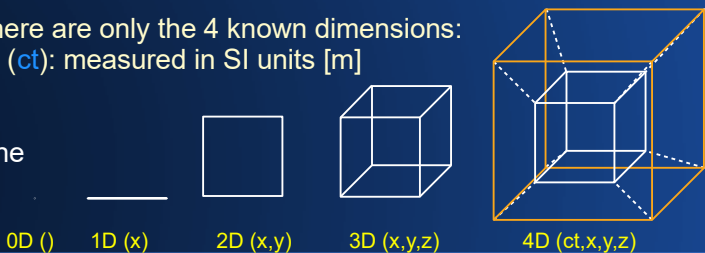
$(\partial \cdot R) = (\partial^\alpha \cdot R^\beta) = (\partial^\alpha \eta_{\alpha\beta} R^\beta) = \eta_{\alpha\beta} (\partial^\alpha R^\beta) = \eta_{\alpha\beta} (\eta^{\alpha\beta}) = \eta_\beta^\alpha = \eta_\alpha^\alpha = \delta_\alpha^\alpha = 4$

This Tensor Invariant Lorentz Scalar relation gives the dimension of SpaceTime. The only way there can more dimensions is if there is another SpaceTime direction available. The 4-Divergence is also used in SR Conservation Laws, ex. $(\partial \cdot \mathbf{J}) = 0$

All empirical evidence to-date indicates that there are only the 4 known dimensions:
 1 temporal (t): measured in SI units = [s], with (ct): measured in SI units [m]
 3 spatial (x, y, z) : measured in SI units = [m]

These are of course the ones that appear in the

4-Position
 $R=(ct, \mathbf{r}) \rightarrow (ct, x, y, z)$



The Tesseract, a 4D cube, symbolizes 4D SpaceTime

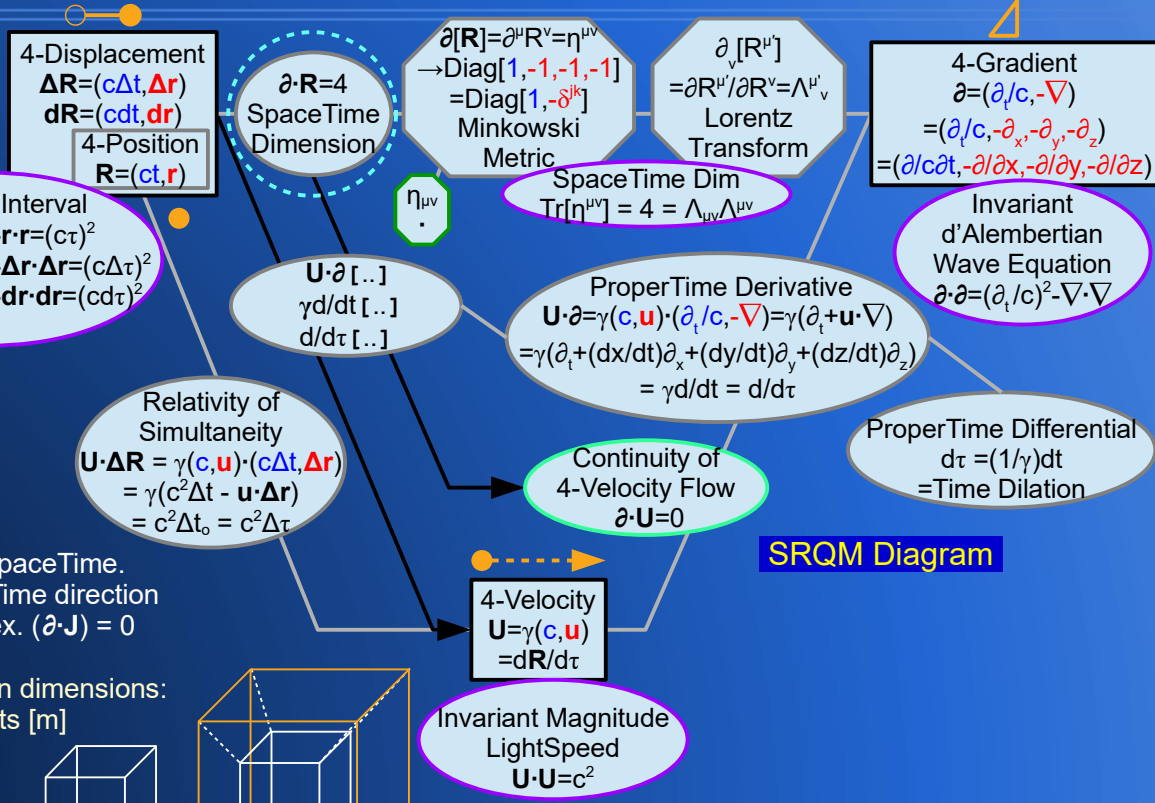
SR : Minkowski SpaceTime is 4D

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 (0,2)-Tensor $T_{\mu\nu}$

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4-Displacement
 $\Delta R=(c\Delta t, \Delta \mathbf{r})$
 $dR=(cdt, d\mathbf{r})$
4-Position
 $R=(ct, \mathbf{r})$

$\partial \cdot R = 4$
 SpaceTime Dimension

$\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu}$
 $\rightarrow \text{Diag}[1, -1, -1, -1]$
 $= \text{Diag}[1, -\delta^{jk}]$
 Minkowski Metric

$\partial_\nu [R^\mu]$
 $= \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu$
 Lorentz Transform

4-Gradient
 $\partial=(\partial/c, -\nabla)$
 $= (\partial/c, -\partial_x, -\partial_y, -\partial_z)$
 $= (\partial/c \partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$

Invariant Interval
 $R \cdot R = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (c\tau)^2$
 $\Delta R \cdot \Delta R = (c\Delta t)^2 - \Delta \mathbf{r} \cdot \Delta \mathbf{r} = (c\Delta \tau)^2$
 $dR \cdot dR = (cdt)^2 - d\mathbf{r} \cdot d\mathbf{r} = (cd\tau)^2$

SpaceTime Dim
 $\text{Tr}[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu} \Lambda^{\mu\nu}$

Invariant d'Alembertian Wave Equation
 $\partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla$

$\mathbf{U} \cdot \partial [..]$
 $\gamma d/dt [..]$
 $d/d\tau [..]$

ProperTime Derivative
 $\mathbf{U} \cdot \partial = \gamma(\mathbf{c}, \mathbf{u}) \cdot (\partial/c, -\nabla) = \gamma(\partial_t + \mathbf{u} \cdot \nabla)$
 $= \gamma(\partial_t + (dx/dt)\partial_x + (dy/dt)\partial_y + (dz/dt)\partial_z)$
 $= \gamma d/dt = d/d\tau$

ProperTime Differential
 $d\tau = (1/\gamma)dt$
 $= \text{Time Dilation}$

Relativity of Simultaneity
 $\mathbf{U} \cdot \Delta R = \gamma(\mathbf{c}, \mathbf{u}) \cdot (c\Delta t, \Delta \mathbf{r})$
 $= \gamma(c^2 \Delta t - \mathbf{u} \cdot \Delta \mathbf{r})$
 $= c^2 \Delta t_0 = c^2 \Delta \tau$

Continuity of 4-Velocity Flow
 $\partial \cdot \mathbf{U} = 0$

4-Velocity
 $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$
 $= dR/d\tau$

Invariant Magnitude LightSpeed
 $\mathbf{U} \cdot \mathbf{U} = c^2$


SRQM Diagram

SRQM Diagram:

The Basis of Classical SR Physics The Minkowski Metric ($\eta^{\mu\nu}$), Measurement

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SR → QM



A Tensor Study of Physical 4-Vectors

4-Gradient
 $\partial = (\partial/c, -\nabla)$

4-Position
 $R = (ct, \mathbf{r}) = \langle \text{Event} \rangle$

The SR Minkowski Metric = "Flat" SR SpaceTime
 $\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1]_{\text{rect}} = \text{Diag}[1, -\delta^k] = \text{Diag}[1, -I_{(3)}]$

Derivation:
 $\partial[R] = \partial^\mu R^\nu = (\partial/c, -\nabla)(ct, \mathbf{r})$

The component representation of the Minkowski Metric $\eta^{\mu\nu}$ will differ with the chosen basis, just like with 4-Vectors.

$\eta^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1]$: Cartesian/Rectangular basis
 $\eta^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1/r^2, -1]$: Polar/Cylindrical basis
 $\eta^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1/r^2, -1/(r \sin[\theta])^2]$: Spherical basis

Generally, components $[\eta^{\mu\mu}] = 1/[\eta_{\mu\mu}]$ and $\eta_\mu^\nu = \delta_\mu^\nu$

SR:Minkowski Metric
 $\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$
 $\text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -I_{(3)}] = \text{Diag}[1, -\delta^k]$
 (in Cartesian form) "Particle Physics" Convention
 $\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu^\nu = \delta_\mu^\nu$ $\text{Tr}[\eta^{\mu\nu}] = 4$

4-UnitTemporal
 $T = T^\mu = \gamma(1, \beta) = \mathbf{U}/c$

SR:Temporal Projection
 "Vertical" $V^{\mu\nu} = T^\mu T^\nu \rightarrow$
 $\text{Diag}[1, 0, 0, 0] = \text{Diag}[1, 0^k]$

SR:Spatial Projection
 "Horizontal" $H^{\mu\nu} = \eta^{\mu\nu} - T^\mu T^\nu \rightarrow$
 $\text{Diag}[0, -1, -1, -1] = \text{Diag}[0, -\delta^k]$

The SR:Minkowski Metric $\eta^{\mu\nu}$ is the fundamental SR (2,0)-Tensor, which shows how intervals are measured in SR SpaceTime. It is itself the low-mass = (Curvature ~ 0) limiting-case of the more general GR metric $g^{\mu\nu}$. It can be divided into temporal and spatial parts. The Minkowski Metric can be used to raise/lower indices on other tensors and 4-Vectors.

Alt. Derivation: $\partial^\mu X^\nu = \eta^{\mu\sigma} \partial_\sigma X^\nu = \eta^{\mu\sigma} (\partial/\partial X^\sigma) X^\nu = \eta^{\mu\sigma} (\partial X^\nu / \partial X^\sigma) = \eta^{\mu\sigma} (\delta_\sigma^\nu) = \eta^{\mu\nu}$

4-Displacement
 $\Delta R = (c\Delta t, \Delta \mathbf{r})$
 $dR = (cdt, d\mathbf{r})$
4-Position
 $R = (ct, \mathbf{r})$

Invariant Interval
 $R \cdot R = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (c\tau)^2$
 $\Delta R \cdot \Delta R = (c\Delta t)^2 - \Delta \mathbf{r} \cdot \Delta \mathbf{r} = (c\Delta \tau)^2$
 $dR \cdot dR = (cdt)^2 - d\mathbf{r} \cdot d\mathbf{r} = (cd\tau)^2$

$\partial \cdot R = 4$
 SpaceTime Dimension

$\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -\delta^k]$
 Minkowski Metric

$\partial_\nu [R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^{\mu\nu}$
 Lorentz Transform

4-Gradient
 $\partial = (\partial/c, -\nabla)$
 $= (\partial/c, -\partial_x, -\partial_y, -\partial_z)$
 $= (\partial/c \partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$

Invariant d'Alembertian Wave Equation
 $\partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla$

SpaceTime Dim
 $\text{Tr}[\eta^{\mu\nu}] = 4 = \Lambda_{\mu\nu} \Lambda^{\mu\nu}$

$\mathbf{U} \cdot \partial [..]$
 $\gamma d/dt [..]$
 $d/d\tau [..]$

ProperTime Derivative
 $\mathbf{U} \cdot \partial = \gamma(\mathbf{c}, \mathbf{u}) \cdot (\partial/c, -\nabla) = \gamma(\partial_t + \mathbf{u} \cdot \nabla)$
 $= \gamma(\partial_t + (dx/dt)\partial_x + (dy/dt)\partial_y + (dz/dt)\partial_z)$
 $= \gamma d/dt = d/d\tau$

ProperTime Differential
 $d\tau = (1/\gamma)dt$
 $= \text{Time Dilation}$

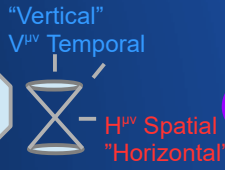
Relativity of Simultaneity
 $\mathbf{U} \cdot \Delta R = \gamma(\mathbf{c}, \mathbf{u}) \cdot (c\Delta t, \Delta \mathbf{r})$
 $= \gamma(c^2 \Delta t - \mathbf{u} \cdot \Delta \mathbf{r})$
 $= c^2 \Delta t_0 = c^2 \Delta \tau$

Continuity of 4-Velocity Flow
 $\partial \cdot \mathbf{U} = 0$

4-Velocity
 $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$
 $= dR/d\tau$

Invariant Magnitude LightSpeed
 $\mathbf{U} \cdot \mathbf{U} = c^2$

SRQM Diagram



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

The SR : Minkowski Metric $\eta^{\mu\nu}$ is the "Flat SpaceTime" low-curvature limiting-case of the more general GR Metric $g^{\mu\nu}$.

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

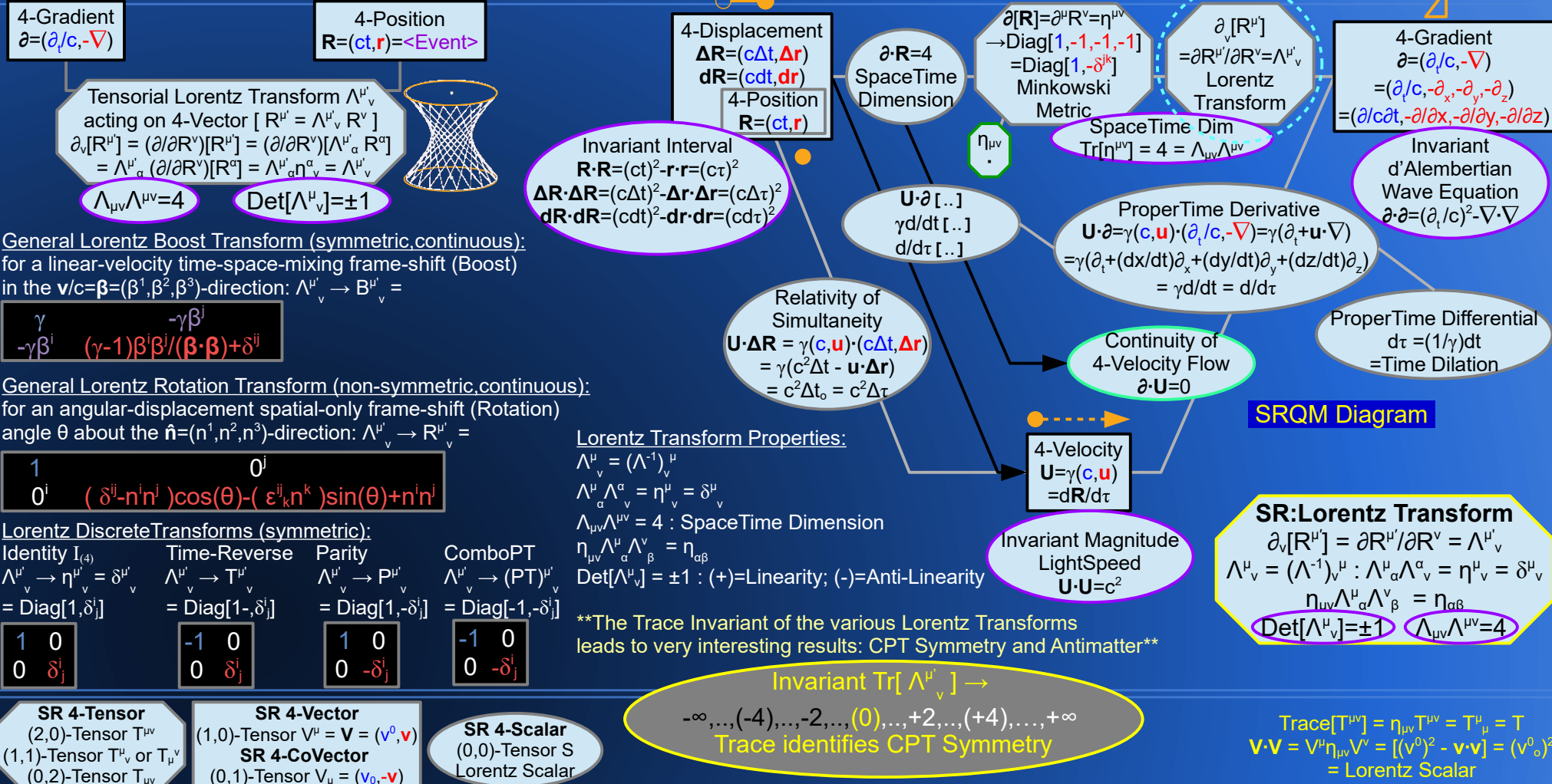
SRQM Diagram:

The Basis of Classical SR Physics The Lorentz Transform $\partial_\nu[R^\mu]=\Lambda^\mu_\nu$

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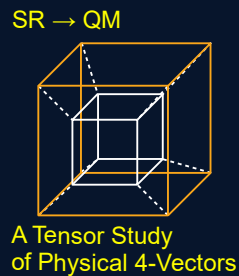


A Tensor Study of Physical 4-Vectors



SRQM Diagram:

The Basis of Classical SR Physics SpaceTime Dimension = 4D, again!



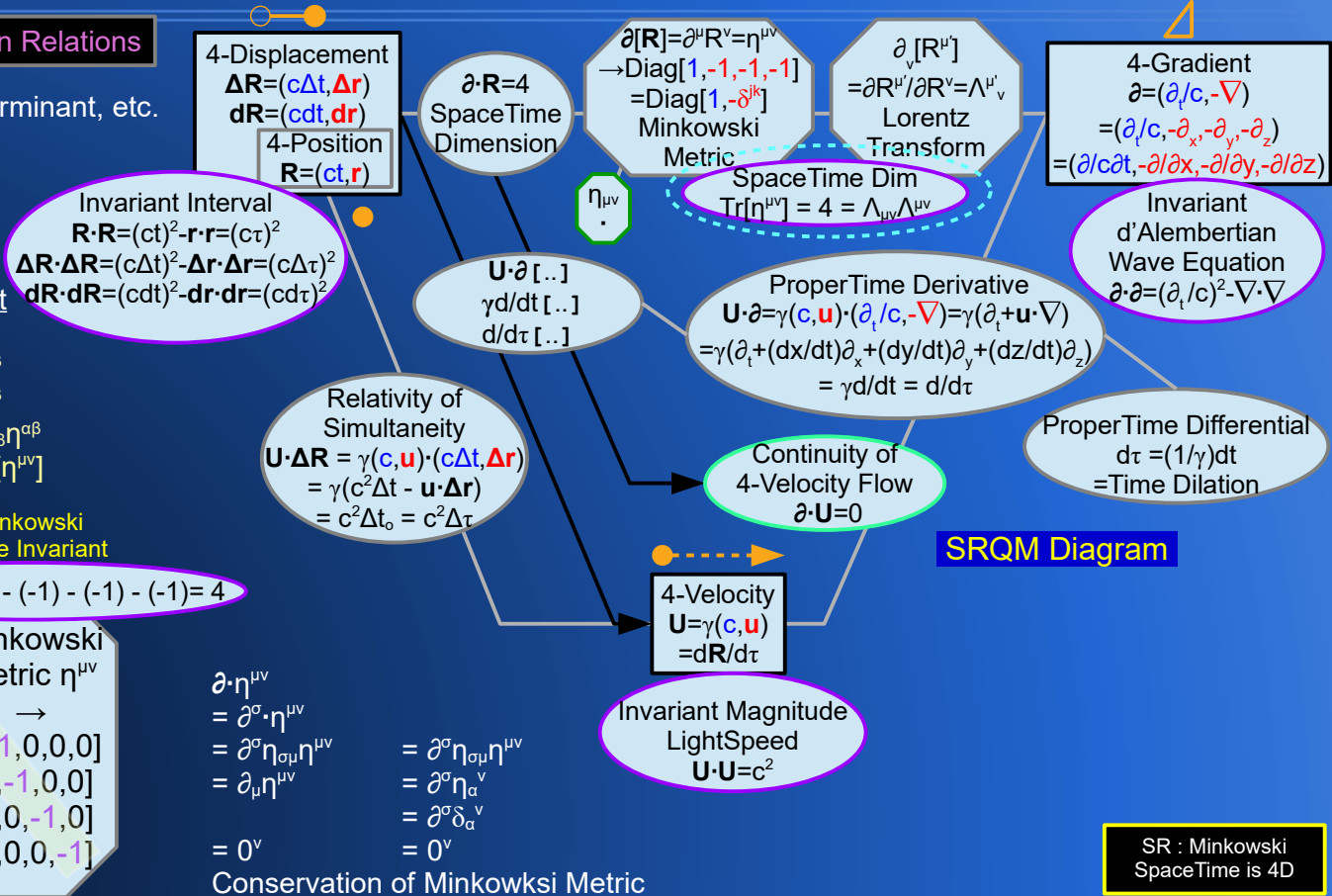
$\partial \cdot \mathbf{R} = \text{Tr}[\eta^{\mu\nu}] = \Lambda^{\mu\beta}\Lambda_{\mu\beta} = 4$: The SpaceTime Dimension Relations

Tensor Invariants include: Trace, InnerProduct, Determinant, etc.
The Trace of the Minkowski Metric & the InnerProduct of any of the Lorentz Transforms give the Dimension of SR SpaceTime = 4D.

Minkowski Metric Trace Invariant
 $\text{Trace}[\eta^{\mu\nu}] = \text{Tr}[\eta^{\mu\nu}] = \eta_{\mu\nu}^{\mu\nu} = \eta_{\mu}^{\mu} = \delta_{\mu}^{\mu} = (1+1+1+1) = 4$

4-Divergence of 4-Position
 $\partial \cdot \mathbf{R} = \partial^{\mu} \cdot \mathbf{R}^{\nu} = \partial^{\mu} \eta_{\mu\nu} \mathbf{R}^{\nu} = \eta_{\mu\nu} \partial^{\mu} \mathbf{R}^{\nu} = \eta_{\mu\nu} \eta^{\mu\nu} = \text{Tr}[\eta^{\mu\nu}] = 4$

Lorentz Transform Inner Prod Invariant
 $\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$
 $\eta^{\alpha\beta} \eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta^{\alpha\beta} \eta^{\alpha\beta}$
 $\eta^{\alpha\beta} \Lambda^{\mu}_{\alpha} (\eta_{\mu\nu} \Lambda^{\nu}_{\beta}) = \eta^{\alpha\beta} \eta^{\alpha\beta}$
 $\Lambda^{\mu\beta} \Lambda_{\mu\beta} = \eta_{\alpha\beta} \eta^{\alpha\beta} = \text{Tr}[\eta^{\mu\nu}] = 4$
 $\Lambda^{\mu\beta} \Lambda_{\mu\beta} = 4$



SRQM Diagram

General Tensor Trace Invariant
 $\text{Tr}[T^{\mu\nu}] = T_{\nu}^{\nu} = (T_0^0 + T_1^1 + T_2^2 + T_3^3) = (T^{00} - T^{11} - T^{22} - T^{33}) = T$

4-Tensor
 $T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$

- SR 4-Tensor**
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
(0,2)-Tensor $T_{\mu\nu}$
- SR 4-Vector**
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
- SR 4-CoVector**
(0,1)-Tensor $V_{\mu} = (\mathbf{v}_0, -\mathbf{v})$
- SR 4-Scalar**
(0,0)-Tensor S
Lorentz Scalar

SR : Minkowski SpaceTime is 4D

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

The Basis of Classical SR Physics Lorentz Scalar (Dot) Product ($\eta_{\mu\nu} = \cdot$)

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SR \rightarrow QM

$\eta_{\mu\nu}$
 \cdot

A Tensor Study of Physical 4-Vectors

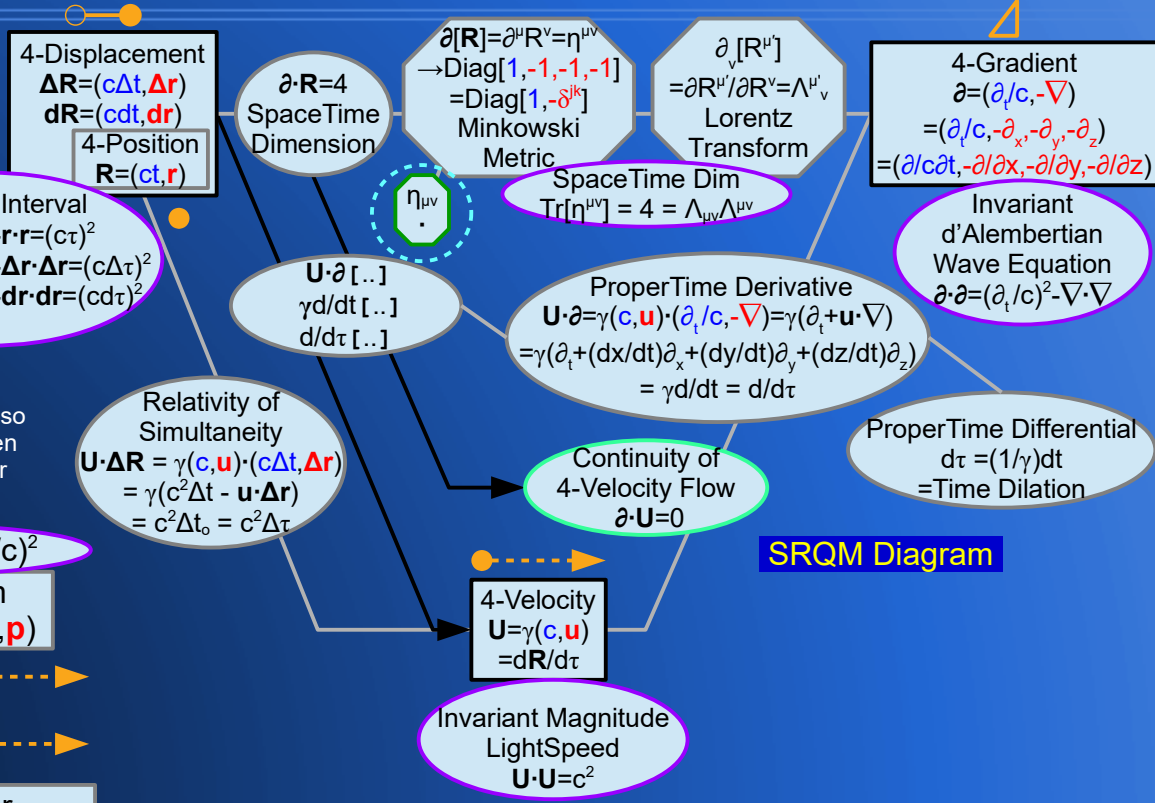
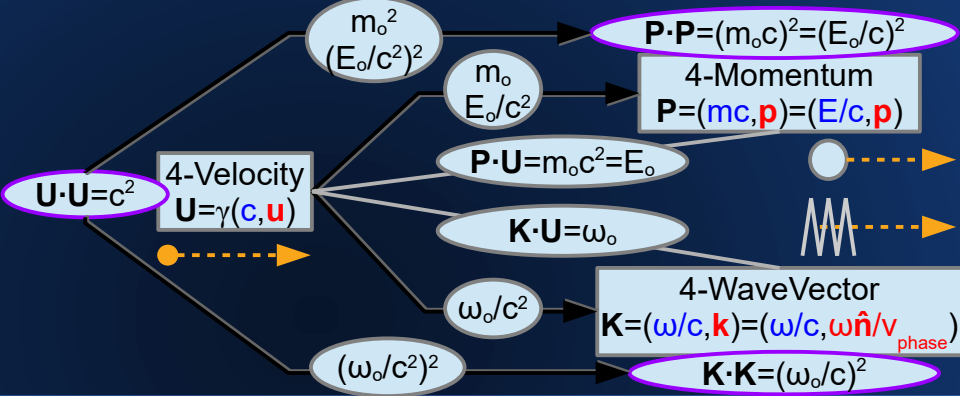
The Lorentz Invariant Lorentz Scalar Product is the SR 4D (Dot) Product. It is used to make Invariant Lorentz Scalars from two 4-Vectors.
 $\mathbf{A} \cdot \mathbf{B} = A^\mu \cdot B^\nu = A^\mu \eta_{\mu\nu} B^\nu = A_\nu B^\nu = A^\mu B_\mu = (a^0 b^0 - \mathbf{a} \cdot \mathbf{b}) = (a^0_\circ b^0_\circ)$
 $\mathbf{A} \cdot \mathbf{A} = A^\mu \cdot A^\nu = A^\mu \eta_{\mu\nu} A^\nu = A_\nu A^\nu = A^\mu A_\mu = (a^0 a^0 - \mathbf{a} \cdot \mathbf{a}) = (a^0_\circ)^2$

$\eta_{\mu\nu}$
 \cdot

$\rightarrow \text{Diag}[+1, -1, -1, -1]_{\text{Cartesian}}$
 with $\hat{\mathbf{e}}_\mu$ and $\hat{\mathbf{e}}_\nu$ as basis vectors
 $\mathbf{A} = A^\mu \hat{\mathbf{e}}_\mu \rightarrow A^\mu_{\text{Cartesian}}$

($\eta_{\mu\nu}$) is itself just the lowered-index form of the SR Minkowski Metric ($\eta^{\mu\nu}$), with individual components [$\eta_{\mu\mu}$] = 1/[$\eta^{\mu\mu}$], else 0. In Cartesian basis, this gives { $\eta_{\mu\nu} = \eta^{\mu\nu}$ }.

It is used in just about every relation between any two interesting 4-Vectors. It also gives the Invariant Magnitude of a single 4-Vector. If the 4-Vector is temporal, then the spatial component can be set to zero, giving the rest-frame invariant value, or the (o)bserver rest value ("naught" = \circ).



a^0 or $a_{0\circ}$: (0)th = temporal component (can relativistically vary)
 a_{\circ} : (o)bserver's rest-frame Invariant value (does not vary)

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_\circ)^2 = \text{Lorentz Scalar}$

SRQM Diagram: The Basis of Classical SR Physics 4-Velocity U , <Event> Motion

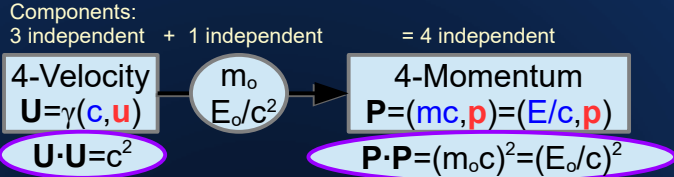
SR → QM
A Tensor Study of Physical 4-Vectors

4-Velocity $U = \gamma(c, \mathbf{u}) = (\gamma c, \gamma \mathbf{u}) = (U \cdot \partial) R = (d/d\tau) R = dR/d\tau = (dt/d\tau)(dR/dt) = \gamma(dR/dt) = \gamma(ct, \dot{\mathbf{r}}) = \gamma(c, \mathbf{u})$

4-Velocity U is the ProperTime Derivative $d/d\tau$ of the 4-Position R or of the 4-Displacement ΔR .

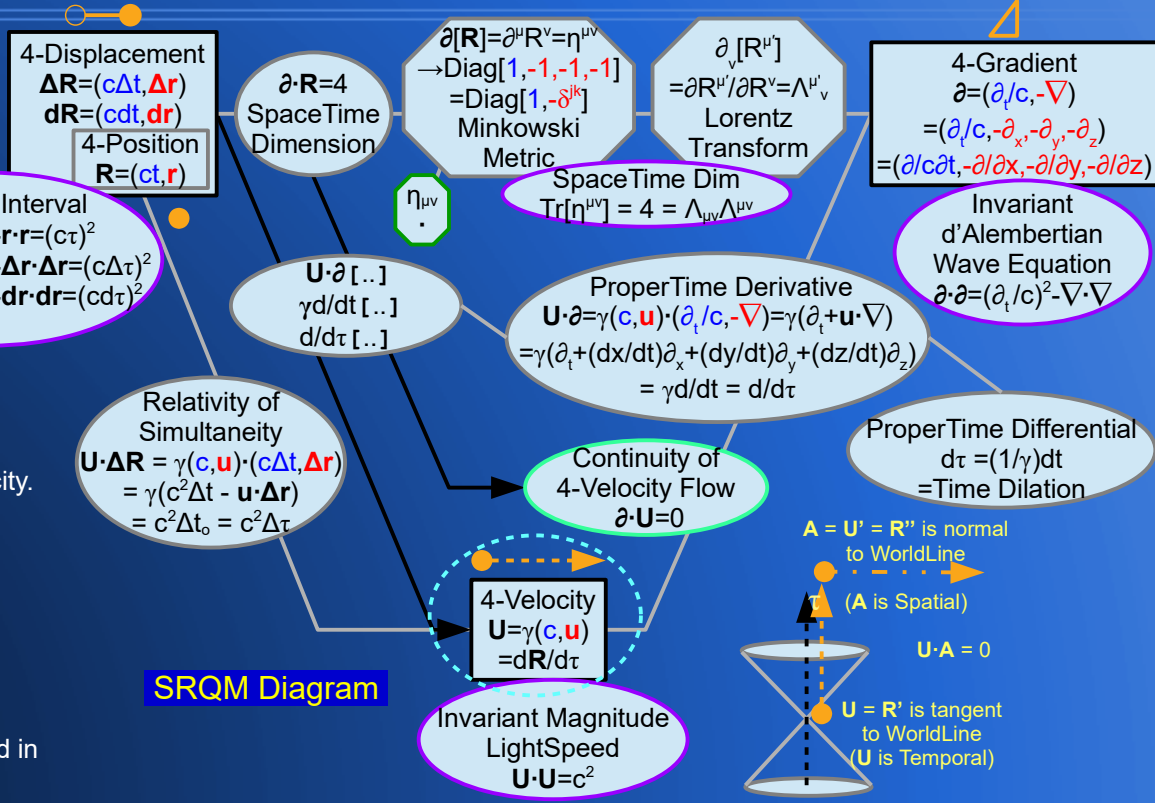
It is the SR 4-Vector that describes the motion of <Event>s through SpaceTime. For an un-accelerated observer, the 4-Velocity is along the WorldLine at all points. For an accelerated observer, the 4-Velocity is still tangent to the WorldLine at each point, but changes direction as the WorldLine bends.

The 4-Velocity is unlike most of the other SR 4-Vectors in that it only has 3 independent components, whereas the others usually have 4. This is due to the constraint placed by the Tensor Invariant of the 4-Velocity. $U \cdot U$ has a constant magnitude, the speed-of-light (c) in SpaceTime.

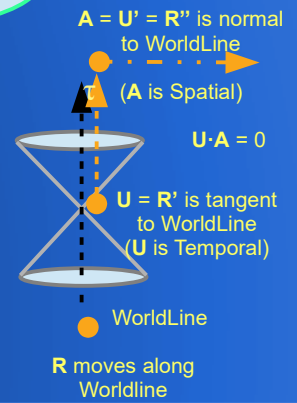


The 4-Velocity also usually has the Relativistic Gamma factor (γ) exposed in component form, whereas most of the other temporal 4-Vectors have it absorbed into the Lorentz 4-Scalar factor that goes into their components.

4-Velocity $U = \gamma(c, \mathbf{u}) = (\gamma c, \gamma \mathbf{u})$
4-Momentum $P = (mc, \mathbf{p}) = m_0 U = \gamma m_0(c, \mathbf{u}) = m(c, \mathbf{u}) = (mc, m\mathbf{u}) = (E/c, \mathbf{p})$



$P = m_0 U$
The temporal components give Einstein's famous $E = mc^2 = \gamma m_0 c^2 = \gamma E_0$
The spatial components give $\mathbf{p} = m\mathbf{u} = \gamma m_0 \mathbf{u}$



SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
--------------------------------------------------------------------------------------------------------------------------	--------------------------------------------------------------------------------------------------------------------------------------------------	--------------------------------------------------------

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

The Basis of Classical SR Physics 4-Velocity Magnitude = Invariant Speed-of-Light (c)

A Tensor Study of Physical 4-Vectors

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$$\text{4-Velocity } \mathbf{U} = \gamma(\mathbf{c}, \mathbf{u}) = (\gamma c, \gamma \mathbf{u}) = (\mathbf{U} \cdot \partial) \mathbf{R} = (d/d\tau) \mathbf{R} = d\mathbf{R}/d\tau = (dt/d\tau)(d\mathbf{R}/dt) = \gamma(d\mathbf{R}/dt) = \gamma(\mathbf{c}, \dot{\mathbf{r}}) = \gamma(\mathbf{c}, \mathbf{u})$$

with Relativistic Gamma $\gamma = 1/\sqrt{1 - \beta \cdot \beta}$, $\beta = \mathbf{u}/c$

The Lorentz Scalar Product of the 4-Velocity gives the Invariant Magnitude Speed-of-Light (c), one the main fundamental SR physical constants of physics. Technically, it is the maximum speed of SR causality, which any massless particles, ex. the photon, travel at.

U · U

$$= \gamma(\mathbf{c}, \mathbf{u}) \cdot \gamma(\mathbf{c}, \mathbf{u})$$
$$= \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u})$$
$$= [1/(1 - \beta \cdot \beta)](c^2 - \mathbf{u} \cdot \mathbf{u}) = [1/(1 - \beta \cdot \beta)]c^2(1 - \beta \cdot \beta)$$
$$= c^2: \text{Invariant Magnitude Speed-of-Light (c)}$$

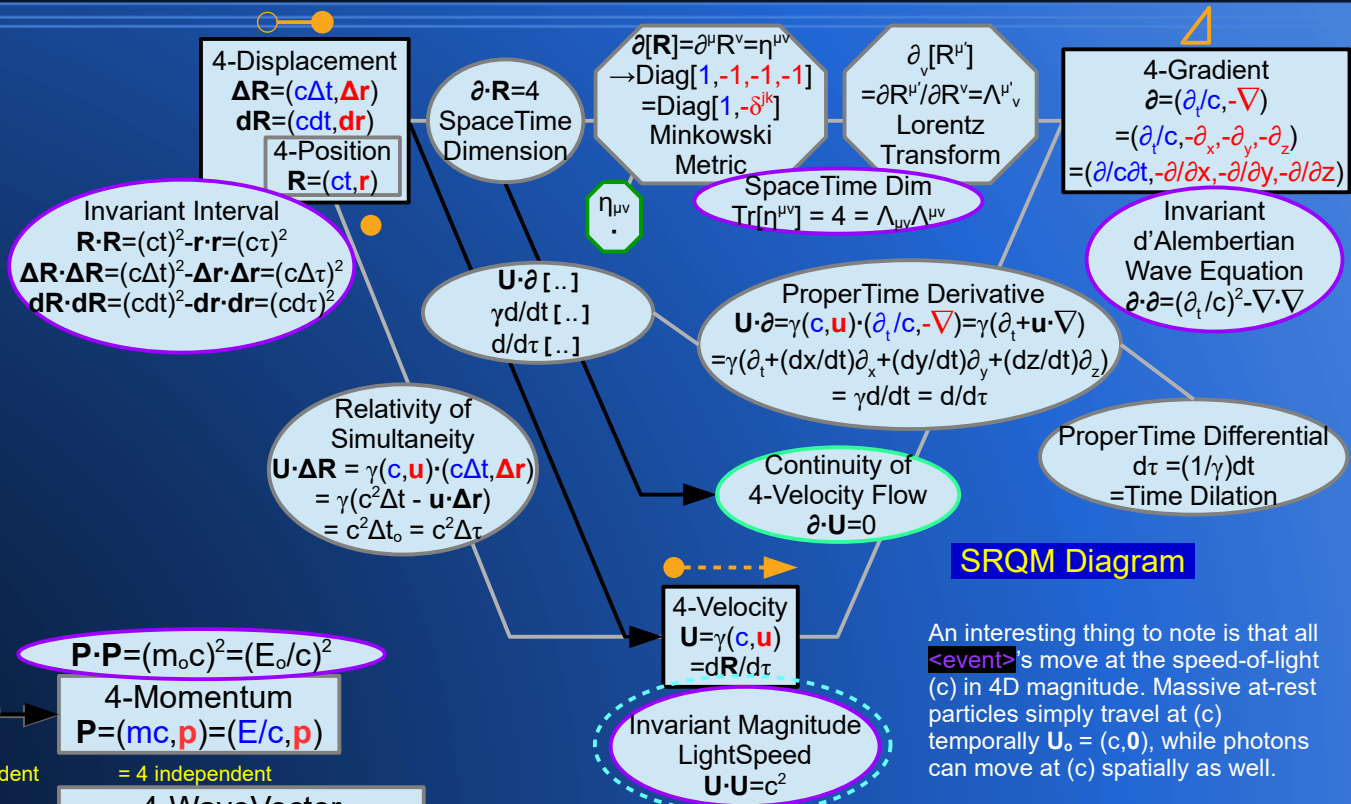
This fundamental constant Invariant (c) provides an extra constraint on the components of 4-Velocity **U**, making it have only 3 independent components (**u**).

This allows one to make new 4-Vectors related to 4-Velocity by multiplying by other Lorentz Scalars.

(Lorentz Scalar) * (4-Velocity) = (New 4-Vector)
Components: 3 independent

$$\mathbf{P} = (mc, \mathbf{p}) = m_0 \mathbf{U}$$
$$\mathbf{K} = (\omega/c, \mathbf{k}) = (\omega_0/c^2) \mathbf{U}$$

The newly made 4-Vector thus has {3+1 = 4} independent components.



SRQM Diagram

An interesting thing to note is that all **<event>**s move at the speed-of-light (c) in 4D magnitude. Massive at-rest particles simply travel at (c) temporally $\mathbf{U}_0 = (c, 0)$, while photons can move at (c) spatially as well.

If (c) was not a constant, but varied somehow, then all 4-Vectors made from the 4-Velocity would have more than 4 independent components, which is not observed. It seems a compelling argument against variable light-speed theories.

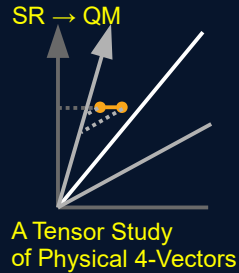
SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^{μ}_{ν} or T_{ν}^{μ} (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$
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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$$
$$\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$

SRQM Diagram:

The Basis of Classical SR Physics Relativity of Simultaneity



Relativity of Simultaneity:

$$\mathbf{U} \cdot \Delta \mathbf{x} = \gamma(c, \mathbf{u}) \cdot (c\Delta t, \Delta \mathbf{x}) = \gamma(c^2\Delta t - \mathbf{u} \cdot \Delta \mathbf{x}) = c^2\Delta t_0 = c^2\Delta \tau$$

If Lorentz Scalar ($\mathbf{U} \cdot \Delta \mathbf{x} = 0 = c^2\Delta \tau$), then the ProperTime displacement ($\Delta \tau$) is zero, and the <Event>'s separation ($\Delta \mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$) is orthogonal to the worldline at \mathbf{U} .

<Event>'s \mathbf{x}_1 and \mathbf{x}_2 are therefore simultaneous ($\Delta \tau = 0$) for the observer on this worldline at \mathbf{U} .

Examining the equation we get $\gamma(c^2\Delta t - \mathbf{u} \cdot \Delta \mathbf{x}) = 0$. The coordinate time difference between the events is ($\Delta t = \mathbf{u} \cdot \Delta \mathbf{x} / c^2$). The condition for simultaneity in an alternate frame (moving at 3-velocity \mathbf{u} wrt. the worldline \mathbf{U}) is $\Delta t = 0$, which implies $(\mathbf{u} \cdot \Delta \mathbf{x}) = 0$.

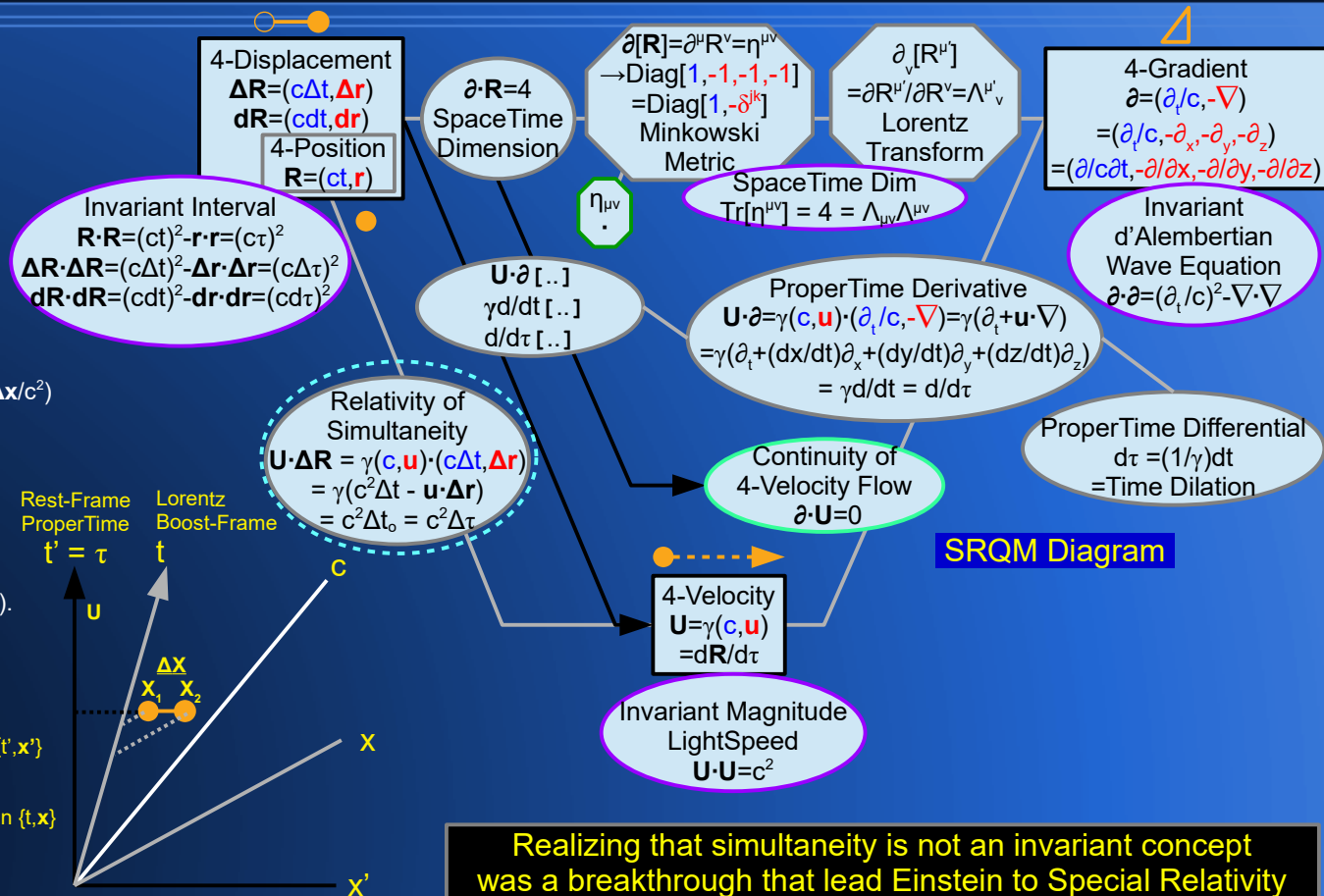
This condition can be met by:
 ($|\mathbf{u}| = 0$), the alternate observer is not moving wrt. the events, i.e. is on worldline \mathbf{U} or on a worldline parallel to \mathbf{U} .
 ($|\Delta \mathbf{x}| = 0$), the events are at the same spatial location (co-local).
 ($\mathbf{u} \cdot \Delta \mathbf{x} = 0 = |\mathbf{u}| |\Delta \mathbf{x}| \cos[\theta]$), the alternate observer's motion is perpendicular (orthogonal $\theta=90^\circ$) to the spatial separation $\Delta \mathbf{x}$ of the events in that frame.

If none of these conditions is met, then the events will not be simultaneous in the alternate reference-frame.

This can be shown on a Minkowski Diagram.

$\Delta \tau = 0$
Simultaneous in $\{t', \mathbf{x}'\}$

 $\Delta t \neq 0$
Not Simultaneous in $\{t, \mathbf{x}\}$



Realizing that simultaneity is not an invariant concept was a breakthrough that lead Einstein to Special Relativity

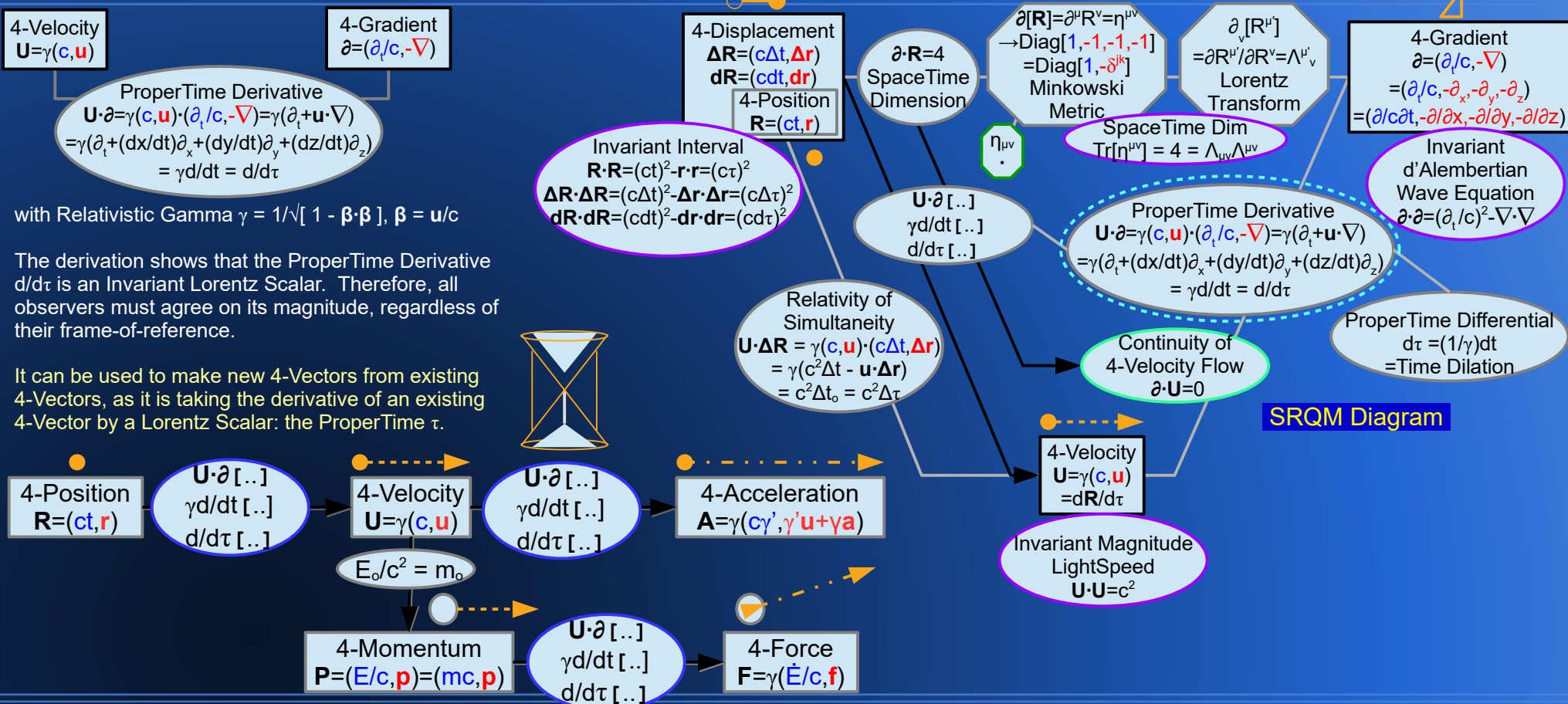
$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$

SRQM Diagram:

The Basis of Classical SR Physics The ProperTime Derivative (d/dτ)

SR → QM
A Tensor Study of Physical 4-Vectors



SRQM Diagram

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (\mathbf{v}_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

The Basis of Classical SR Physics

ProperTime Derivative on SR 4-Vectors and Scalars



SR → QM

A Tensor Study of Physical 4-Vectors

The ProperTime Derivative acting on SR 4-Vectors:

$$\mathbf{U} \cdot \partial = \gamma(\mathbf{c}, \mathbf{u}) \cdot (\partial_t/c, -\nabla) = \gamma(\partial_t + \mathbf{u} \cdot \nabla) = \gamma d/dt = d/d\tau$$

4-Vectors:

- 4-Position $\mathbf{R} = \langle \text{Event} \rangle$
- 4-Velocity $\mathbf{U} = d\mathbf{R}/d\tau$
- 4-Acceleration $\mathbf{A} = d\mathbf{U}/d\tau$
- ...
- 4-Momentum $\mathbf{P} = m_0 \mathbf{U}$
- 4-Force $\mathbf{F} = d\mathbf{P}/d\tau$

As one can see from the list, the ProperTime Derivative gives the 4-Vectors that are the change in status of the 4-Vector that ProperTime Derivative acts on. It can also act on Scalar Values to give deep SR results.

$\partial \cdot \mathbf{R} = 4$: SpaceTime Dimension is 4

$$d/d\tau(\partial \cdot \mathbf{R}) = d/d\tau(4) = 0$$

$$d/d\tau(\partial \cdot \mathbf{R}) = d/d\tau[\partial] \cdot \mathbf{R} + \partial \cdot \mathbf{U} = 0$$

$\partial \cdot \mathbf{U} = 0$: Conservation of the SR 4-Velocity Flow

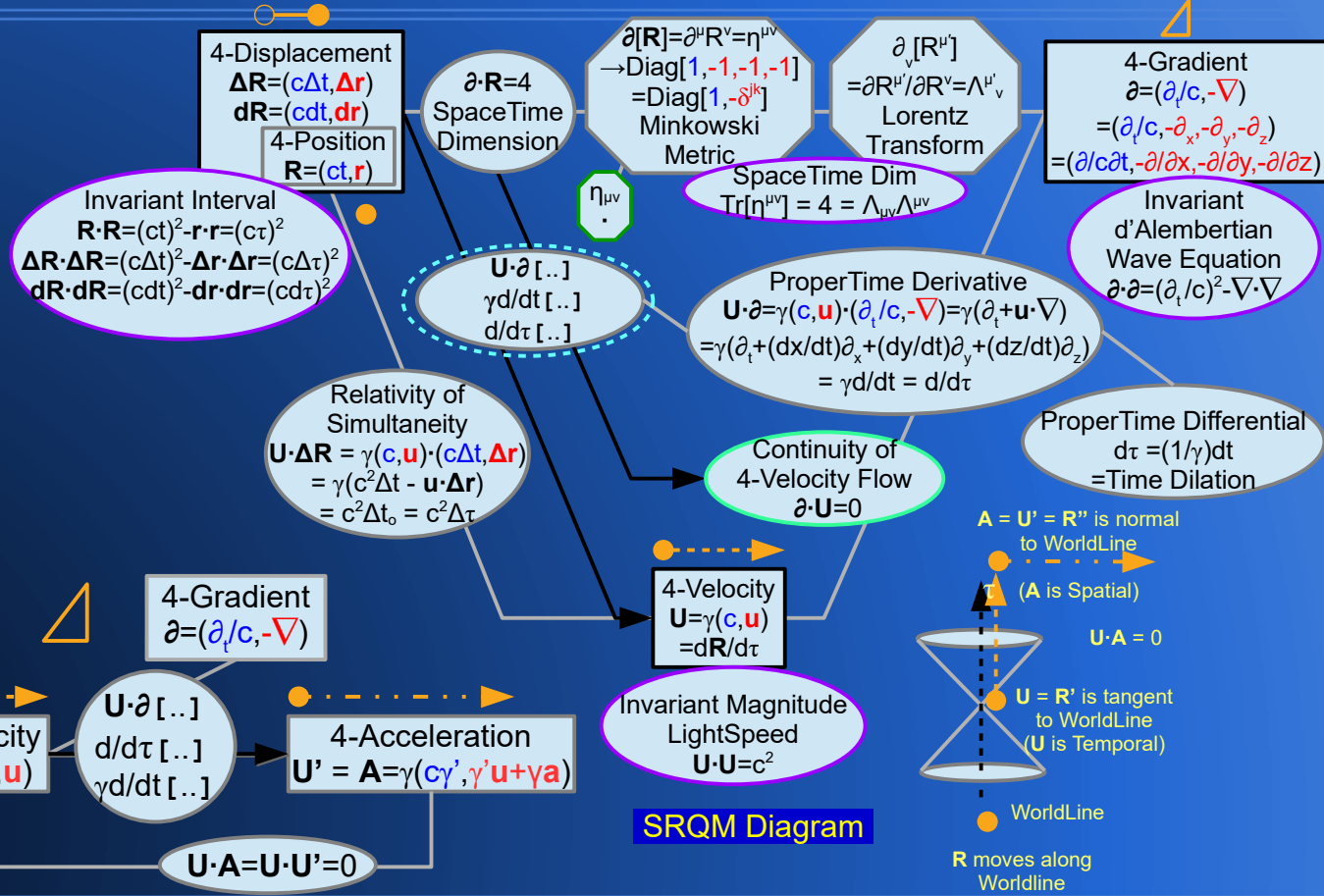
$$\mathbf{U} \cdot \mathbf{U} = c^2$$

$$d/d\tau[\mathbf{U} \cdot \mathbf{U}] = d/d\tau[c^2] = 0$$

$$d/d\tau[\mathbf{U} \cdot \mathbf{U}] = d/d\tau[\mathbf{U}] \cdot \mathbf{U} + \mathbf{U} \cdot d/d\tau[\mathbf{U}] = 2(\mathbf{U} \cdot \mathbf{A}) = 0$$

$$\mathbf{U} \cdot \mathbf{A} = \mathbf{U} \cdot \mathbf{U}' = 0$$

The 4-Velocity is SpaceTime orthogonal to it's own 4-Acceleration

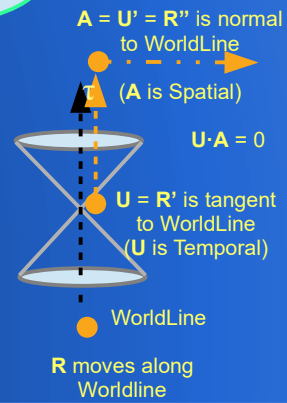


SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

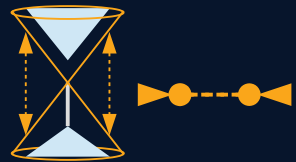
SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

SRQM Diagram



$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$



The Basis of Classical SR Physics

ProperTime Differential (dτ) → Time Dilation & Length Contraction

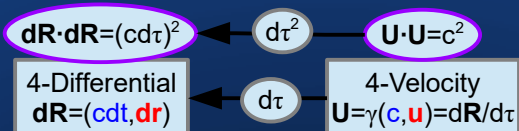
A Tensor Study of Physical 4-Vectors

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John B. Wilson

There are several ways to derive Time Dilation.

ProperTime Derivative (Lorentz 4-Scalar):
 $U \cdot \partial = \gamma(\mathbf{c}, \mathbf{u}) \cdot (\partial_t/c, -\nabla) = \gamma(\partial_t + \mathbf{u} \cdot \nabla) = \gamma d/dt = d/d\tau$

ProperTime Differential (Lorentz 4-Scalar): $d\tau = (1/\gamma)dt$



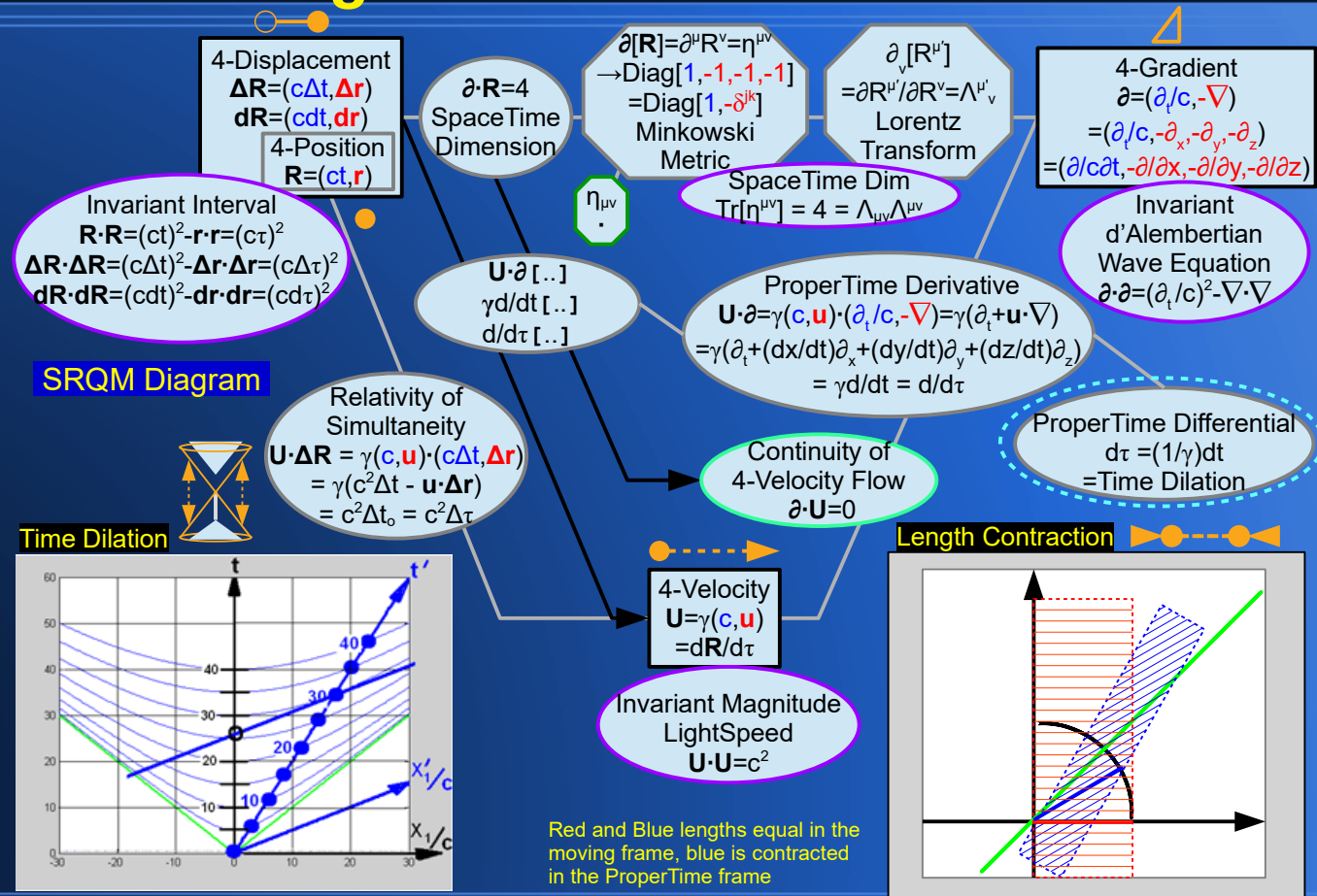
Take the temporal component of the 4-Vector relation.

$dt = \gamma d\tau = \gamma dt_0$
 $\Delta t = \gamma \Delta \tau = \gamma \Delta t_0$: Time Dilation!

The coordinate time Δt measured by an observer is "dilated", compared to the ProperTime as measured by a clock moving with the object. This has the effect that moving objects appear to age more slowly than at-rest objects. The effect is reciprocal as well. Since velocity is relative, each observer will see the other as ageing more slowly, similarly to the effect that each will appear smaller to the other when seen at a distance.

Now multiply both sides by the moving-frame speed $[v]$.

$v \Delta t = \gamma v \Delta \tau$
 $v \Delta t = \text{distance } L_0 \text{ the moving clock travels wrt. frame, which is a proper (fixed-to-frame) displacement length.}$
 $L_0 = \gamma L$
 $L = (1/\gamma)L_0$: Length Contraction!



SR 4-Tensor
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 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Invariant ProperTime (clock at rest), Invariant ProperLength (ruler at rest)
 Time Dilation (clock moving), Length Contraction (ruler moving)

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

The Basis of Classical SR Physics 4-Gradient ∂ , SR 4-Vector Function: Operator

A Tensor Study of Physical 4-Vectors

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4-Gradient
 $\partial = \partial^\mu = (\partial_t/c, -\nabla)$
 $= (\partial_t/c, -\partial_x, -\partial_y, -\partial_z)$
 $= (\partial/c\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$



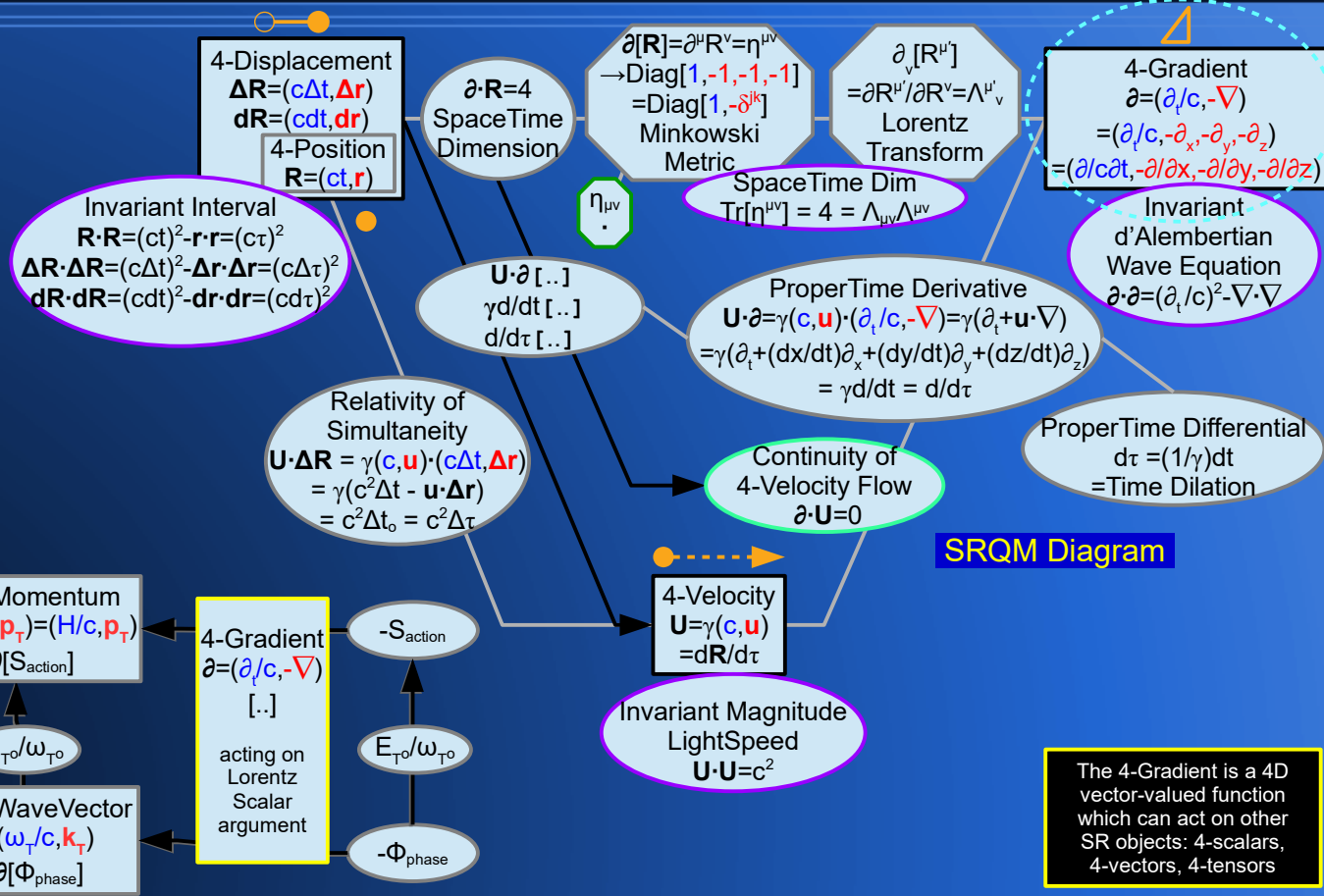
The 4-Gradient ($\partial^\mu = (\partial_t/c, -\nabla)$) is the index-raised version of the SR Gradient One-Form ($\partial_\mu = (\partial_t/c, \nabla)$). It is the 4D version of the partial derivative function of calculus.

It is a 4-Vector function that can act on other 4-Vectors and 4-Scalars. The 4-Gradient tells how things change wrt. time and space.

It is instrumental in creating the ProperTime Derivative $\mathbf{U} \cdot \partial = \gamma d/dt = d/d\tau$.

The 4-Gradient plays a major role in advanced physics, showing how SR waves are formed, creating the Hamilton-Jacobi equations, the Euler-Lagrange equations, Conservation equations, Maxwell's Equations, the Lorenz Gauge, etc. It gives the Dimension of SpaceTime, the Minkowski Metric, and the Lorenz Transformations. In QM, it provides the Schrödinger relations.

It is fundamental in connecting SR to QM.



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
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SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (\mathbf{v}_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Hamilton-Jacobi Equation: $\mathbf{P}_T = -\partial[S_{action}]$
 SR Plane-Wave Equation: $\mathbf{K}_T = -\partial[\Phi_{phase}]$

Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

The Basis of Classical SR Physics Invariant d'Alembertian Wave Equation ($\partial \cdot \partial$)

SR → QM
A Tensor Study of Physical 4-Vectors

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The Lorentz Scalar Invariant of the 4-Gradient gives the Invariant d'Alembertian Wave Equation, describing SR wave motion. It is seen in the SR Maxwell Equation for EM light waves.

$\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla$
d'Alembertian

$(\partial \cdot \partial) \mathbf{A} - \partial(\partial \cdot \mathbf{A}) = \mu_0 \mathbf{J}$
Maxwell EM Wave Eqn

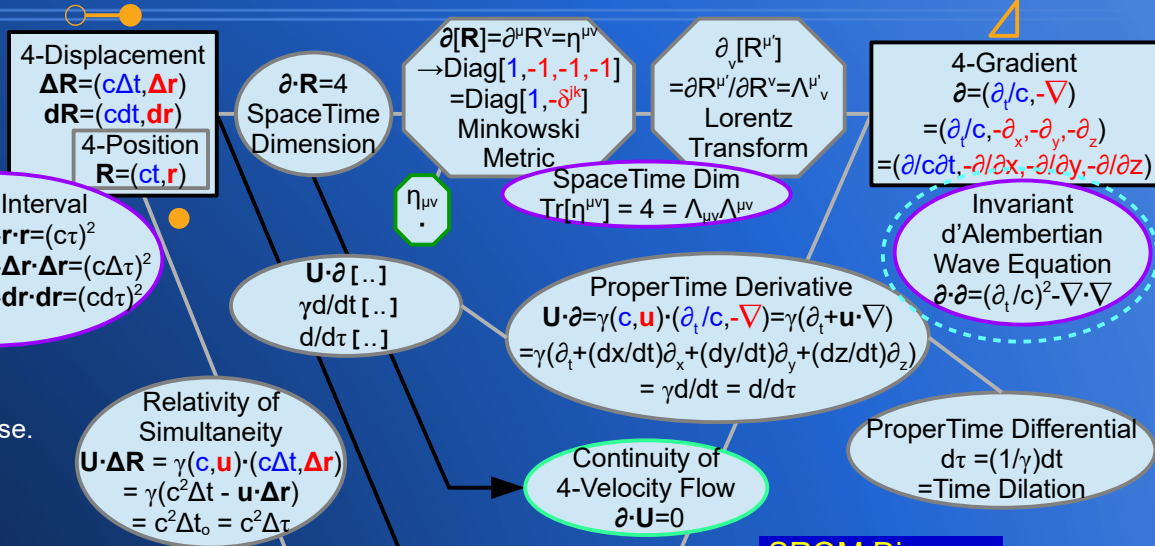
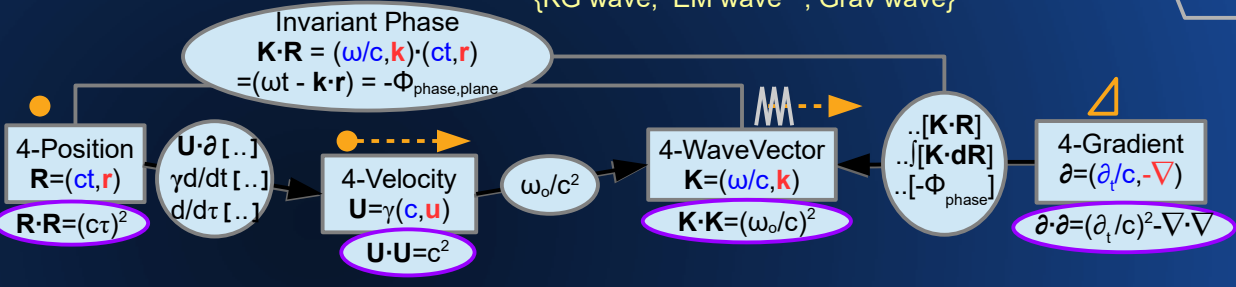
Lorentz Gauge Conservation of EM Potential: $\partial \cdot \mathbf{A} = 0$

Importantly, the d'Alembertian is fully from basic SR rules, with no quantum axioms required. However, it will be seen again in the Klein-Gordon RQM wave equation.

It provides for the introduction of an SR Wave 4-Vector \mathbf{K} , which can also be given by the negative Gradient of a Lorentz Scalar Phase.

4-WaveVector $\mathbf{K} = (\omega/c, \mathbf{k}) = \mathbf{U} = (\omega/c, \mathbf{k}) = -\partial[\Phi_{\text{phase}}] = \partial[\mathbf{K} \cdot \mathbf{R}]$

The usual mathematical (complex) plane-wave solutions apply in SR: $f = (a)^* e^{i[\pm i(\mathbf{K} \cdot \mathbf{R})]}$, with (a) mplitude possibly {4-Scalar S, 4-Vector V^μ , 4-Tensor $T^{\mu\nu}$ } {KG wave, EM wave, Grav wave}



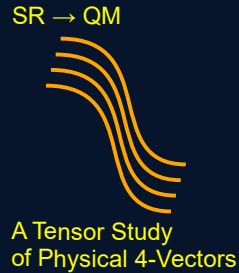
SR is the "natural" 4D arena for the description of waves, using the d'Alembertian $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla$

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (\mathbf{v}_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

The Basis of Classical SR Physics Continuity of 4-Velocity Flow ($\partial \cdot \mathbf{U} = 0$)



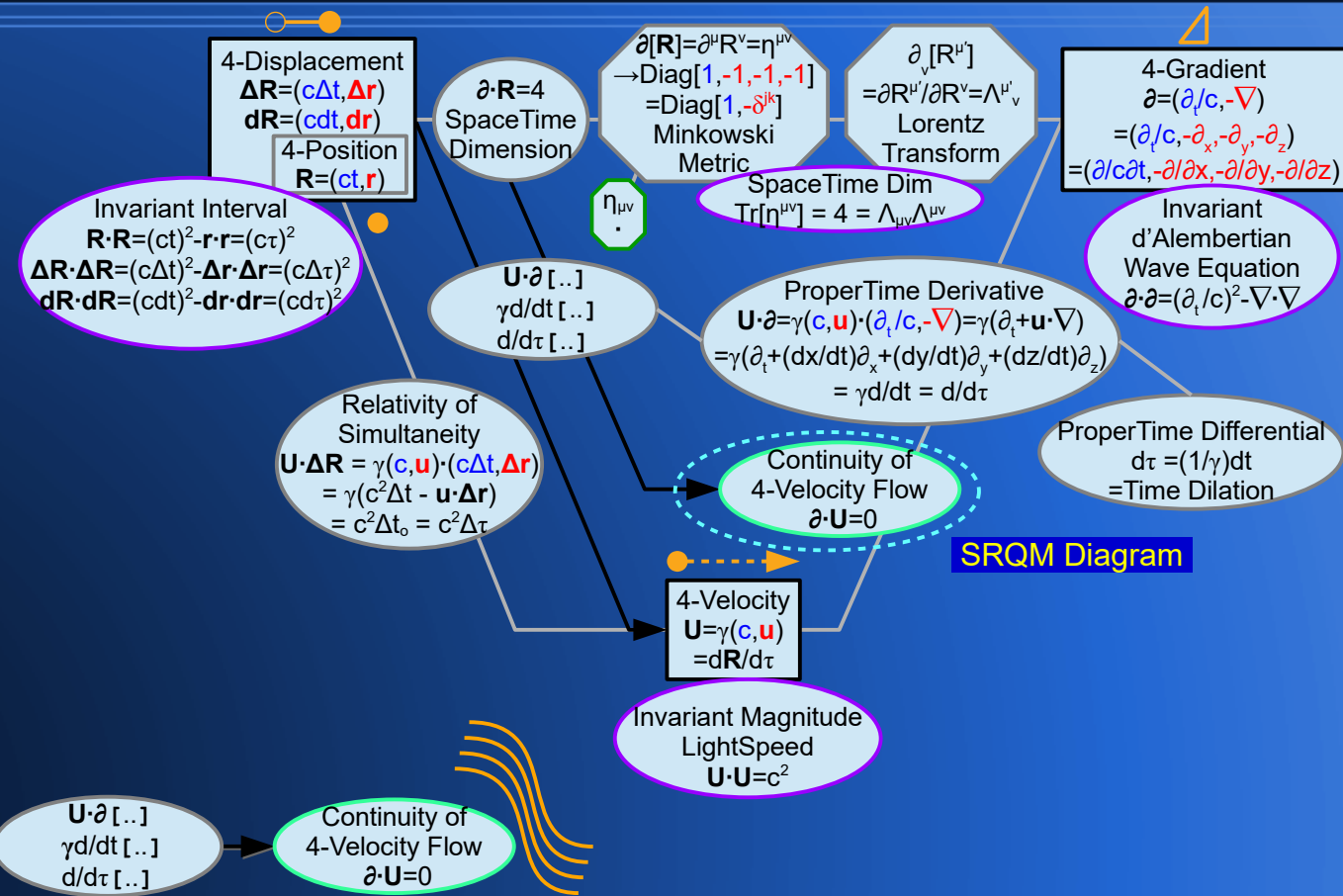
Continuity of 4-Velocity Flow $\partial \cdot \mathbf{U} = 0$
This leads to all the SR Conservation Laws.

- $\partial \cdot \mathbf{R} = 4$
- $d/d\tau(\partial \cdot \mathbf{R}) = d/d\tau(4) = 0$
- $d/d\tau(\partial \cdot \mathbf{R}) = d/d\tau(\partial) \cdot \mathbf{R} + \partial \cdot d/d\tau(\mathbf{R}) = 0$
- $d/d\tau(\partial \cdot \mathbf{R}) = d/d\tau[\partial] \cdot \mathbf{R} + \partial \cdot \mathbf{U} = 0$
- $\partial \cdot \mathbf{U} = -d/d\tau[\partial] \cdot \mathbf{R}$
- $\partial \cdot \mathbf{U} = -(\mathbf{U} \cdot \partial)[\partial] \cdot \mathbf{R}$
- $\partial \cdot \mathbf{U} = -(\mathbf{U}_\nu \partial^\nu)[\partial_\mu] \mathbf{R}^\mu$
- $\partial \cdot \mathbf{U} = -\mathbf{U}_\nu \partial^\nu \partial_\mu \mathbf{R}^\mu$
- $\partial \cdot \mathbf{U} = -\mathbf{U}_\nu \partial_\mu \partial^\nu \mathbf{R}^\mu$: I believe this is legit, partials commute
- $\partial \cdot \mathbf{U} = -\mathbf{U}_\nu \partial_\mu \eta^{\nu\mu}$
- $\partial \cdot \mathbf{U} = -\mathbf{U}_\nu (0^\nu)$
- $\partial \cdot \mathbf{U} = 0$
- Conservation of the 4-Velocity Flow (4-Velocity Flow-Field)

All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change of a quantity is balanced by the flow of that quantity into or out-of a local region.

Conservation of Charge:
 $\rho_0 \partial \cdot \mathbf{U} = \partial \cdot \rho_0 \mathbf{U} = \partial \cdot \mathbf{J} = (\partial_t \rho + \nabla \cdot \mathbf{j}) = 0$



SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
--------------------------------------------------------------------------------------------------------------------------	--------------------------------------------------------------------------------------------------------------------------------------------------	----------------------------------------------------------

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SRQM Diagram:

The Basis of Classical SR Physics <Event> Substantiation

SR → QM



A Tensor Study of Physical 4-Vectors

Now focus on a few more of the main SR 4-Vectors.

- 4-Position R^μ**
 $R=(ct, \mathbf{r}) = \langle \text{Event} \rangle$
● <Event> Location
- 4-Velocity U^μ**
 $U=\gamma(\mathbf{c}, \mathbf{u})$
● <Event> Motion
- 4-Gradient ∂^μ**
 $\partial=(\partial/c, -\nabla)$
△ <Event> Alteration
- 4-Momentum P^μ**
 $P=(E/c, \mathbf{p})=(m\mathbf{c}, \mathbf{p})=(m\mathbf{c}, m\mathbf{u})$
 $= (E_0/c^2)\mathbf{U}=m_0\mathbf{U}$
● <Event> Substantiation (particle:mass)
- 4-WaveVector K^μ**
 $K=(\omega/c, \mathbf{k})=(\omega/c, \omega\hat{\mathbf{n}}/v_{\text{phase}})$
 $= (1/cT, \hat{\mathbf{n}}/\lambda)=(\omega_0/c^2)\mathbf{U}$
〰 <Event> Substantiation (wave)
- 4-CurrentDensity:ChargeFlux J^μ**
 $J=(pc, \mathbf{j})=(pc, \mathbf{p}\mathbf{u})$
 $= (\rho_0)\mathbf{U}=(qn_0)\mathbf{U}=(q)\mathbf{N}$
☀ <Event> Substantiation (charge Q)
- 4-(Dust)NumberFlux N^μ**
 $N=(nc, \mathbf{n})=(nc, \mathbf{n}\mathbf{u})$
 $= (n_0)\mathbf{U}$
☁ <Event> Substantiation (dust:number N)

4-Displacement
 $\Delta R=(c\Delta t, \Delta \mathbf{r})$
 $dR=(cdt, d\mathbf{r})$
 4-Position
 $R=(ct, \mathbf{r})$

SRQM Diagram

4-Gradient
 $\partial=(\partial/c, -\nabla)$
 $=(\partial/c, -\partial_x, -\partial_y, -\partial_z)$
 $=(\partial/c\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$

4-Velocity
 $U=\gamma(\mathbf{c}, \mathbf{u})$
 $=dR/d\tau$

These 4-Vectors give more of the main classical results of Special Relativity, including SR concepts like:
 SR Particles and Waves, Matter-Wave Dispersion
 Einstein's $E = mc^2 = \gamma m_0 c^2 = \gamma E_0$, Rest Mass, Rest Energy
 Conservation of Charge (Q), Conservation of Particle Number (N), Continuity Equations

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
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 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

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SRQM Diagram:

The Basis of Classical SR Physics 4-Momentum, Einstein's $E = mc^2$

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A Tensor Study
of Physical 4-Vectors

4-Position $\mathbf{R}=(ct, \mathbf{r})$
4-Gradient $\partial=(\partial_t/c, -\nabla)$
4-Velocity $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$

4-Momentum $\mathbf{P} = (E/c, \mathbf{p}) = m_0\mathbf{U} = \gamma m_0(\mathbf{c}, \mathbf{u}) = m(\mathbf{c}, \mathbf{u})$

Temporal part:
{energy}

$$E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = m_0 c^2 + (\gamma - 1) m_0 c^2$$

(rest) + (kinetic)

Spatial part:
{3-momentum}

$$\mathbf{p} = \gamma m_0 \mathbf{u} = m \mathbf{u}$$

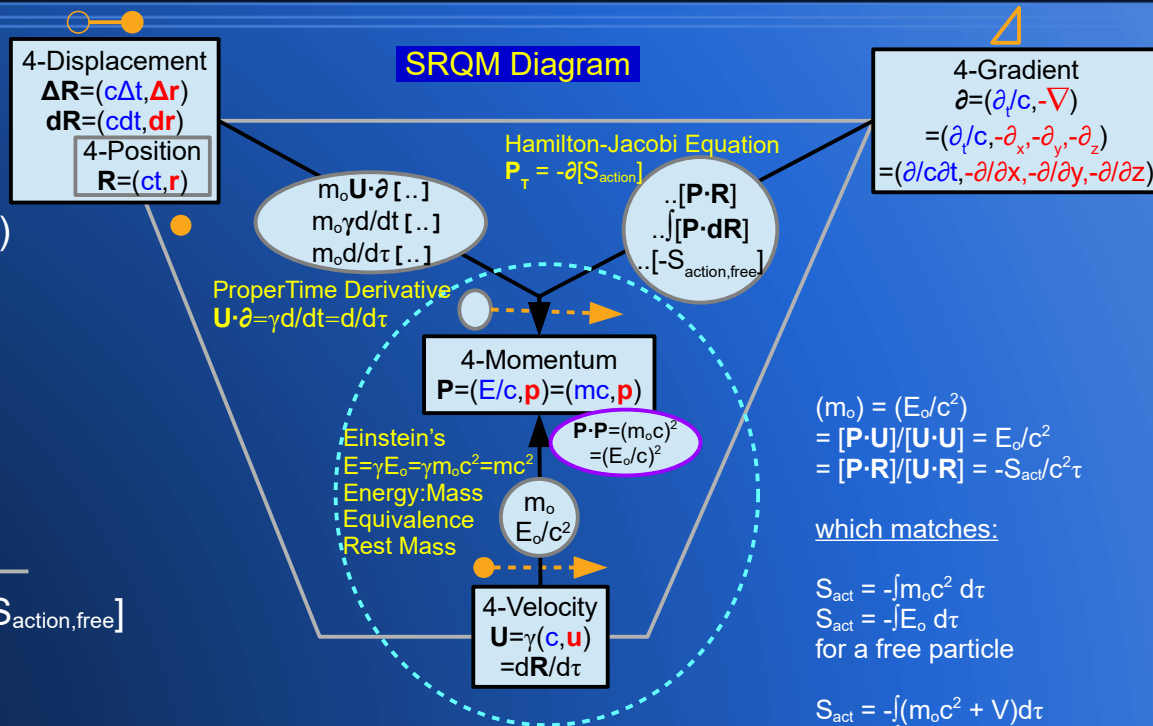
4-Momentum $\mathbf{P} = (E/c, \mathbf{p}) = -\partial[S_{\text{action, free}}] = -(\partial_t/c, -\nabla)[S_{\text{action, free}}]$

Temporal part:
{energy}

$$E = -\partial_t[S_{\text{action, free}}]$$

Spatial part:
{3-momentum}

$$\mathbf{p} = +\nabla[S_{\text{action, free}}]$$



$$(m_0) = (E_0/c^2)$$

$$= [\mathbf{P} \cdot \mathbf{U}] / [\mathbf{U} \cdot \mathbf{U}] = E_0/c^2$$

$$= [\mathbf{P} \cdot \mathbf{R}] / [\mathbf{U} \cdot \mathbf{R}] = -S_{\text{act}}/c^2 \tau$$

which matches:

$$S_{\text{act}} = -\int m_0 c^2 d\tau$$

$$S_{\text{act}} = -\int E_0 d\tau$$

for a free particle

$$S_{\text{act}} = -\int (m_0 c^2 + V) d\tau$$

$$S_{\text{act}} = -\int (E_0 + V) d\tau$$

in a potential

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SRQM Diagram:

The Basis of Classical SR Physics

4-WaveVector, $u * v_{\text{phase}} = c^2$

A Tensor Study of Physical 4-Vectors

4-Position $R=(ct, \mathbf{r})$
4-Gradient $\partial=(\partial_t/c, -\nabla)$
4-Velocity $U = \gamma(c, \mathbf{u})$

4-WaveVector $K = (\omega/c, \mathbf{k}) = (\omega_0/c^2)U = \gamma(\omega_0/c^2)(c, \mathbf{u})$

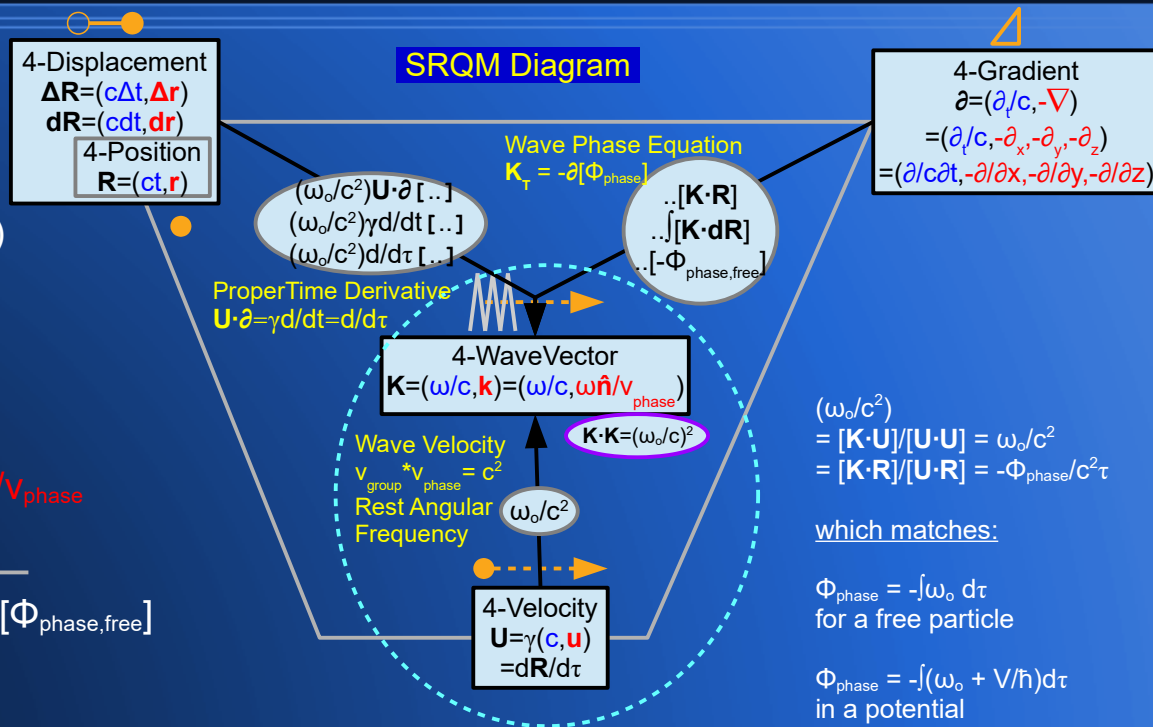
Temporal part: $\omega = \gamma\omega_0$
 {angular frequency}

Spatial part: $\mathbf{k} = \gamma(\omega_0/c^2)\mathbf{u} = (\omega/c^2)\mathbf{u} = \omega\hat{\mathbf{n}}/v_{\text{phase}}$
{3-wavevector} $|\mathbf{u} * v_{\text{phase}}| = c^2$

4-WaveVector $K = (\omega/c, \mathbf{k}) = -\partial[\Phi_{\text{phase, free}}] = -(\partial_t/c, -\nabla)[\Phi_{\text{phase, free}}]$

Temporal part: $\omega = -\partial_t[\Phi_{\text{phase, free}}]$
 {angular frequency}

Spatial part: $\mathbf{k} = +\nabla[\Phi_{\text{phase, free}}]$
{3-wavevector}



$(K \cdot K) = (\omega/c)^2 - (\mathbf{k} \cdot \mathbf{k}) = (\omega_0/c^2)^2$
 $\omega^2 = (|\mathbf{k}|c)^2 + (\omega_0)^2$: Matter-Wave Dispersion Relation
 Relativistic AngFreq(ω) vs Invariant Rest AngFreq(ω_0)

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 (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$



SRQM Diagram:

The Basis of Classical SR Physics 4-CurrentDensity, Charge Conservation

A Tensor Study of Physical 4-Vectors

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4-Position $\mathbf{R}=(ct, \mathbf{r})$
4-Gradient $\partial=(\partial_t/c, -\nabla)$
4-Velocity $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$

4-CurrentDensity $\mathbf{J} = (\rho c, \mathbf{j}) = \rho_0 \mathbf{U} = \gamma \rho_0 (\mathbf{c}, \mathbf{u}) = \rho(\mathbf{c}, \mathbf{u})$
4-ChargeFlux \mathbf{J}

Temporal part: $\rho = \gamma \rho_0$
 {charge-density}

Spatial part: $\mathbf{j} = \gamma \rho_0 \mathbf{u} = \rho \mathbf{u}$
 {3-current-density}

Conservation of Charge (Q)

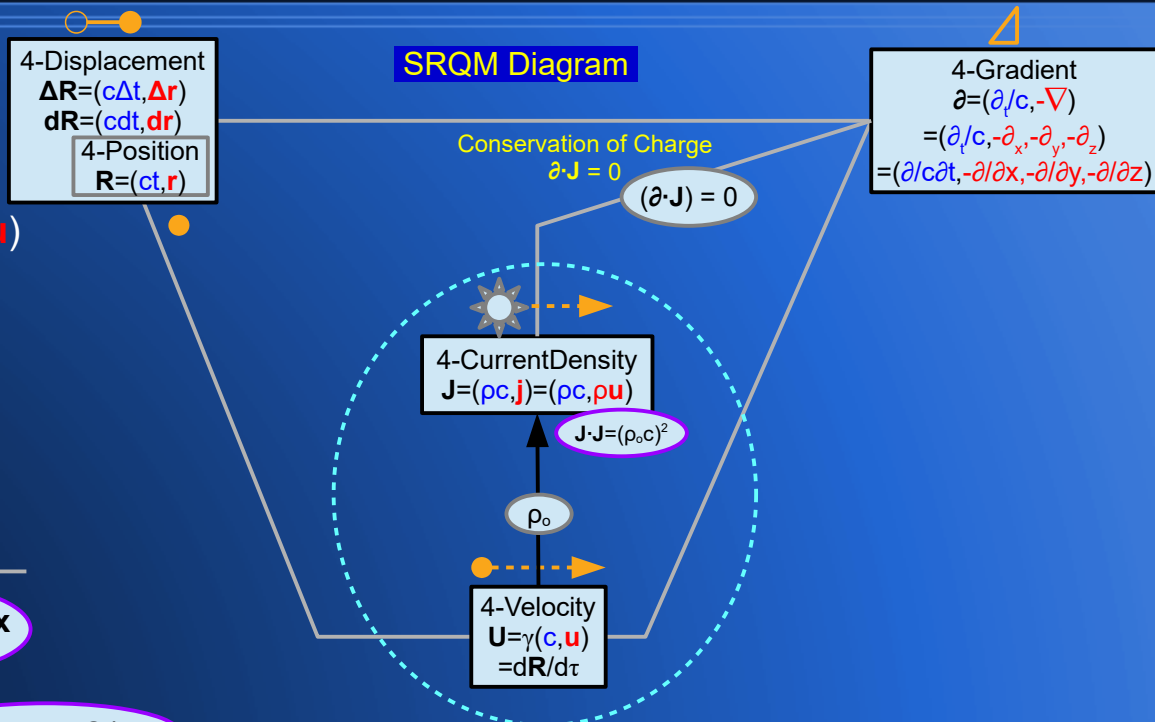
$$Q = \int \rho d^3\mathbf{x} = \int \gamma \rho_0 d^3\mathbf{x} \rightarrow \rho_0 V_0$$

$$\partial \cdot \mathbf{J} = (\partial_t/c, -\nabla) \cdot (\rho c, \mathbf{j}) = (\partial_t \rho + \nabla \cdot \mathbf{j}) = 0$$

Continuity Equation: Noether's Theorem

The temporal change in charge density is balanced by the spatial change in current density.
 Charge is neither created nor destroyed
 It just moves around as charge currents...

$$\int d\mathbf{T} \cdot \mathbf{J} = -cQ/V_0$$



Relativistic ChargeDensity(ρ) vs Invariant Rest ChargeDensity(ρ_0)

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 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or T_{μ}^ν
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$

SRQM Diagram:

The Basis of Classical SR Physics 4-(Dust)NumberFlux, Particle # Conservation

A Tensor Study of Physical 4-Vectors

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- 4-Position $R=(ct, \mathbf{r})$
- 4-Gradient $\partial=(\partial_t/c, -\nabla)$
- 4-Velocity $U = \gamma(c, \mathbf{u})$
- 4-NumberFlux $N = (nc, \mathbf{n}) = n_o U = \gamma n_o(c, \mathbf{u}) = n(c, \mathbf{u})$

Temporal part: $n = \gamma n_o$
{number-density}

Spatial part: $\mathbf{n} = \gamma n_o \mathbf{u} = \mathbf{n}u$
{3-number-flux}

Conservation of Particle # (N)

$$N = \int n d^3\mathbf{x} = \int \gamma n_o d^3\mathbf{x} \rightarrow n_o V_o$$

$$\partial \cdot N = (\partial_t/c, -\nabla) \cdot (nc, \mathbf{n}) = (\partial_t n + \nabla \cdot \mathbf{n}) = 0$$

Continuity Equation: Noether's Theorem

The temporal change in number density is balanced by the spatial change in number-flux.

Particle # is neither created nor destroyed
It just moves around as number currents...

$$\int dT \cdot N = -cN/V_o$$

4-Displacement
 $\Delta R=(c\Delta t, \Delta \mathbf{r})$
 $dR=(cdt, d\mathbf{r})$
4-Position
 $R=(ct, \mathbf{r})$

SRQM Diagram

Conservation of Particle #

$$\partial \cdot N = 0$$

4-Gradient
 $\partial=(\partial_t/c, -\nabla)$
 $=(\partial_t/c, -\partial_x, -\partial_y, -\partial_z)$
 $=(\partial/c\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$

4-NumberFlux
 $N=(nc, \mathbf{n})=(nc, \mathbf{n}u)$

$$N \cdot N = (n_o c)^2$$

n_o

4-Velocity
 $U=\gamma(c, \mathbf{u})$
 $=dR/d\tau$

$$(N \cdot N) = (nc)^2 - (\mathbf{n} \cdot \mathbf{n}) = (n_o c)^2$$

$$n^2 = (|\mathbf{n}|/c)^2 + (n_o)^2$$

Relativistic NumberDensity(n) vs Invariant Rest NumberDensity(n_o)

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$$

Lorentz Transforms $\Lambda^{\mu}_{\nu} = \partial_{\nu}[X^{\mu}]$ (Continuous) vs (Discrete) (Proper Det=+1) vs (Improper Det=-1)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The main idea that makes a generic 4-Vector into an SR 4-Vector is that it must transform correctly according to an SR Lorentz Transformation $\{\Lambda^{\mu}_{\nu} = \partial X^{\mu}/\partial X^{\nu} = \partial_{\nu}[X^{\mu}]\}$, which is basically any linear, unitary or antiunitary, transform (Determinant $[\Lambda^{\mu}_{\nu}] = \pm 1$) which leaves the Invariant Interval unchanged.

The SR continuous transforms (variable with some parameter) have {Det = +1, Proper} and include:

“Rotation” {a mixing of space-space coordinates} and “Boost” {a mixing of time-space coordinates}.

The SR discrete transforms can be {Det = +1, Proper} or {Det = -1, Improper} and include:

“Space Parity-Inversion” {reversal of the space coordinates}, “Time-Reversal” {reversal of the temporal coordinate},

The “Identity” {no change}, and various single dimension Flips and their combinations.



SR:Lorentz Transform

$$\partial_{\nu}[R^{\mu}] = \partial R^{\mu}/\partial R^{\nu} = \Lambda^{\mu}_{\nu}$$

$$\Lambda^{\mu}_{\nu} = (\Lambda^{-1})^{\mu}_{\nu} : \Lambda^{\mu}_{\alpha}\Lambda^{\alpha}_{\nu} = \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu}$$

$$\eta_{\mu\nu}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$$

$$\text{Det}[\Lambda^{\mu}_{\nu}] = \pm 1 \quad \Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$$

Typical Lorentz Boost Transformation, for a linear-velocity frame-shift (x,t)-Boost in the \hat{x} -direction:

$$A^{\nu} = (a^t, a^x, a^y, a^z)$$

$$A^{\mu} = (a^t, a^x, a^y, a^z)$$

$$= B^{\mu}_{\nu}A^{\nu}$$

$$= (\gamma a^t - \gamma\beta a^x, -\gamma\beta a^t + \gamma a^x, a^y, a^z)$$

{for \hat{x} -boost Lorentz Transform}

Lorentz Parity-Inversion Transformation:

$$A^{\nu} = (a^t, a^x, a^y, a^z)$$

$$A^{\mu} = (a^t, a^x, a^y, a^z)$$

$$= P^{\mu}_{\nu}A^{\nu}$$

$$= (a^t, -a^x, -a^y, -a^z)$$

{for Parity Inverse Lorentz Transform}

Continuous: ex. Boost depends on variable parameter β , with $\gamma=1/\sqrt{1-\beta^2}$

Lorentz Boost Transform $\Lambda^{\mu}_{\nu} \rightarrow B^{\mu}_{\nu} =$

$$\begin{matrix} \hat{t} & \hat{x} & \hat{y} & \hat{z} \\ \hat{t} & \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \hat{x} & \\ \hat{y} & \\ \hat{z} & \end{matrix}$$

Boosted 4-Vector

$$A' = A^{\mu} = \Lambda^{\mu}_{\nu} A^{\nu} \rightarrow B^{\mu}_{\nu} A^{\nu} = (a^{0'}, \mathbf{a}')$$

ex. for \hat{x} -boost

$$\rightarrow (\gamma a^t - \gamma\beta a^x, -\gamma\beta a^t + \gamma a^x, a^y, a^z)$$

$$\text{Det}[B^{\mu}_{\nu}] = +1, \text{ Proper} \\ \gamma^2 - \beta^2\gamma^2 = +1$$

Proper: preserves orientation of basis

Discrete: ex. Parity has no variable parameters

Lorentz Parity Transform $\Lambda^{\mu}_{\nu} \rightarrow P^{\mu}_{\nu} =$

$$\begin{matrix} \hat{t} & \hat{x} & \hat{y} & \hat{z} \\ \hat{t} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ \hat{x} & \\ \hat{y} & \\ \hat{z} & \end{matrix}$$

Parity-Inversed 4-Vector

$$A' = A^{\mu} = \Lambda^{\mu}_{\nu} A^{\nu} \rightarrow P^{\mu}_{\nu} A^{\nu} = (a^{0'}, \mathbf{a}')$$

$$\rightarrow (a^t, -a^x, -a^y, -a^z)$$

$$\text{Det}[P^{\mu}_{\nu}] = -1, \text{ Improper} \\ (-1)^3 = -1$$

Improper: reverses orientation of basis

SR 4-Tensor

(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector

(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
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(0,0)-Tensor S
Lorentz Scalar

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}_{\mu} = T \\ \mathbf{V} \cdot \mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$

Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Proper Lorentz Transforms (Det=+1): Continuous: (Boost) vs (Rotation)

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$\beta = v/c$: dimensionless Velocity Beta Factor { $\beta=(0..1)$, with speed-of-light (c) at ($\beta=1$) }
 $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\beta\cdot\beta}$: dimensionless Lorentz Relativistic Gamma Factor { $\gamma=(1..\infty)$ }

Typical Lorentz Boost Transform (symmetric):
for a linear-velocity frame-shift (x,t)-Boost in the \hat{x} -direction:

$$\Lambda^{\mu'}_{\nu} \rightarrow B^{\mu'}_{\nu}[\zeta] = e^{\Lambda \cdot (\zeta \cdot \mathbf{K})} =$$

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh[\zeta] & -\sinh[\zeta] & 0 & 0 \\ -\sinh[\zeta] & \cosh[\zeta] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e^{\Lambda(\zeta_x)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^{\nu} = (a^t, a^x, a^y, a^z)$$

$$A^{\mu'} = (a^t, a^x, a^y, a^z)' = B^{\mu'}_{\nu} A^{\nu} = (\gamma a^t - \gamma\beta a^x, -\gamma\beta a^t + \gamma a^x, a^y, a^z)$$

Lorentz Transforms:
Lambda (Λ) for Lorentz
(B) for Boost
(R) for Rotation

Proper Transforms
Determinant = +1

$$\{\cos^2 + \sin^2 = +1\}$$

$$\{\gamma^2 - \beta^2\gamma^2 = +1\}$$

$$\{\cosh^2 - \sinh^2 = +1\}$$

ζ = rapidity = hyperbolic angle
 $\gamma = \cosh[\zeta] = 1/\sqrt{1-\beta^2}$
 $\beta\gamma = \sinh[\zeta]$
 $\beta = \tanh[\zeta]$

Typical Lorentz Rotation Transform (non-symmetric):
for an angular-displacement frame-shift (x,y)-Rotation about the \hat{z} -direction:

$$\Lambda^{\mu'}_{\nu} \rightarrow R^{\mu'}_{\nu}[\theta] = e^{\Lambda(\theta \cdot \mathbf{J})} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[\theta] & -\sin[\theta] & 0 \\ 0 & \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e^{\Lambda(\theta_z)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

SR: Lorentz Transform

$$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'} / \partial R^{\nu} = \Lambda^{\mu'}_{\nu}$$

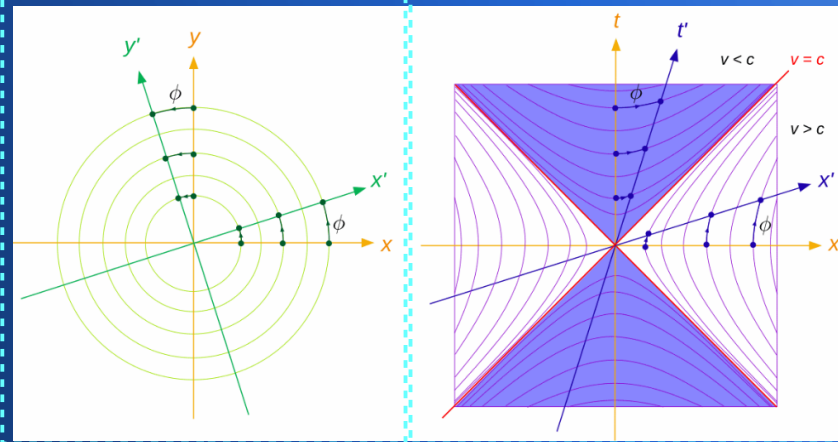
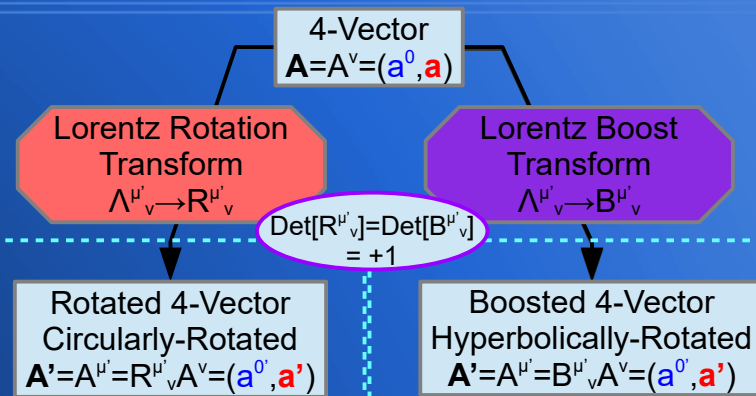
$$\Lambda^{\mu'}_{\nu} = (\Lambda^{-1})^{\mu}_{\nu'} : \Lambda^{\mu}_{\alpha} \Lambda^{\alpha}_{\nu} = \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu}$$

$$\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$$

$$\text{Det}[\Lambda^{\mu'}_{\nu}] = \pm 1 \quad \Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$$

$$A^{\nu} = (a^t, a^x, a^y, a^z)$$

$$A^{\mu'} = (a^t, a^x, a^y, a^z)' = R^{\mu'}_{\nu} A^{\nu} = (a^t, \cos[\theta] a^x - \sin[\theta] a^y, \sin[\theta] a^x + \cos[\theta] a^y, a^z)$$



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
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Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Proper Lorentz Transforms (Det=+1): (Boost) vs (Rotation) vs (Identity)

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General Lorentz Boost Transform (symmetric, continuous):

for a linear-velocity frame-shift (Boost) in the $\mathbf{v}/c = \boldsymbol{\beta} = (\beta^1, \beta^2, \beta^3)$ -direction:

$$\Lambda^{\mu'}_{\nu} \rightarrow B^{\mu'}_{\nu} = \begin{pmatrix} \gamma & & & -\gamma\beta_j \\ -\gamma\beta^i & (\gamma-1)\beta^i\beta^j / (\boldsymbol{\beta} \cdot \boldsymbol{\beta}) + \delta^i_j & & \end{pmatrix}$$

$$\Lambda^{\mu'}_{\nu} = \begin{bmatrix} \Lambda^{\mu'}_0 & \Lambda^{\mu'}_j \\ \Lambda^{\nu'}_0 & \Lambda^{\nu'}_j \end{bmatrix} \quad \Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$$

$$\text{Tr}[\eta^{\mu'}_{\nu}] = 4 \quad \text{Det}[\eta^{\mu'}_{\nu}] = +1$$

No mixing
Lorentz Identity Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow \eta^{\mu'}_{\nu} = I_{(4)}$

Space-Space
Lorentz Rotation Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow R^{\mu'}_{\nu}$

$$\text{Tr}[R^{\mu'}_{\nu}] = \{0..4\} \quad \text{Det}[R^{\mu'}_{\nu}] = +1$$

Time-Space
Lorentz Boost Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow B^{\mu'}_{\nu}$

$$\text{Tr}[B^{\mu'}_{\nu}] = \{4..Infinity\} \quad \text{Det}[B^{\mu'}_{\nu}] = +1$$

General Lorentz Rotation Transform (non-symmetric, continuous):

for an angular-displacement frame-shift (Rotation) angle θ about the $\hat{\mathbf{n}} = (n^1, n^2, n^3)$ -direction:

$$\Lambda^{\mu'}_{\nu} \rightarrow R^{\mu'}_{\nu} = \begin{pmatrix} 1 & & & 0_j \\ 0^i & (\delta^i_j - n^i n_j) \cos(\theta) - (\epsilon_{ijk} n^k) \sin(\theta) + n^i n_j & & \end{pmatrix}$$

Identical 4-Vector Un-Rotated
 $\mathbf{A}' = \mathbf{A}^{\mu'} = \eta^{\mu'}_{\nu} \mathbf{A}^{\nu} = (\mathbf{a}^0, \mathbf{a}') = \mathbf{A}$

The Lorentz Identity Transform is the limit of both the Rotation and Boost Transforms when the "rotation angle" is 0

Rotated 4-Vector Circularly-Rotated
 $\mathbf{A}' = \mathbf{A}^{\mu'} = R^{\mu'}_{\nu} \mathbf{A}^{\nu} = (\mathbf{a}^0, \mathbf{a}')$

Boosted 4-Vector Hyperbolically-Rotated
 $\mathbf{A}' = \mathbf{A}^{\mu'} = B^{\mu'}_{\nu} \mathbf{A}^{\nu} = (\mathbf{a}^0, \mathbf{a}')$

Lorentz Identity Transform (symmetric, "discrete, continuous"):

for a non-frame-shift (Identity) in any direction

$$\Lambda^{\mu'}_{\nu} \rightarrow \eta^{\mu'}_{\nu} = \delta^{\mu'}_{\nu} = \text{Diag}[1, \delta^i_j] = I_{(4)}$$

$$\begin{pmatrix} 1 & 0_j \\ 0^i & \delta^i_j \end{pmatrix}$$

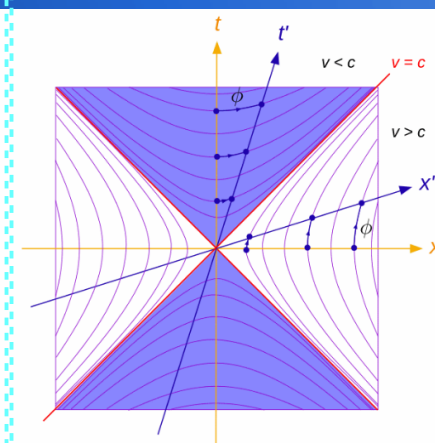
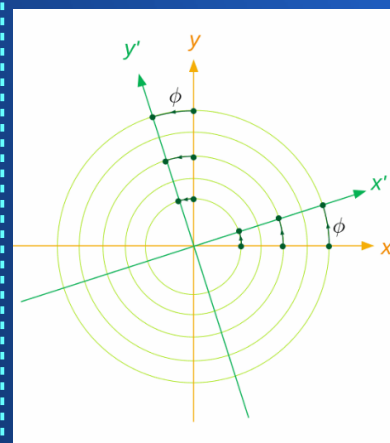
SR: Lorentz Transform

$$\partial_{\nu}[R^{\mu}] = \partial R^{\mu} / \partial R^{\nu} = \Lambda^{\mu'}_{\nu}$$

$$\Lambda^{\mu'}_{\nu} = (\Lambda^{-1})^{\mu}_{\nu'} : \Lambda^{\mu}_{\alpha} \Lambda^{\alpha}_{\nu} = \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu}$$

$$\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$$

$$\text{Det}[\Lambda^{\mu'}_{\nu}] = \pm 1 \quad \Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$$



$\beta = v/c$: dimensionless Velocity Beta Factor { $\beta = (0..1)$, with speed-of-light (c) at ($\beta=1$) }
 $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\boldsymbol{\beta} \cdot \boldsymbol{\beta}}$: dimensionless Lorentz Relativistic Gamma Factor { $\gamma = (1..∞)$ }

Identity transformation for zero relative motion/rotation: $B[0] = R[0] = I_{(4)}$

Proper Transformation = positive unit determinant: $\text{det}[B] = \text{det}[R] = \text{det}[\eta] = +1$.

Inverses: $B(\mathbf{v})^{-1} = B(-\mathbf{v})$ (relative motion in the opposite direction), and $R(\theta)^{-1} = R(-\theta)$ (rotation in the opposite sense about the same axis)

Matrix symmetry: B is symmetric (equals transpose, $B=B^T$), while R is nonsymmetric but orthogonal (transpose equals inverse, $R^T = R^{-1}$)

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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$

Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Discrete (non-continuous) (Parity-Inversion) vs (Time-Reversal) vs (Identity)

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General Lorentz Parity-Inversion Transform:
 $\Lambda^{\mu'}_{\nu} \rightarrow P^{\mu'}_{\nu}$ (Improper, symmetric, discrete)

$$\begin{bmatrix} 1 & 0_j \\ 0^i & -\delta^i_j \end{bmatrix}$$

General Lorentz Time-Reversal Transform:
 $\Lambda^{\mu'}_{\nu} \rightarrow T^{\mu'}_{\nu}$ (Improper, symmetric, discrete)

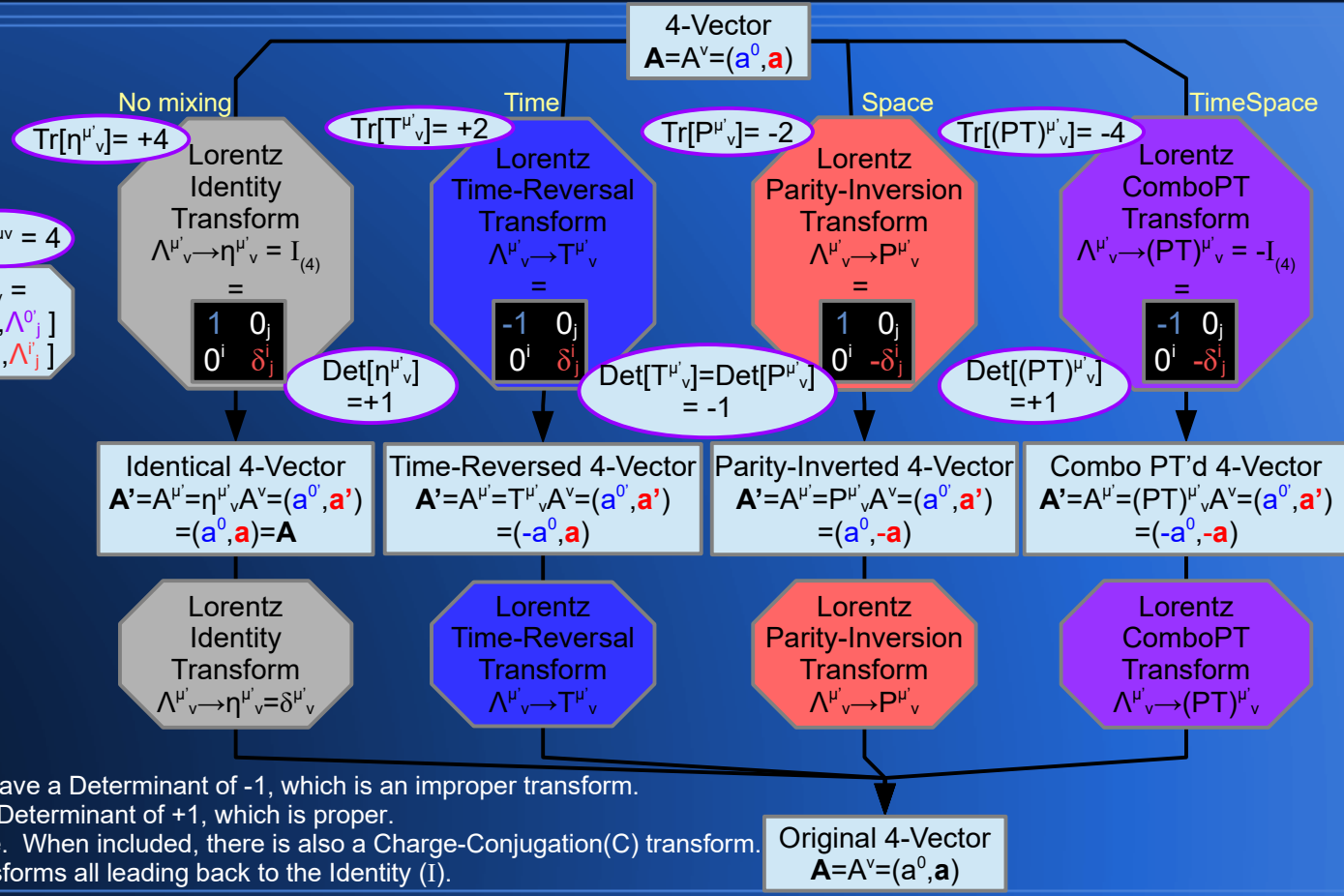
$$\begin{bmatrix} -1 & 0_j \\ 0^i & \delta^i_j \end{bmatrix}$$

General Lorentz Identity Transform:
 $\Lambda^{\mu'}_{\nu} \rightarrow \eta^{\mu'}_{\nu} = \delta^{\mu'}_{\nu} = I_{(4)}$ (Proper, symmetric)

$$\begin{bmatrix} 1 & 0_j \\ 0^i & \delta^i_j \end{bmatrix}$$



SR:Lorentz Transform
 $\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'} / \partial R^{\nu} = \Lambda^{\mu'}_{\nu}$
 $\Lambda^{\mu}_{\nu} = (\Lambda^{-1})^{\mu}_{\nu} : \Lambda^{\mu}_{\alpha} \Lambda^{\alpha}_{\nu} = \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu}$
 $\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$
 $\text{Det}[\Lambda^{\mu}_{\nu}] = \pm 1$ $\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$



Both the Parity-Inversion (P) and Time-Reversal (T) have a Determinant of -1, which is an improper transform. However, combinations (PP), (TT), (PT) have overall Determinant of +1, which is proper. Classical SR Time Reversal neglects spin and charge. When included, there is also a Charge-Conjugation(C) transform. Then one gets (CC),(PP),(TT),(PT)(PT) & (CPT) transforms all leading back to the Identity (I).

SR 4-Tensor
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 (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
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SR 4-CoVector
 (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Note that the Trace of Discrete Lorentz Transforms goes in steps from $\{-4, -2, 2, 4\}$. As we will see in a bit, this is a major hint for SR antimatter.

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Discrete & Fixed Rotation → Particle Exchange Lorentz Coordinate-Flip Transforms

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Lorentz Flip-t Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow Ft^{\mu'}_{\nu} = T^{\mu'}_{\nu}$

t	x	y	z
t	-1	0	0
x	0	1	0
y	0	0	1
z	0	0	0

Tr[Ft^{μ'ν}]= 2
Det[Ft^{μ'ν}]= -1

Tr[R^{μ'ν}]=2+2cos[θ]={0..4}
Det[R^{μ'ν}]=cos[θ]² + sin[θ]²= +1

Lorentz z-Rotation Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow R^{\mu'}_{\nu}$

t	x	y	z
t	1	0	0
x	0	cos[θ]	-sin[θ]
y	0	sin[θ]	cos[θ]
z	0	0	1



SR:Lorentz Transform

$\partial_{\nu}[R^{\mu}] = \partial R^{\mu} / \partial R^{\nu} = \Lambda^{\mu}_{\nu}$
 $\Lambda^{\mu}_{\nu} = (\Lambda^{-1})^{\mu}_{\nu} : \Lambda^{\mu}_{\alpha} \Lambda^{\alpha}_{\nu} = \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu}$
 $\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$
 Det[Λ^{μν}]=±1 Λ_{μν}Λ^{μν}=4

Any single Lorentz Flip Transform is Improper, with a Determinant of -1. However, pairwise combinations are Proper, with a Determinant of +1.

Lorentz Flip-x Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow Fx^{\mu'}_{\nu}$

t	x	y	z
t	1	0	0
x	0	-1	0
y	0	0	1
z	0	0	0

Tr[Fx^{μ'ν}]= 2
Det[Fx^{μ'ν}]= -1

Tr[Fxy^{μ'ν}]= 0
Det[Fxy^{μ'ν}]= +1

Lorentz Flip-xy Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow Fxy^{\mu'}_{\nu}$
Exchange

t	x	y	z
t	1	0	0
x	0	-1	0
y	0	0	-1
z	0	0	0

The combination of any two Spatial Flips is the equivalent of a Spatial Rotation by (π) about the associated rotational axis. Since this is a Proper transform, it is also the equivalent of a particle location exchange.

Lorentz Flip-y Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow Fy^{\mu'}_{\nu}$

t	x	y	z
t	1	0	0
x	0	1	0
y	0	0	-1
z	0	0	0

Tr[Fy^{μ'ν}]= 2
Det[Fy^{μ'ν}]= -1

Λ_{μν}Λ^{μν} = 4

Λ^{μ'ν} =
 $\begin{bmatrix} \Lambda^{0'}_0 & \Lambda^{0'}_j \\ \Lambda^{i'}_0 & \Lambda^{i'}_j \end{bmatrix}$

The combination of all three Spatial Flips, Flip-xyz, gives the Lorentz Parity Transform, which is again Improper.

Lorentz Flip-z Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow Fz^{\mu'}_{\nu}$

t	x	y	z
t	1	0	0
x	0	1	0
y	0	0	1
z	0	0	-1

Tr[Fz^{μ'ν}]= 2
Det[Fz^{μ'ν}]= -1

The Flip-t is the standard Lorentz Time-Reversal, Improper.

Tr[Fxyz^{μ'ν}]= -2
Det[Fxyz^{μ'ν}]= -1

Lorentz Parity Transform
 $\Lambda^{\mu'}_{\nu} \rightarrow P^{\mu'}_{\nu}$
Fxyz^{μ'ν}

t	x	y	z
t	1	0	0
x	0	-1	0
y	0	0	-1
z	0	0	-1

SR 4-Tensor
 (2,0)-Tensor T^{μν}
 (1,1)-Tensor T^μ_ν or T_μ^ν
 (0,2)-Tensor T_{μν}

SR 4-Vector
 (1,0)-Tensor V^μ = V = (v⁰, v)
SR 4-CoVector
 (0,1)-Tensor V_μ = (v₀, -v)

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

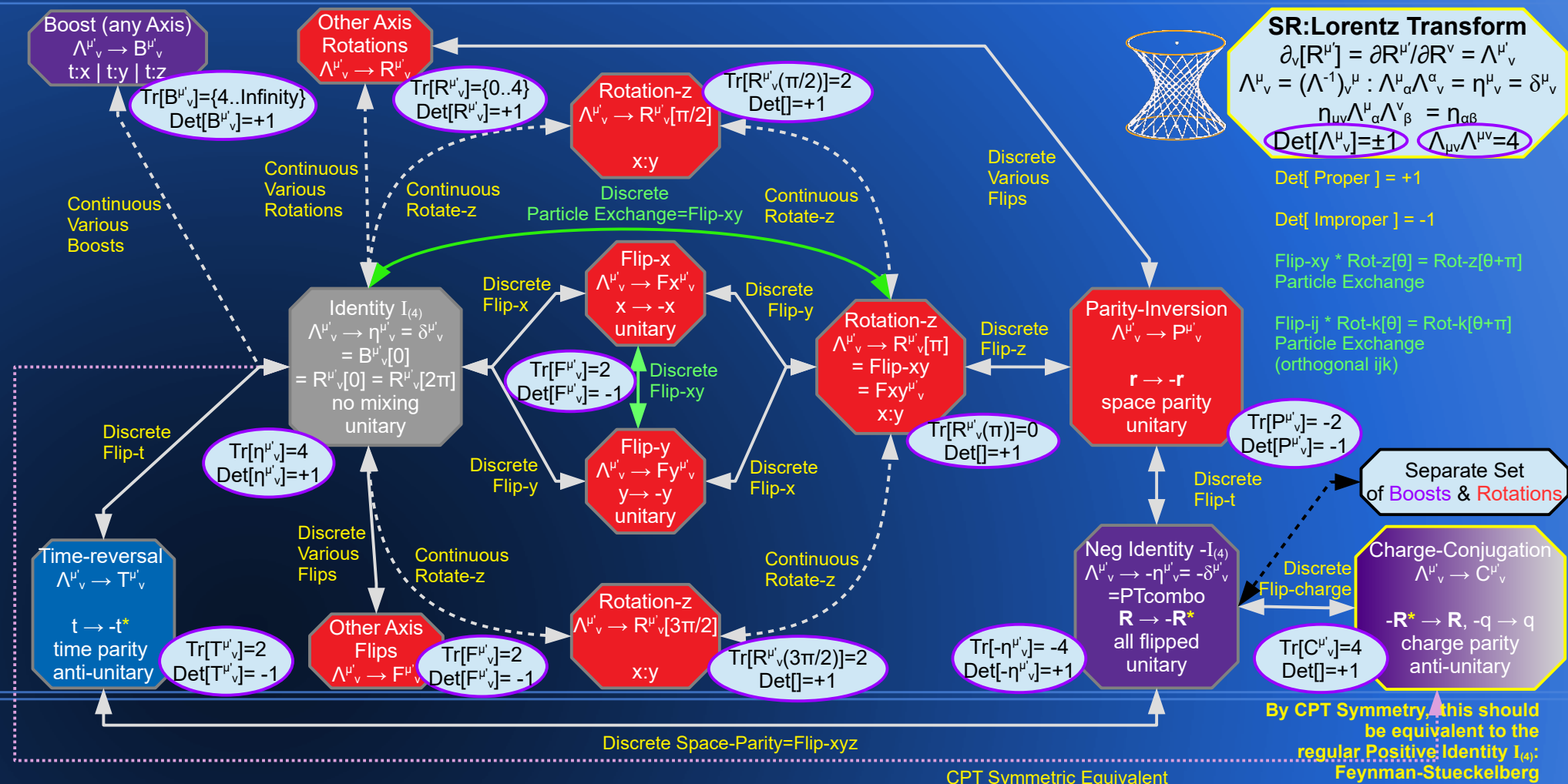
Trace[T^{μν}] = η_{μν}T^{μν} = T^μ_μ = T
 V·V = V^μη_{μν}V^ν = [(v⁰)² - v·v] = (v⁰)² - v²
 = Lorentz Scalar

Lorentz Transforms $\Lambda^{\mu}_{\nu} = \partial_{\nu}[X^{\mu}]$

Lorentz Transform Connection Map

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Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Lorentz Transform Connection Map – Discrete Transforms CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

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Examine all possible combinations of Discrete Lorentz Transformations which are Linear (Determinant of ±1).

A lot of the standard SR texts only mention (P)arity-Inverse and (T)ime-Reversal. However, there are many others, including (F)lips and (R)otations of a fixed amount. However, the (T)imeReversal and Combo(P)arity(T)ime take one into a separate section of the chart. Taking into account all possible discrete Lorentz Transformations fills in the rest of the chart. The resulting interpretation is that there is **CPT Symmetry** (Charge:Parity:Time) and **Dual TimeSpace** (with reversed timeflow). In other words, one can go from the Identity Transform (all +1) to the Negative Identity Transform (all -1) by doing a Combo PT Lorentz Transform or by Negating the Charge (Matter→AntiMatter). The Feynman-Stueckelberg Interpretation aligns with this as the AntiMatter Side.

This is similar to Dirac's prediction of AntiMatter, but without the formal need of Quantum Mechanics, or Spin. In fact, it is more general than Dirac's work, which was about the electron. This is from general Lorentz Transforms for any kind of particle.

SR:Lorentz Transform

$$\partial_{\nu}[R^{\mu}] = \partial R^{\mu} / \partial R^{\nu} = \Lambda^{\mu'}_{\nu}$$

$$\Lambda^{\mu}_{\nu} = (\Lambda^{-1})^{\mu}_{\nu} : \Lambda^{\mu}_{\alpha} \Lambda^{\alpha}_{\nu} = \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu}$$

$$\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$$

Det[Λ^{μ}_{ν}]=±1 **$\Lambda_{\mu\nu} \Lambda^{\mu\nu}=4$**



Tao – I Ching – YinYang
fantastic metaphors for
SR SpaceTime...
Tao: "Flow of the Universe"
"way, path, route, road"
I Ching: "Book of Changes"
"Transformations"
YinYang: "Positive/Negative"
"complementary opposites"



Matter-AntiMatter
Dual balance along Temporal
Binary Spatial states
for 3 units dimensions
Discrete Lorentz
Transform (1,1)-Tensor
{ octagon representation }
Pair production (+ -)
in little circles (• •)

<u>t</u>	<u>x</u>	<u>y</u>	<u>z</u>	<u>Discrete NormalMatter (NM) Lorentz Transform Type</u>	<u>Trace : Determinant</u>
+1	+1	+1	+1	Minkowski-Identity : AM-Flip-txyz=AM-ComboPT	Tr = +4 : Det = +1 Proper
+1	+1	+1	-1	Flip-z	Tr = +2 : Det = -1 Improper
+1	+1	-1	+1	Flip-y	Tr = +2 : Det = -1 Improper
+1	+1	-1	-1	Flip-yz=Rotate-yz(π)	Tr = 0 : Det = +1 Proper
+1	-1	+1	+1	Flip-x	Tr = +2 : Det = -1 Improper
+1	-1	+1	-1	Flip-xz=Rotate-xz(π)	Tr = 0 : Det = +1 Proper
+1	-1	-1	+1	Flip-xy=Rotate-xy(π)	Tr = 0 : Det = +1 Proper
+1	-1	-1	-1	Flip-xyz=ParityInverse : AM-Flip-t=AM-TimeReversal	Tr = -2 : Det = -1 Improper
-1	+1	+1	+1	Flip-t=TimeReversal : AM-Flip-xyz=AM-ParityInverse	Tr = +2 : Det = -1 Improper
-1	+1	+1	-1	AM-Flip-xy=AM-Rotate-xy(π)	Tr = 0 : Det = +1 Proper
-1	+1	-1	+1	AM-Flip-xz=AM-Rotate-xz(π)	Tr = 0 : Det = +1 Proper
-1	+1	-1	-1	AM-Flip-x	Tr = -2 : Det = -1 Improper
-1	-1	+1	+1	AM-Flip-yz=AM-Rotate-yz(π)	Tr = 0 : Det = +1 Proper
-1	-1	+1	-1	AM-Flip-y	Tr = -2 : Det = -1 Improper
-1	-1	-1	+1	AM-Flip-z	Tr = -2 : Det = -1 Improper
-1	-1	-1	-1	AM-Minkowski-Identity : Flip-txyz=ComboPT	Tr = -4 : Det = +1 Proper
<u>t</u>	<u>x</u>	<u>y</u>	<u>z</u>	<u>Discrete AntiMatter (AM) Lorentz Transform Type</u>	<u>Trace : Determinant</u>

Note that the
(T)imeReversal
and
Combo
(P)arityInverse &
(T)imeReversal
take
NormalMatter
↕
AntiMatter

Lorentz Transforms $\Lambda^{\mu}_{\nu} = \partial_{\nu}[X^{\mu}]$

Lorentz Transform Connection Map – Trace Identification CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

All Lorentz Transforms have Tensor Invariants: Determinant of ± 1 and Inner Product of 4. However, one can use the Tensor Invariant Trace to Identify CPT Symmetry

$$\begin{aligned} \text{Tr}[\text{NM-Rotate}] &= \{0...+4\} & \text{Tr}[\text{NM-Identity}] &= +4 & \text{Tr}[\text{NM-Boost}] &= \{+4...+\infty\} \\ \text{Tr}[\text{AM-Rotate}] &= \{0....-4\} & \text{Tr}[\text{AM-Identity}] &= -4 & \text{Tr}[\text{AM-Boost}] &= \{-4.....-\infty\} \end{aligned}$$

Discrete NormalMatter (NM) Lorentz Transform Type
Minkowski-Identity : AM-Flip-txyz=AM-ComboPT

Flip-t=TimeReversal, Flip-x, Flip-y, Flip-z
 AM-Flip-xyz=AM-ParityInverse

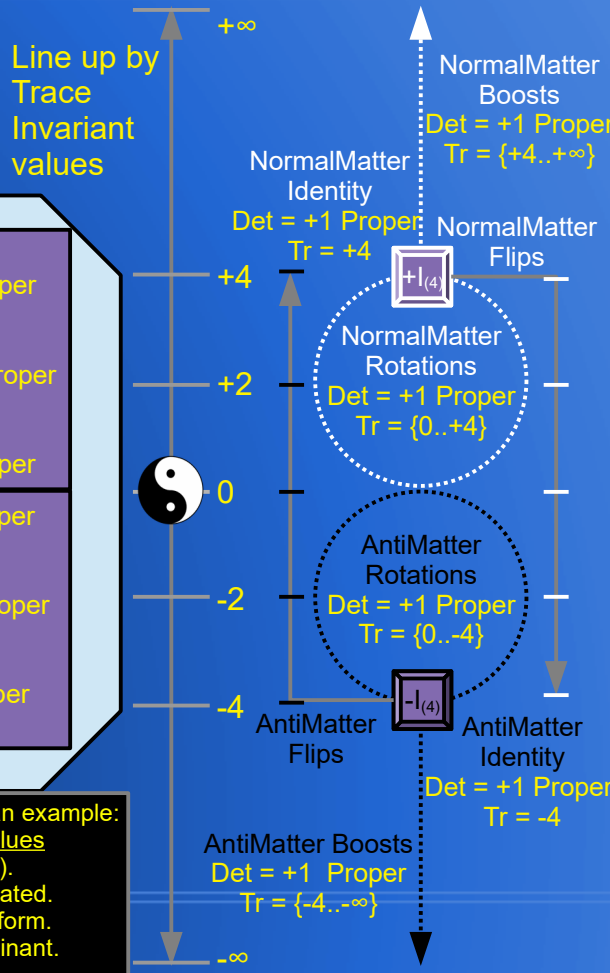
Flip-xy=Rotate-xy(π), Flip-xz=Rotate-xz(π), Flip-yz=Rotate-yz(π)

AM-Flip-xy=AM-Rotate-xy(π), AM-Flip-xz=AM-Rotate-xz(π), AM-Flip-yz=AM-Rotate-yz(π)

Flip-xyz=ParityInverse
 AM-Flip-t=AM-TimeReversal, AM-Flip-x, AM-Flip-y, AM-Flip-z

AM-Minkowski-Identity : Flip-txyz=ComboPT
Discrete AntiMatter (AM) Lorentz Transform Type

<u>Trace : Determinant</u> Tr = +4 : Det = +1 Proper
Tr = +2 : Det = -1 Improper
Tr = 0 : Det = +1 Proper
Tr = 0 : Det = +1 Proper
Tr = -2 : Det = -1 Improper
<u>Trace : Determinant</u> Tr = -4 : Det = +1 Proper



SR:Lorentz Transform

$$\partial_{\nu}[R^{\mu}] = \partial R^{\mu} / \partial R^{\nu} = \Lambda^{\mu}_{\nu}$$

$$\Lambda^{\mu}_{\nu} = (\Lambda^{-1})^{\nu}_{\mu} : \Lambda^{\mu}_{\alpha} \Lambda^{\alpha}_{\nu} = \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu}$$

$$\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$$

Det $[\Lambda^{\mu}_{\nu}] = \pm 1$ **$\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$**



Two interesting properties of (1,1)-Tensors, of which the Lorentz Transform is an example:
Trace = Sum (Σ) of EigenValues : Determinant = Product (Π) of EigenValues
 As Rank 4 Tensors, each Lorentz Transform has 4 EigenValues (EV's).
 Create an Anti-Transform which has all EigenValue Tensor Invariants negated.
 $\Sigma[-(\text{EV's})] = -\Sigma[\text{EV's}]$: The Anti-Transform has negative Trace of the Transform.
 $\Pi[-(\text{EV's})] = (-1)^4 \Pi[\text{EV's}] = \Pi[\text{EV's}]$: The Anti-Transform has equal Determinant.
 The Trace Invariant identifies a "Dual" Negative-Side for all Lorentz Transforms.

Lorentz Transform Connection Map - Interpretations CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

A Tensor Study of Physical 4-Vectors

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Based on the Lorentz Transform properties of the last few pages, here is interesting observation about Lorentz Transforms: They all have Determinant of ± 1 , and Inner Product of 4, but the Trace varies depending on the particular Transform.

The Trace of the Identity is at 4. Assume this applies to normal matter particles.
The Trace of normal matter particle Rotations varies from (0..4)
The Trace of the normal matter particle Boosts varies from (4..Infinity)
So, one can think of Trace = 4 being the connection point between normal matter Rotations and Boosts.

Now, various Flip Transforms (inc. the Time Reversal and Parity Transforms, and their combination as PT transform), take the Trace in steps from (-4,-2,0,+2+4). Applying a bit of symmetry:

The Trace of the Negative Identity is at -4. Assume this applies to anti-matter particles.
The Trace of anti-matter particle Rotations varies from (0..-4)
The Trace of the anti-matter particle Boosts varies from (-4..-Infinity)
So, one can think of Trace = -4 being the connection point between anti-matter Rotations and Boosts.


This observation would be in agreement with the CPT Theorem (Feynman-Stueckelberg) idea that normal matter particles moving backward in time are CPT symmetrically equivalent to antimatter particles moving forward in time.

Now, scale this up to Universe size: The Baryon Asymmetry problem (aka. The Matter-AntiMatter Asymmetry Problem). If the Universe was created as a huge chunk of energy, and matter-creating energy is always transformed into matter-antimatter mirrored pairs, then where is all the antimatter???? Turns out this is directly related to the Arrow-of-Time Problem as well.

Answer: It is temporally on the "Other/Dual side" of the Big-Bang! The antimatter created at the Big-Bang is travelling in the negative time (-t) direction from the Big-Bang creation point, and the normal matter is travelling in the positive time direction (+t).
Universal CPT Symmetry. So, what happened "before" the Big-Bang? It "is" the AntiMatter Dual to our normal matter universe! Pair-production is creation of AM-NM mirrored pairs within SpaceTime. The Big-Bang is the creation of SpaceTime itself.

This also resolves the Arrow-of-Time Problem. If all known physical microscopic processes are time-symmetric, why is the flow of Time experienced as uni-directional???? {see Wikipedia "CPT Symmetry", "CP Violation", "Andrei Sakharov"}

Answer: Time flow on this side of the Universe is in the (+t) direction, while time flow on the dual side of the Universe is in the (-t) direction. The math all works out. Time flow is bi-directional, but on opposite sides of the Big-Bang! **Universal CPT Symmetry.**



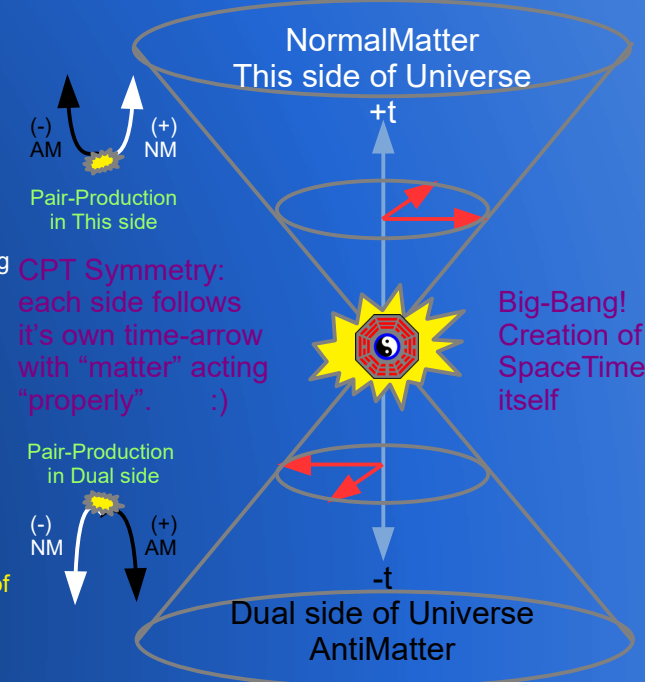
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$$\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$$

Det $[\Lambda^{\mu}_{\nu}] = \pm 1$ **$\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$**



CPT Symmetry: each side follows it's own time-arrow with "matter" acting "properly". :)

Big-Bang! Creation of SpaceTime itself

This gives total CPT Symmetry to all of the possible Lorentz Transforms (AM=Antimatter, NM=Normal Matter):
 Various (AM_Flips) : Various (NM_Flips)
 -Infinity...(AM_Boosts)...(AM_Identity=-4)...(AM_Rotations)...0...(NM_Rotations)...(+4=NM_Identity)...(NM_Boosts)...+Infinity

This solves the:
Baryon (Matter-AntiMatter) Asymmetry Problem & Arrow(s)-of-Time Problem (+ / -)

Lorentz Transform Connection Map – Interpretations 2

CPT, Big-Bang, (Matter-AntiMatter), Arrows-of-Time

A Tensor Study of Physical 4-Vectors

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This idea of Universal CPT Symmetry also gives a Universal Dimensional Symmetry as well.

Consider the well known “balloon” analogy of the universe expansion. The “spatial” coordinates are on the surface of the balloon, and the expansion is in the +t direction. There is symmetry in the +/- directions of the spatial coordinates, but the time flow is always uni-directional, +t, as the balloon gets bigger.

By allowing a “dual side”, it provides a universal dimensional symmetry. One now has +/- symmetry for the temporal directions.

The “center” of the Universe is literally, the Big Bang Singularity. It is the “center=zero” point of both time and space directions.

The expansion gives time flow away from Big Bang singularity in both the Normal Side (+) and the Dual “Side (-). The spatial coordinates expand in both the (+/-) directions on both sides.

Note that this gives an unusual interpretation of what came “before” the Big Bang. The “past” on either side extends only to the BB singularity, not beyond. Time flow is always away from this creation singularity.

This is also in accord with known black hole physics, in that all matter entering a BH ends at the BH singularity. Time and space coordinates both come to a stop at either type of singularity, from the point of view of an observer that is in the spacetime but not at the singularity.

So, the Big Bang is a “starting” singularity, and black holes are “ending” singularities. Also provides for idea of “white holes” actually just being black holes on the alternate side. White hole=time-reversed black hole. This way, the mass is still attractive. Time flow is simply reversed on the alternate side so stuff still goes into the hole...

So, **Universal CPT Symmetry = Universal Dimensional Symmetry.**

And, going even further, I suspect this is the reason there is a duality in Metric conventions. In other words, physicists have wondered why one can use {+, -, -, -} or {-, +, +, +}. I submit that one of these metrics applies to the Normal Matter side, while the other complementarily applies to the Dual side. This would allow correct causality conditions to apply on either side. Again, this is similar to the Dirac prediction of antimatter based on a duality of possible solutions.



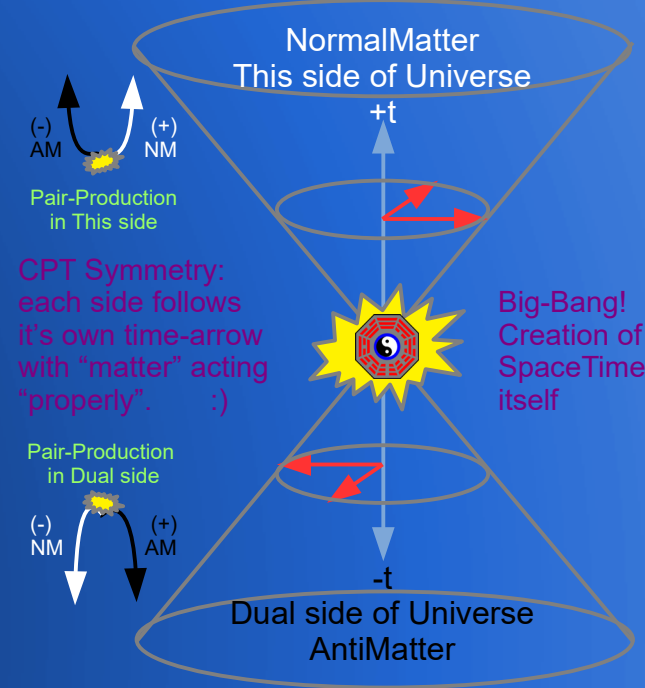
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 -Infinity...(AM_Boosts)...(AM_Identity=-4)...(AM_Rotations)...0...(NM_Rotations)...(+4=NM_Identity)...(NM_Boosts)...+Infinity

This solves the:
 Baryon (Matter-AntiMatter) Asymmetry Problem
 & Arrow(s)-of-Time Problem (+ / -)

SRQM Study: Model SpaceTimes

A Tensor Study
of Physical 4-Vectors

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Model SpaceTimes	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
Klein Geometry G/H			
Lorentzian pseudo-Riemannian	Anti de Sitter SO(3,2)/SO(3,1)	Minkowski ISO(3,1)/SO(3,1) $ds^2 = (cdt)^2 - \mathbf{dx} \cdot \mathbf{dx}$	De Sitter SO(4,1)/SO(3,1)
Riemannian	Hyperbolic SO(4,1)/SO(4)	Euclidean ISO(4)/SO(4) $ds^2 = (cdt)^2 + \mathbf{dx} \cdot \mathbf{dx}$	Spherical SO(5)/SO(4)

Lie Groups

de Sitter Group SO(1,4)
de Sitter invariant relativity
(maybe?)

Poincaré Group ISO(1,3)
{ $r \ll r_{ds} = \text{de Sitter Radius}$ }
 $r_{ds} = \sqrt{3/\Lambda} = L_H/\sqrt{|\Omega_\Lambda|}$

SR & GR Physics
(** *currently thought correct* **)

$$\Lambda^{\mu'}_{\nu} \rightarrow B^{\mu'}_{\nu} = \text{Boost}$$

\underline{t}	\underline{x}	\underline{y}	\underline{z}
\underline{t}	\underline{x}	\underline{y}	\underline{z}
γ	$-\beta\gamma$	0	0
$-\beta\gamma$	γ	0	0
0	0	1	0
0	0	0	1

Galilei Group

{ $|\mathbf{v}| \ll c$ }
Classical Physics

$$\Lambda^{\mu'}_{\nu} \rightarrow S^{\mu'}_{\nu} = \text{Motion: Shear}$$

\underline{t}	\underline{x}	\underline{y}	\underline{z}
\underline{t}	\underline{x}	\underline{y}	\underline{z}
1	0	0	0
$-\beta$	1	0	0
0	0	1	0
0	0	0	1

SRQM Transforms: Venn Diagram

Poincaré = Lorentz + Translations

(10)

(6)

(4)

A Tensor Study of Physical 4-Vectors

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Transformations

(# of independent parameters = # continuous symmetries = # Lie Dimensions)

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation

Lie group of all affine isometries of SR:Minkowski TimeSpace (preserve quadratic form $\eta_{\mu\nu}$)

General Linear, Affine Transform $X^\mu = \Lambda^\mu_\nu X^\nu + \Delta X^\mu$ with $\text{Det}[\Lambda^\mu_\nu] = \pm 1$

(6+4=10)

Lorentz Transform Λ^μ_ν

(3+3=6)

4-Tensor {mixed type-(1,1)}

Discrete

Continuous

Time-reversal
 $\Lambda^\mu_\nu \rightarrow T^\mu_\nu$
(0)
 $t \rightarrow -t^*$
time parity
anti-unitary

Spatial Flip Combs
 $\Lambda^\mu_\nu \rightarrow F^\mu_\nu$
(0)
 $\{x|y|z\} \rightarrow -\{x|y|z\}$
unitary

Rotation
 $\Lambda^\mu_\nu \rightarrow R^\mu_\nu$
(3)
 $x:y | x:z | y:z$

Parity-Inversion
 $\Lambda^\mu_\nu \rightarrow P^\mu_\nu$
(0)
 $\mathbf{r} \rightarrow -\mathbf{r}$
space parity
unitary

Identity $I_{(4)}$
 $\Lambda^\mu_\nu \rightarrow \eta^\mu_\nu = \delta^\mu_\nu$
(0)
no mixing
unitary

Boost
 $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$
(3)
 $t:x | t:y | t:z$

Charge-Conjugation
 $\Lambda^\mu_\nu \rightarrow C^\mu_\nu$
(0)
 $\mathbf{R} \rightarrow -\mathbf{R}^*, q \rightarrow -q$
charge parity
anti-unitary

CPT Symmetry
{Charge}
{Parity}
{Time}

Isotropy
{same all directions}

Translation Transform ΔX^μ

(1+3=4)

4-Vector

Discrete

Continuous

4-Zero
 $\Delta X^\mu \rightarrow (0,0)$
(0)
no motion

Temporal
 $\Delta X^\mu \rightarrow (c\Delta t, \mathbf{0})$
(1)
 Δt

Spatial
 $\Delta X^\mu \rightarrow (0, \Delta \mathbf{x})$
(3)
 $\Delta x | \Delta y | \Delta z$

Homogeneity
{same all points}

	M^{01}	M^{02}	M^{03}		P^0
M^{10}		M^{12}	M^{13}		P^1
M^{20}	M^{21}		M^{23}		P^2
M^{30}	M^{31}	M^{32}			P^3

4-AngularMomentum $M^{\mu\nu} = X^\mu \wedge P^\nu = X^\mu P^\nu - X^\nu P^\mu$
= Generator of Lorentz Transformations (6)
= { $\Lambda^\mu_\nu \rightarrow R^\mu_\nu$ Rotations (3) + $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$ Boosts (3) }

4-LinearMomentum P^μ
= Generator of Translation Transformations (4)
= { $\Delta X^\mu \rightarrow (c\Delta t, \mathbf{0})$ Time (1) + $\Delta X^\mu \rightarrow (0, \Delta \mathbf{x})$ Space (3) }

$\text{Det}[\Lambda^\mu_\nu] = +1$ for Proper Lorentz Transforms
 $\text{Det}[\Lambda^\mu_\nu] = -1$ for Improper Lorentz Transforms

Lorentz Matrices can be generated by a matrix M with $\text{Tr}[M]=0$ which gives:
{ $\Lambda = e^\wedge M = e^\wedge (+\theta \cdot \mathbf{J} - \zeta \cdot \mathbf{K})$ }
{ $\Lambda^T = (e^\wedge M)^T = e^\wedge M^T$ }
{ $\Lambda^{-1} = (e^\wedge M)^{-1} = e^\wedge -M$ }



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$$\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$$

$$\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$$

$$\text{Det}[\Lambda^\mu_\nu] = \pm 1 \quad \Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$$

Rotations $J_i = -\epsilon_{imn} M^{mn} / 2$, Boosts $K_i = M_{i0}$

[$(\mathbf{R} \rightarrow -\mathbf{R}^*)$] or [$(t \rightarrow -t^*)$ & $(\mathbf{r} \rightarrow -\mathbf{r})$] imply $q \rightarrow -q$
Feynman-Stueckelberg Interpretation
Amusingly, Inhomogeneous Lorentz adds homogeneity.

Classical Transforms: Venn Diagram

Full Galilean = Galilean + Translations

(10)

(6)

(4)

Transformations

(# of independent parameters = # continuous symmetries = # Lie Dimensions)

Galilean Transformation Group aka. Inhomogeneous Galilean Transformation

Lie group of all affine isometries of Classical:Euclidean Time + Space (preserve quadratic form δ_{ij})

General Linear, Affine Transform $X^{\mu'} = \Lambda^{\mu'}_{\nu} X^{\nu} + \Delta X^{\mu'}$ with $\text{Det}[\Lambda^{\mu'}_{\nu}] = \pm 1$

(6+4=10)

Galilean Transform $\Lambda^{\mu'}_{\nu}$

(3+3=6)

4-Tensor {mixed type-(1,1)}

Discrete

Continuous

Time-reversal
 $\Lambda^{\mu'}_{\nu} \rightarrow T^{\mu'}_{\nu}$
(0)
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time parity
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 $\Lambda^{\mu'}_{\nu} \rightarrow F^{\mu'}_{\nu}$
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 $\Lambda^{\mu'}_{\nu} \rightarrow R^{\mu'}_{\nu}$
(3)
 $x:y | x:z | y:z$

Motion: Shear
 $\Lambda^{\mu'}_{\nu} \rightarrow S^{\mu'}_{\nu}$
(3)
 $t:x | t:y | t:z$

Isotropy
{same all directions}

Translation Transform $\Delta X^{\mu'}$

(1+3=4)

4-Vector

Discrete

Continuous

4-Zero
 $\Delta X^{\mu'} \rightarrow (0,0)$
(0)
no motion

Temporal
 $\Delta X^{\mu'} \rightarrow (c\Delta t, 0)$
(1)
 Δt

Spatial
 $\Delta X^{\mu'} \rightarrow (0, \Delta \mathbf{x})$
(3)
 $\Delta x | \Delta y | \Delta z$

Homogeneity
{same all points}

Lie Groups

de Sitter Group SO(1,4)
de Sitter invariant relativity
(maybe?)

Poincaré Group ISO(1,3)
{ $r \ll r_{ds} = \text{de Sitter Radius}$ }
 $r_{ds} = \sqrt{[3/\Lambda]} = L_H/\sqrt{[\Omega_\Lambda]}$

SR & GR Physics
(** currently thought correct **)

$\Lambda^{\mu'}_{\nu} \rightarrow B^{\mu'}_{\nu} =$ Boost

t	x	y	z
t	γ	$-\beta\gamma$	0
x	$-\beta\gamma$	γ	0
y	0	0	1
z	0	0	1

Galilei Group
{ $|\mathbf{v}| \ll c$ }
Classical Physics

$\Lambda^{\mu'}_{\nu} \rightarrow S^{\mu'}_{\nu} =$ Motion: Shear

t	x	y	z
t	1	0	0
x	$-\beta$	1	0
y	0	0	1
z	0	0	1

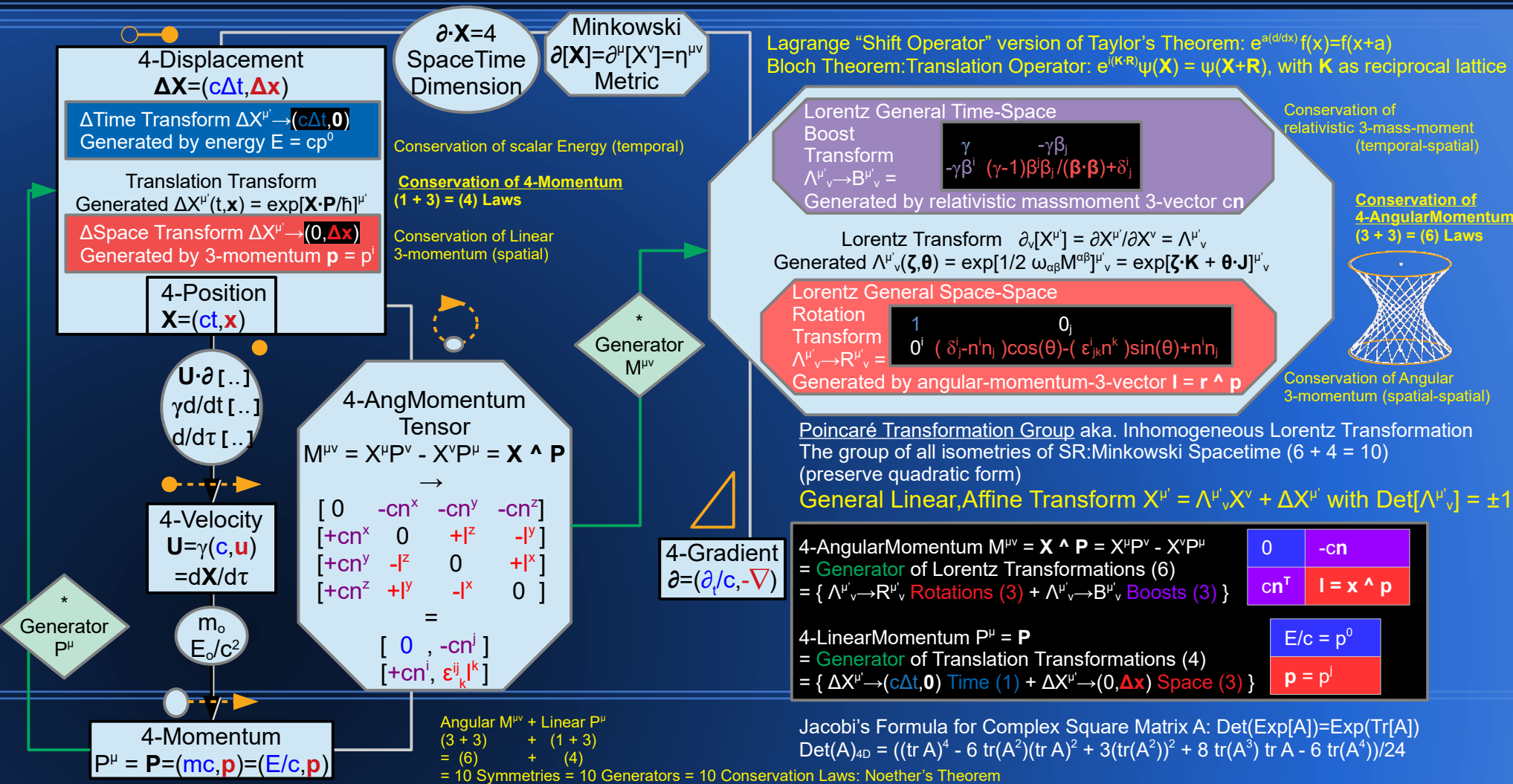
Review of SR Transforms

10 Poincaré Symmetries, 10 Conservation Laws

10 Generators : Noether's Theorem

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Review of SR Transforms

Poincaré Algebra & Generators

Casimir Invariants

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The (10) one-parameter groups can be expressed directly as exponentials of the generators:

- $U[1, (a^0, \mathbf{0})] = e^{i(a^0 \cdot H)} = e^{i(a^0 \cdot P^0)}$: (1) Hamiltonian = Energy = Temporal Momentum H
- $U[1, (0, \lambda \hat{\mathbf{a}})] = e^{i(-i\lambda \hat{\mathbf{a}} \cdot \mathbf{p})}$: (2) Linear Momentum p
- $U[\Lambda(i\lambda \hat{\boldsymbol{\theta}}/2), 0] = e^{i(\lambda \hat{\boldsymbol{\theta}} \cdot \mathbf{j})}$: (3) Angular Momentum j
- $U[\Lambda(\lambda \hat{\boldsymbol{\phi}}/2), 0] = e^{i(\lambda \hat{\boldsymbol{\phi}} \cdot \mathbf{k})}$: (3) Lorentz Boost k

The Poincaré Algebra is the Lie Algebra of the Poincaré Group:

Total of (1+3+3+3 = 4+6 = 10) Invariances from Poincaré Symmetry

Poincaré Algebra is the Lie Algebra of the Poincaré Group.

	$M^{01} = -cn^1$	$M^{02} = -cn^2$	$M^{03} = -cn^3$	P^0
$M^{10} = cn^1$		$M^{12} = l^3$	$M^{13} = -l^2$	P^1
$M^{20} = cn^2$	$M^{21} = -l^3$		$M^{23} = l^1$	P^2
$M^{30} = cn^3$	$M^{31} = l^2$	$M^{32} = -l^1$		P^3

Covariant form:

These are the commutators of the the Poincaré Algebra :

- $[X^\mu, X^\nu] = 0^{\mu\nu}$
- $[P^\mu, P^\nu] = -i\hbar q(F^{\mu\nu})$ if interacting with EM field; otherwise = $0^{\mu\nu}$ for free particles
- $M^{\mu\nu} = (X^\mu P^\nu - X^\nu P^\mu) = i\hbar(X^\mu \partial^\nu - X^\nu \partial^\mu)$
- $[M^{\mu\nu}, P^\rho] = i\hbar(\eta^{\rho\nu} P^\mu - \eta^{\rho\mu} P^\nu)$
- $[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar(\eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\sigma\nu} M^{\rho\mu} + \eta^{\rho\mu} M^{\sigma\nu})$

$$M^{\mu\nu} = \mathbf{X} \wedge \mathbf{P} = X^\mu P^\nu - X^\nu P^\mu$$

$$\mathbf{P} = \mathbf{P}$$

0	-cn
cn ^T	$\mathbf{l} = \mathbf{x} \wedge \mathbf{p}$

$E/c = p^0$
$\mathbf{p} = p^j$

M = Generator of Lorentz Transformations (6) = { Rotations (3) + Boosts (3) }
 P = Generator of Translation Transformations (4) = { Time (1) + Space (3) }

Rotations $J_i = -\epsilon_{imn} M^{mn}/2$, Boosts $K_i = M_{i0}$

The set of all Lorentz Generators $V = \{\boldsymbol{\zeta} \cdot \mathbf{K} + \boldsymbol{\theta} \cdot \mathbf{J}\}$ forms a vector space over the real numbers. The generators $\{J_x, J_y, J_z, K_x, K_y, K_z\}$ form a basis set of V. The components of the axis-angle vector and rapidity vector $\{\theta_x, \theta_y, \theta_z, \zeta_x, \zeta_y, \zeta_z\}$ are the coordinates of a Lorentz generator wrt this basis.

Very importantly, the Poincaré group has Casimir Invariant Eigenvalues = { Mass m, Spin j }, hence Mass *and* Spin are purely SR phenomena, no QM axioms required!

This Representation of the Poincaré Group or Representation of the Lorentz Group is known as Wigner's Classification in Representation Theory of Particle Physics

Component form: Rotations $J_i = -\epsilon_{imn} M^{mn}/2$, Boosts $K_i = M_{i0}$

- $[J_m, P_n] = i\epsilon_{mnp} P^k$
- $[J_m, P_0] = 0$
- $[K_j, P_k] = i\eta_{jk} P^0$
- $[K_j, P_0] = -iP_j$
- $[J_m, J_n] = i\epsilon_{mnp} J^k$
- $[J_m, K_n] = i\epsilon_{mnp} K^k$
- $[K_m, K_n] = -i\epsilon_{mnp} J^k$, a Wigner Rotation resulting from consecutive boosts
- $[J_m + iK_m, J_n - iK_n] = 0$

Poincaré Algebra has 2 Casimir Invariants = Operators that commute with all of the Poincaré Generators

These are $\{P^2 = P^\mu P_\mu = (m_0 c)^2, W^2 = W^\mu W_\mu = -(m_0 c)^2 j(j+1)\}$, with $W^\mu = (-1/2)\epsilon^{\mu\nu\rho\sigma} J_\nu P_\rho$ as the Pauli-Lubanski Pseudovector

- $[P^2, P^0] = [P^2, P^i] = [P^2, J^i] = [P^2, K^i] = 0$: Hence the 4-Momentum Magnitude squared commutes with all Poincaré Generators
- $[W^2, P^0] = [W^2, P^i] = [W^2, J^i] = [W^2, K^i] = 0$: Hence the 4-SpinMomentum Magnitude squared commutes with all Poincaré Generators

10 Poincaré Symmetry Invariances

Noether's Theorem: 10 SR Conservation Laws

A Tensor Study of Physical 4-Vectors

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d'Alembertian Invariant Wave Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = (\partial_t/c)^2$

Time Translation:

Let $\mathbf{X}_T = (ct+c\Delta t, \mathbf{x})$, then $\partial[\mathbf{X}_T] = (\partial_t/c, -\nabla)(ct+c\Delta t, \mathbf{x}) = \text{Diag}[1, -1] = \partial[\mathbf{X}] = \eta^{\mu\nu}$

so $\partial[\mathbf{X}_T] = \partial[\mathbf{X}]$ and $\partial[\mathbf{K}] = [[0]]$

$(\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_T] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}_T]) = \partial[\mathbf{K}] \cdot \mathbf{X}_T + \mathbf{K} \cdot \partial[\mathbf{X}_T] = 0 + \mathbf{K} \cdot \partial[\mathbf{X}] = \partial[\mathbf{K}] \cdot \mathbf{X} + \mathbf{K} \cdot \partial[\mathbf{X}] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}]) = (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}]$:

Space Translation:

Let $\mathbf{X}_S = (ct, \mathbf{x} + \Delta \mathbf{x})$, then $\partial[\mathbf{X}_S] = (\partial_t/c, -\nabla)(ct, \mathbf{x} + \Delta \mathbf{x}) = \text{Diag}[1, -1] = \partial[\mathbf{X}] = \eta^{\mu\nu}$

so $\partial[\mathbf{X}_S] = \partial[\mathbf{X}]$ and $\partial[\mathbf{K}] = [[0]]$

$(\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_S] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}_S]) = \partial[\mathbf{K}] \cdot \mathbf{X}_S + \mathbf{K} \cdot \partial[\mathbf{X}_S] = 0 + \mathbf{K} \cdot \partial[\mathbf{X}] = \partial[\mathbf{K}] \cdot \mathbf{X} + \mathbf{K} \cdot \partial[\mathbf{X}] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}]) = (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}]$:

Lorentz Space-Space Rotation:

Let $\mathbf{X}_R = (ct, R[\mathbf{x}])$, then $\partial[\mathbf{X}_R] = (\partial_t/c, -\nabla)(ct, R[\mathbf{x}]) = \text{Diag}[1, -1] = \partial[\mathbf{X}] = \eta^{\mu\nu}$

so $\partial[\mathbf{X}_R] = \partial[\mathbf{X}]$ and $\partial[\mathbf{K}] = [[0]]$

$(\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_R] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}_R]) = \partial[\mathbf{K}] \cdot \mathbf{X}_R + \mathbf{K} \cdot \partial[\mathbf{X}_R] = 0 + \mathbf{K} \cdot \partial[\mathbf{X}] = \partial[\mathbf{K}] \cdot \mathbf{X} + \mathbf{K} \cdot \partial[\mathbf{X}] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}]) = (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}]$:

Lorentz Time-Space Boost:

Let $\mathbf{X}_B = \gamma(ct - \beta \cdot \mathbf{x}, -\beta ct + \mathbf{x})$, then $\partial[\mathbf{X}_B] = (\partial_t/c, -\nabla)\gamma(ct - \beta \cdot \mathbf{x}, -\beta ct + \mathbf{x}) = [[\gamma, -\gamma\beta], [-\gamma\beta, \gamma]] = \Lambda^{\mu\nu}$

$\partial[\mathbf{K} \cdot \mathbf{X}_B] = \partial[\mathbf{K}] \cdot \mathbf{X}_B + \mathbf{K} \cdot \partial[\mathbf{X}_B] = \Lambda^{\mu\nu} \mathbf{K} = \mathbf{K}_B =$ a Lorentz Boosted \mathbf{K} , as expected

$\partial \cdot \mathbf{K}_B = \partial \cdot \Lambda^{\mu\nu} \mathbf{K} = \Lambda_{\mu\nu}(\partial \cdot \mathbf{K}) = \Lambda^{\mu\nu}(0) = 0 = \partial \cdot \mathbf{K} =$ Divergence of $\mathbf{K} = 0$, as expected

$(\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_B] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}_B]) = \partial \cdot \mathbf{K}_B = \partial \cdot \mathbf{K} = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}]) = (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}]$:

SR Waves:

Let $\Psi = ae^{\Lambda \cdot i(\mathbf{K} \cdot \mathbf{X})}$, $\Psi_T = ae^{\Lambda \cdot i(\mathbf{K} \cdot \mathbf{X}_T)}$, $\Psi_S = ae^{\Lambda \cdot i(\mathbf{K} \cdot \mathbf{X}_S)}$, $\Psi_R = ae^{\Lambda \cdot i(\mathbf{K} \cdot \mathbf{X}_R)}$, $\Psi_B = ae^{\Lambda \cdot i(\mathbf{K} \cdot \mathbf{X}_B)}$

$(\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_T] = (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_S] = (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_R] = (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_B] = (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}]$: Wave Equation Invariant under all Poincaré transforms

Total of (1+3+3+3 = 10) Invariances from Poincaré Symmetry

4-Gradient
 $\partial = (\partial_t/c, -\nabla)$
 $= (\partial_t/c, -\partial_x, -\partial_y, -\partial_z)$
 $= (\partial/c \partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$

Invariant d'Alembertian Wave Equation
 $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla$

Time Translation Invariance (1)
 Conservation of Energy = (Temporal) Momentum E
Temporal part of $P^\mu = (E/c, \mathbf{p})$

Space Translation Invariances (3)
 Conservation of Linear (Spatial) Momentum \mathbf{p}
Spatial part of $P^\mu = (E/c, \mathbf{p})$

Lorentz Space-Space Rotation Invariances (3)
 Conservation of Angular (Spatial) Momentum \mathbf{l}
Spatial-Spatial part of $M^{\mu\nu} = \mathbf{X} \wedge \mathbf{P}$

Lorentz Time-Space Boost Invariances (3)
 Conservation of Relativistic Mass-Moment \mathbf{n}
Temporal-Spatial part of $M^{\mu\nu} = \mathbf{X} \wedge \mathbf{P}$
 see Wikipedia: Relativistic Angular Momentum

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 =$ Lorentz Scalar

SR 4-Vector Magnitudes

Dot Product, Lorentz Scalar Product

Einstein Summation Convention

A Tensor Study of Physical 4-Vectors

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An example of the magnitude of a 3-vector is the length of a 3-displacement $\Delta \mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_0)$.


Examine 3-position $\mathbf{r}_1 \rightarrow \mathbf{r} = (x,y,z)$, which is a 3-displacement with the base at the origin $\mathbf{r}_0 \rightarrow \mathbf{0} = (0,0,0)$.

The Dot Product of \mathbf{r} , $\{\mathbf{r} \cdot \mathbf{r} = r^j \delta_{jk} r^k = r_k r^k = r_j r_j = (x^2 + y^2 + z^2) = (r^2)\}$ is the Pythagorean Theorem.

The Kronecker Delta $\delta_{jk} = \text{Diag}[1,1,1] = I_{(3)}$.

The magnitude is $\sqrt{[\mathbf{r} \cdot \mathbf{r}]} = \sqrt{[r^2]} = |\mathbf{r}|$. 3D magnitudes are always positive.

Galilean Invariant
 $\mathbf{r} \cdot \mathbf{r} = (x)^2 + (y)^2 + (z)^2 = (r^2)$
length r

 3-position
 $\mathbf{r} = r^i \rightarrow (x,y,z) = \langle \text{location} \rangle$

The magnitude of a 4-Vector is very similar to the magnitude of a 3-vector, but there are some interesting differences.

One uses the Lorentz Scalar Product, a 4D Dot Product, which includes a time component, and is based on the

SR:Minkowski Metric Tensor. I typically use the "Particle Physics" convention of the Minkowski Metric

$\eta_{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1]$ {Cartesian form}, with the other entries zero.

SR:Minkowski Metric
 $\partial[\mathbf{R}] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$
 $\text{Diag}[1,-1,-1,-1] = \text{Diag}[1,-I_{(3)}] = \text{Diag}[1,-\delta^{jk}]$
{in Cartesian form} "Particle Physics" Convention
 $\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu^\nu = \delta_\mu^\nu$ $\text{Tr}[\eta^{\mu\nu}] = 4$

$$\mathbf{A} \cdot \mathbf{A}' = \mathbf{A} \cdot \mathbf{A} = A^\mu \eta_{\mu\nu} A^\nu = A_\nu A^\nu = A^\mu A_\mu = \sum_{\nu=0..3} [a_\nu a^\nu] = (a_0 a^0 + a_1 a^1 + a_2 a^2 + a_3 a^3) = \sum_{\nu=0..3} [a^\nu a_\nu]$$

$$= (a^0 a^0 - a^1 a^1 - a^2 a^2 - a^3 a^3) = (a^0 a^0 - \mathbf{a} \cdot \mathbf{a})$$

using the Einstein summation convention where upper-lower paired indices are summed over.

Lorentz Invariant
 $\mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct)^2$
Interval $c\tau$

4-Position
 $\mathbf{R} = R^\mu = (r^\mu) = (ct, \mathbf{r}) = \langle \text{Event} \rangle$

SR:Lorentz Transform
 $\partial_\nu [R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu$
 $\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$
 $\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$
 $\text{Det}[\Lambda^\mu_\nu] = \pm 1$ $\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$

$\mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct)^2 - (x^2 + y^2 + z^2) = (c\Delta\tau)^2$
for 4-Position $\mathbf{R} = (ct, \mathbf{r})$
4D magnitudes can be negative, zero, positive

SpaceTime Dimension
 $\partial \cdot \mathbf{R} = \partial_\mu R^\mu = 4$
Dimension

The 4-Vector version has the Pythagorean element in the spatial components, the temporal component is of opposite sign. This gives a "causality condition", where SpaceTime intervals (in the [+,-,-,-] metric) can be:

$\Delta \mathbf{R} \cdot \Delta \mathbf{R} = [(c\Delta t)^2 - \Delta \mathbf{r} \cdot \Delta \mathbf{r}] =$	$(c\Delta\tau)^2$ Time-like: Temporal	(+) {causal = temporally-ordered}
	(0) Light-like: Null: Photonic	(0) {causal, maximum signal speed ($ \Delta \mathbf{r} / \Delta t = c$)}
	$-(\Delta r_0)^2$ Space-like: Spatial	(-) {non-causal, spatially-extended}

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Classical (scalar)
Galilean Invariant

3-vector
Not Lorentz Invariant

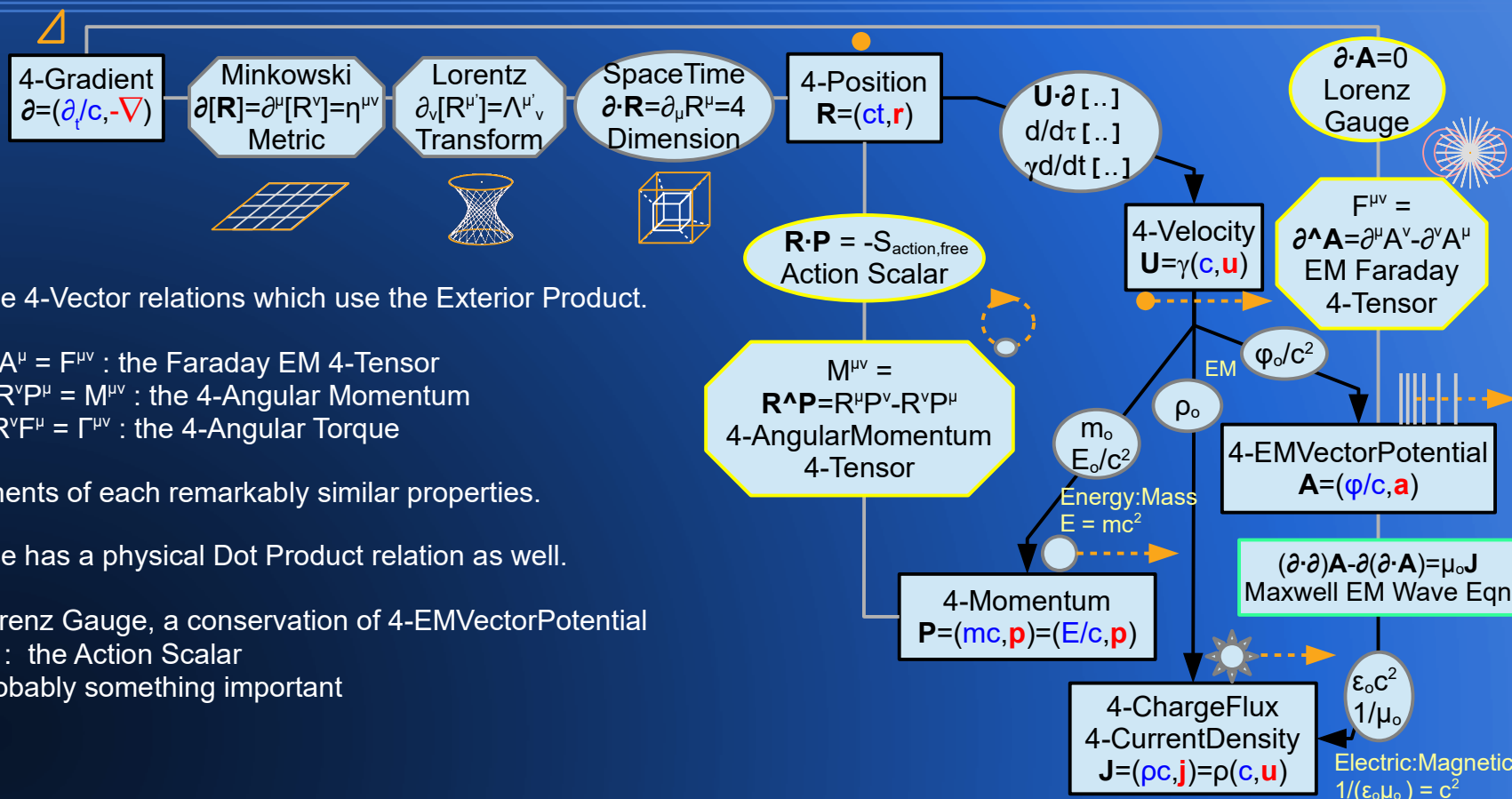
$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)$
= Lorentz Scalar

SRQM Study:

Lorentz Scalar Product $A \cdot B = A_\mu B^\mu$ Exterior Product $A \wedge B = A^\mu B^\nu - A^\nu B^\mu$

A Tensor Study of Physical 4-Vectors

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There are at least three 4-Vector relations which use the Exterior Product.

- $\partial \wedge A = \partial^\mu \wedge A^\nu = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$: the Faraday EM 4-Tensor
- $R \wedge P = R^\mu \wedge P^\nu = R^\mu P^\nu - R^\nu P^\mu = M^{\mu\nu}$: the 4-Angular Momentum
- $R \wedge F = R^\mu \wedge F^\nu = R^\mu F^\nu - R^\nu F^\mu = \Gamma^{\mu\nu}$: the 4-Angular Torque

This gives the components of each remarkably similar properties.

Likewise, each of these has a physical Dot Product relation as well.

- $\partial \cdot A = \partial_\mu A^\mu = 0$: the Lorenz Gauge, a conservation of 4-EMVectorPotential
- $R \cdot P = R_\mu P^\mu = -S_{\text{action, free}}$: the Action Scalar
- $R \cdot F = R_\mu F^\mu = ???$: probably something important

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Study:

4-Momentum, 4-Force

4-AngularMomentum, 4-Torque

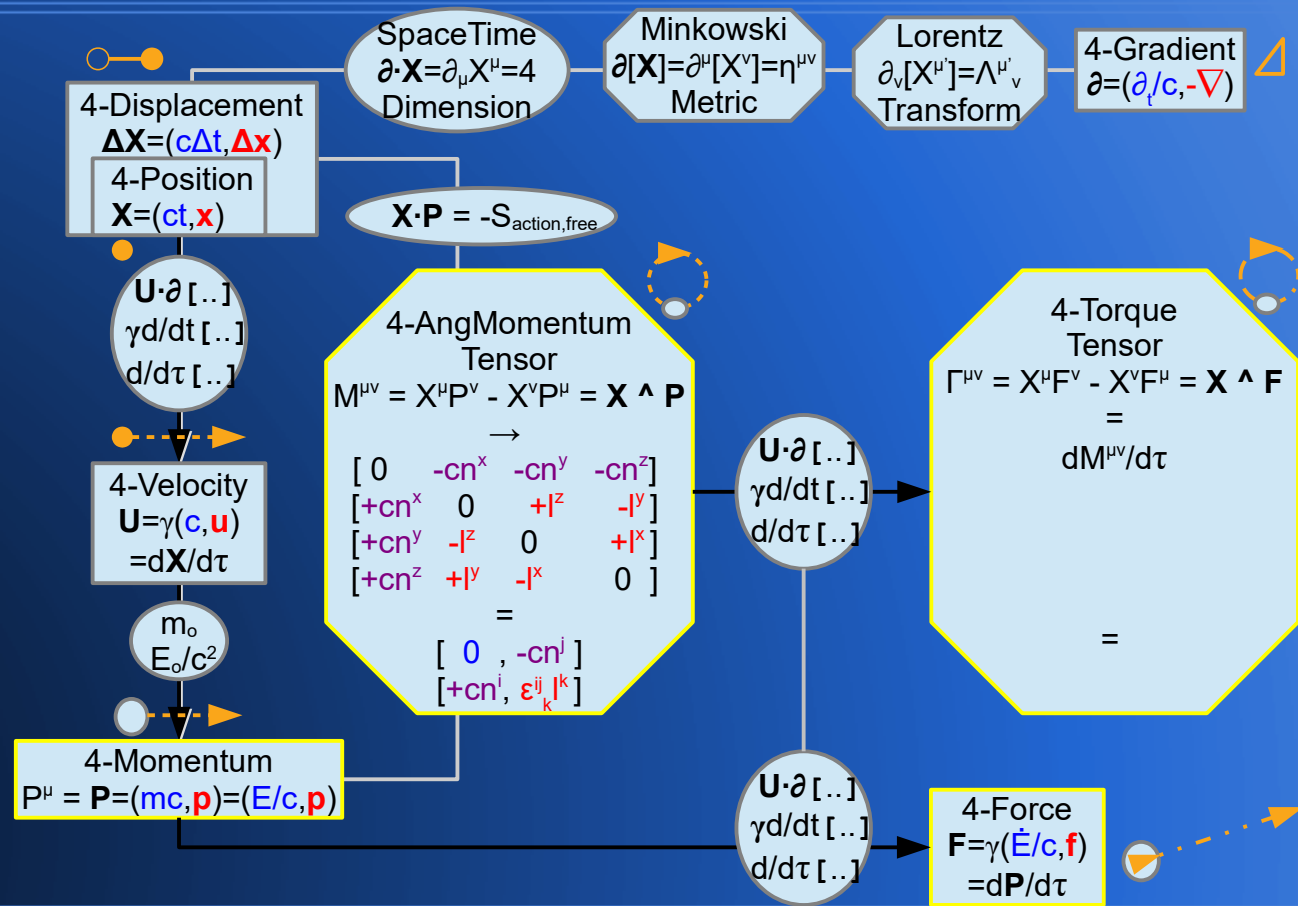
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Linear:
4-Force is the ProperTime Derivative of 4-Momentum.

Angular:
4-Torque is the ProperTime Derivative of 4-AngularMomentum.

$$\begin{aligned}
 & d/d\tau [M^{\mu\nu}] \\
 &= d/d\tau [X^\mu P^\nu - X^\nu P^\mu] \\
 &= [U^\mu P^\nu + X^\mu F^\nu - U^\nu P^\mu - X^\nu F^\mu] \\
 &= [U^\mu m_0 U^\nu + X^\mu F^\nu - U^\nu m_0 U^\mu - X^\nu F^\mu] \\
 &= [U^\mu m_0 U^\nu - U^\nu m_0 U^\mu + X^\mu F^\nu - X^\nu F^\mu] \\
 &= [m_0 (U^\mu U^\nu - U^\nu U^\mu) + X^\mu F^\nu - X^\nu F^\mu] \\
 &= [m_0 (0^{\mu\nu}) + X^\mu F^\nu - X^\nu F^\mu] \\
 &= [X^\mu F^\nu - X^\nu F^\mu] \\
 & d/d\tau [M^{\mu\nu}] = \Gamma^{\mu\nu} = [X^\mu F^\nu - X^\nu F^\mu] = \mathbf{X} \wedge \mathbf{F}
 \end{aligned}$$



SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or T_{μ}^ν (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
---------------------------------------------------------------------------------------------------------------------------	--------------------------------------------------------------------------------------------------------------------------------------------------	----------------------------------------------------------

$$\begin{aligned}
 \text{Trace}[T^{\mu\nu}] &= \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T \\
 \mathbf{V} \cdot \mathbf{V} &= V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 \\
 &= \text{Lorentz Scalar}
 \end{aligned}$$

SR Minkowski SpaceTime

4-Vectors, 4-CoVectors, Scalars, Tensors

Invariant Lorentz Scalar Product

A Tensor Study of Physical 4-Vectors

4-Vectors are actually tensorial entities of Minkowski SpaceTime, (1,0)-Tensors, which maintain covariance for inertial observers, meaning that they may have different components for different observers, but describe the same physical object. (like viewing a sculpture from different angles – snapshots look different but it's actually the same object)
There are also 4-CoVectors, or One-Forms, which are (0,1)-Tensors and dual to 4-Vectors.

Both GR and SR use a metric tensor $g^{\mu\nu}$ to describe measurements in SpaceTime.
SR uses the “flat” Minkowski Metric $g^{\mu\nu} \rightarrow \eta^{\mu\nu} = \eta_{\mu\nu} \rightarrow \text{Diag}[1, -\mathbf{I}_3] = \text{Diag}[1, -\delta^{jk}] = \text{Diag}[1, -1, -1, -1]$ {Cartesian form}, which is the {curvature ~ 0 limit = low-mass limit} of the GR metric $g^{\mu\nu}$.

4-Vectors = (1,0)-Tensors

$$\mathbf{A} = A^\mu = (a^\mu) = (a^0, \mathbf{a}') = (a^0, \mathbf{a}) = (a^0, a^1, a^2, a^3) \rightarrow (a^t, \mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^z)$$

$$\mathbf{B} = B^\mu = (b^\mu) = (b^0, \mathbf{b}') = (b^0, \mathbf{b}) = (b^0, b^1, b^2, b^3) \rightarrow (b^t, \mathbf{b}^x, \mathbf{b}^y, \mathbf{b}^z)$$

4-CoVectors = (0,1)-Tensors

$$A_\mu = (a_\mu) = (a_0, \mathbf{a}_\mu) = (a_0, -\mathbf{a}) = (a_0, a_1, a_2, a_3) \rightarrow (a_t, \mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z) \quad \text{where } A_\mu = \eta_{\mu\nu} A^\nu \text{ and } A^\mu = \eta^{\mu\nu} A_\nu$$

$$= (a_0, \mathbf{a}_\mu) = (a_0, -\mathbf{a}) = (a^0, -\mathbf{a}') = (a^0, -a^1, -a^2, -a^3) \rightarrow (a^t, -\mathbf{a}^x, -\mathbf{a}^y, -\mathbf{a}^z)$$

$$B_\mu = (b_\mu) = (b_0, \mathbf{b}_\mu) = (b_0, -\mathbf{b}) = (b_0, b_1, b_2, b_3) \rightarrow (b_t, \mathbf{b}_x, \mathbf{b}_y, \mathbf{b}_z) \quad \text{where } B_\mu = \eta_{\mu\nu} B^\nu \text{ and } B^\mu = \eta^{\mu\nu} B_\nu$$

$$= (b_0, \mathbf{b}_\mu) = (b^0, -\mathbf{b}') = (b^0, -b^1, -b^2, -b^3) \rightarrow (b^t, -\mathbf{b}^x, -\mathbf{b}^y, -\mathbf{b}^z)$$

Index raising & lowering

$$\mathbf{A} \cdot \mathbf{B}' = \mathbf{A} \cdot \mathbf{B} = A^\mu \eta_{\mu\nu} B^\nu = A_\nu B^\nu = A^\mu B_\mu = \sum_{\nu=0..3} [a_\nu b^\nu] = \sum_{\mu=0..3} [a^\mu b_\mu] = (a^0 b^0 - \mathbf{a} \cdot \mathbf{b}) = (a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3)$$

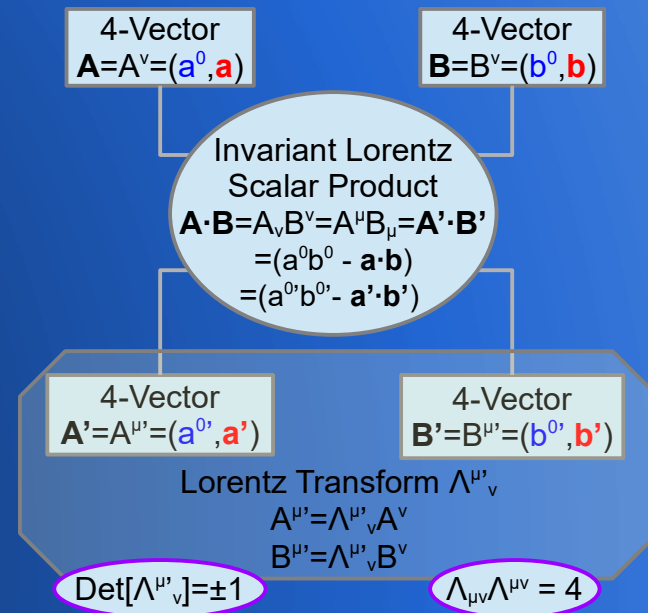
using the Einstein summation convention where upper-lower paired indices are summed over

Proof that this is an invariant:

$$\mathbf{A} \cdot \mathbf{B}' = A^\mu \eta_{\mu\nu} B'^\nu = (\Lambda^\mu_\alpha A^\alpha) \eta_{\mu\nu} (\Lambda^\nu_\beta B^\beta) = (\Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta) A^\alpha B^\beta = (\Lambda^\nu_\alpha \Lambda^\mu_\nu) A^\alpha B^\beta = (\eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\mu_\nu) A^\alpha B^\beta = (\eta_{\alpha\beta} \delta^\alpha_\nu) A^\alpha B^\beta = (\eta_{\alpha\beta}) A^\alpha B^\beta = A^\alpha (\eta_{\alpha\beta}) B^\beta = \mathbf{A} \cdot \mathbf{B}$$

Lorentz Scalar Product → Lorentz Invariant Scalar = Same value for all inertial observers
Lorentz Invariants are also tensorial entities: (0,0)-Tensors

Einstein & Lorentz “saw” the physics of SR, Minkowski & Poincaré “saw” the mathematics of SR. We are indebted to all of them for the simplicity, beauty, and power of how SR and 4-vectors work...



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
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(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$

SR 4-Vectors & Lorentz Scalars

Rest Values ("naughts"=₀) are Lorentz Scalars

A Tensor Study of Physical 4-Vectors

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$\mathbf{A} \cdot \mathbf{A} = (a^0 a^0 - \mathbf{a} \cdot \mathbf{a}) = (a^0_0)^2$, where (a^0_0) is the rest-value, the value of the temporal coordinate when the spatial coordinate is zero.
The "rest-values" of several physical properties are all Lorentz scalars.

$\mathbf{P} = (mc, \mathbf{p})$ $\mathbf{K} = (\omega/c, \mathbf{k})$
 $\mathbf{P} \cdot \mathbf{P} = (mc)^2 - \mathbf{p} \cdot \mathbf{p}$ $\mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k}$
 $(\mathbf{P} \cdot \mathbf{P})$ and $(\mathbf{K} \cdot \mathbf{K})$ are Lorentz Scalars. We can choose a frame that may simplify the expressions.

4-Vector
 $\mathbf{A} = (a^0, \mathbf{a}) = (a^0, a^1, a^2, a^3)$
→ $(a^0_0, \mathbf{0})$ {in spatial rest frame}

$\mathbf{A} \cdot \mathbf{A} = (a^0_0)^2$

Notation:
"o" for rest values (naughts)
"0" for temporal components (0th index)

Choose a frame in which the spatial component is zero.
This is known as the "rest-frame" of the 4-Vector. It is not moving spatially.

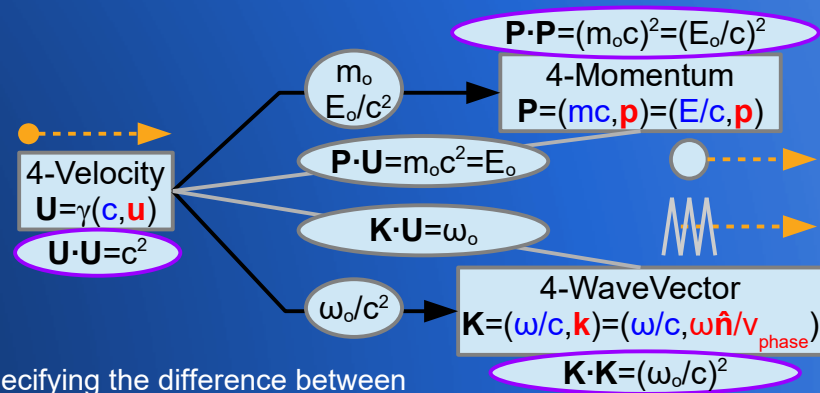
$\mathbf{P} \cdot \mathbf{P} = (mc)^2 - \mathbf{p} \cdot \mathbf{p} = (m_0 c)^2$ $\mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_0/c)^2$
The resulting simpler expressions then give the "rest values", indicated by (o).
RestMass (m_0) and RestAngularFrequency (ω_0)
They are Invariant Lorentz Scalars by construction.

This leads to simple relations between 4-Vectors.
 $\mathbf{P} = (m_0 c) \mathbf{U} = (E_0/c^2) \mathbf{U}$ $\mathbf{K} = (\omega_0/c^2) \mathbf{U}$

And gives nice Scalar Product relations between 4-Vectors as well.
 $\mathbf{P} \cdot \mathbf{U} = (m_0 c) \mathbf{U} \cdot \mathbf{U} = (m_0 c)^2 = (E_0)$ $\mathbf{K} \cdot \mathbf{U} = (\omega_0/c^2) \mathbf{U} \cdot \mathbf{U} = (\omega_0/c^2) c^2 = (\omega_0)$

This property of SR equations is a very good reason to use the "naught" convention for specifying the difference between relativistic component values which can vary, like (m), versus Rest Value Invariant Scalars, like (m_0), which do not vary.
They are usually related via a Lorentz Factor: { $m = \gamma m_0$ } and { $E = \gamma E_0$ }, as seen in the relation of \mathbf{P} and \mathbf{U} .

$\mathbf{P} = (mc, \mathbf{p}) = (m_0) \mathbf{U} = (m_0) \gamma(c, \mathbf{u}) = (\gamma m_0 c, \gamma m_0 \mathbf{u}) = (mc, m\mathbf{u}) = (mc, \mathbf{p})$
 $\mathbf{P} = (E/c, \mathbf{p}) = (E_0/c^2) \mathbf{U} = (E_0/c^2) \gamma(c, \mathbf{u}) = (\gamma E_0/c, \gamma E_0 \mathbf{u}/c^2) = (E/c, E\mathbf{u}/c^2) = (E/c, \mathbf{p})$



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(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SR 4-Vectors & 4-Tensors

Lorentz Scalar Product & Tensor Trace

Invariants: Similarities

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

All {4-Vectors:4-Tensors} have an associated {Lorentz Scalar Product:Trace}

Each 4-Vector has a “magnitude” given by taking the Lorentz Scalar Product of itself.

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = V^\mu V_\mu = V_\nu V^\nu = (v_0 v^0 + v_1 v^1 + v_2 v^2 + v_3 v^3) = (v^0 v^0 - \mathbf{v} \cdot \mathbf{v}) = (v^0_o)^2$$

The absolute magnitude of \mathbf{V} is $\sqrt{|\mathbf{V} \cdot \mathbf{V}|}$

Each 4-Tensor has a “magnitude” given by taking the Tensor Trace of itself.

$$\text{Trace}[T^{\mu\nu}] = \text{Tr}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T_\nu^\nu = (T^0_0 + T^1_1 + T^2_2 + T^3_3) = (T^{00} - T^{11} - T^{22} - T^{33}) = T$$

Note that the Trace runs down the diagonal of the 4-Tensor.

Notice the similarities. In both cases there is a tensor contraction with the Minkowski Metric Tensor $\eta_{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1]$ {Cartesian basis}

ex. $\mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_o/c)^2 = (m_o c)^2$

which says that the “magnitude” of the 4-Momentum is the RestEnergy/c = RestMass*c

ex. $\text{Trace}[\eta^{\mu\nu}] = (\eta^{00} - \eta^{11} - \eta^{22} - \eta^{33}) = 1 - (-1) - (-1) - (-1) = 1 + 1 + 1 + 1 = 4$

which says that the “magnitude” of the Minkowski Metric = SpaceTime Dimension = 4

Lorentz Scalar Invariant

$$\mathbf{V} \cdot \mathbf{V} = V^\mu V_\mu = (v^0 v^0 - \mathbf{v} \cdot \mathbf{v}) = (v^0_o)^2$$

4-Vector

$$\mathbf{V} = V^\mu = (v^0, \mathbf{v})$$

Trace Tensor Invariant

$$\text{Tr}[T^{\mu\nu}] = T^\mu_\mu = (T^{00} - T^{11} - T^{22} - T^{33}) = T$$

4-Tensor

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

$$\mathbf{P} \cdot \mathbf{P} = (m_o c)^2 = (E_o/c)^2$$

4-Momentum

$$\mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p})$$

$$\text{Tr}[\eta^{\mu\nu}] = 4$$

Minkowski Metric

$$\partial[\mathbf{R}] = \eta^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1]$$

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SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

Phase Space Integration

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Some 4-Vectors have an alternate form of Tensor Invariant: $d\mathbf{v}'/v'^0 = d\mathbf{v}/v^0$, in addition to the standard Lorentz Invariant $\mathbf{V}\cdot\mathbf{V} = V^\mu V_\mu = (v^0 v^0 - \mathbf{v}\cdot\mathbf{v}) = (v^0_0)^2$

If $\mathbf{V}\cdot\mathbf{V} = (\text{constant})$; with $\mathbf{V} = (v^0, \mathbf{v})$
 then $d(\mathbf{V}\cdot\mathbf{V}) = 2^*(\mathbf{V}\cdot d\mathbf{V}) = d(\text{constant}) = 0$
 hence $(\mathbf{V}\cdot d\mathbf{V}) = 0 = v^0 dv^0 - \mathbf{v}\cdot d\mathbf{v}$
 $dv^0 = \mathbf{v}\cdot d\mathbf{v}/v^0$

Generally; with $\Lambda = \Lambda^\mu_{\nu}$ = Lorentz Boost Transform in the β -direction
 $\mathbf{V}' = \Lambda\mathbf{V}$: from which the temporal component $v'^0 = (\gamma v^0 - \gamma\beta\cdot\mathbf{v})$
 $d\mathbf{V}' = \Lambda d\mathbf{V}$: from which the spatial component $d\mathbf{v}' = (\gamma d\mathbf{v} - \gamma\beta dv^0)$

Combining:

$$d\mathbf{v}' = (\gamma d\mathbf{v} - \gamma\beta(\mathbf{v}\cdot d\mathbf{v}/v^0))$$

$$d\mathbf{v}' = (1/v^0)^*(\gamma v^0 d\mathbf{v} - \gamma\beta(\mathbf{v}\cdot d\mathbf{v}))$$

$$d\mathbf{v}' = (1/v^0)^*(\gamma v^0 - \gamma\beta\cdot\mathbf{v})d\mathbf{v}$$

$$d\mathbf{v}' = (\gamma v^0 - \gamma\beta\cdot\mathbf{v})^*(1/v^0)^*d\mathbf{v}$$

$$d\mathbf{v}' = (v^0/v^0)d\mathbf{v}$$

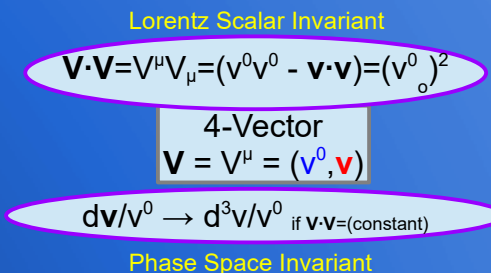
$$d\mathbf{v}'/v'^0 = d\mathbf{v}/v^0 = \text{Invariant of } \mathbf{V} = (v^0, \mathbf{v}) \text{ for } \mathbf{V}\cdot\mathbf{V} = (\text{constant})$$

So, for example:

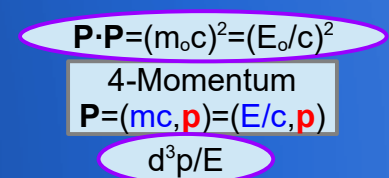
$$\mathbf{P}\cdot\mathbf{P} = (m_0 c)^2 = (\text{constant})$$

Thus, $d\mathbf{p}'/(E'/c) = d\mathbf{p}/(E/c) = \text{Invariant}$

Or: $d\mathbf{p}'/E' = d\mathbf{p}/E \rightarrow d^3p/E = dp^x dp^y dp^z/E = \text{Invariant}$, usually seen as $\int F(\text{various invariants})^* d^3p/E = \text{Invariant}$



An alternate approach is:
 $\int d^4p \delta[p^2 - (m_0 c)^2]$
 $= \int d^4p (1/2|m_0 c|) (\delta[p^+ + m_0 c] + \delta[p^- - m_0 c])$
 $= c d^3p/2E$
 $= \text{Invariant}$



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$d^4\mathbf{X} = -(V_o)\mathbf{dT}\cdot\mathbf{dX} = -(dV_o)\mathbf{T}\cdot\mathbf{dX} = cdt\,d^3\mathbf{x} = cdt\,dx\,dy\,dz$
The 4D Position coords that are integrated to give a 4D volume: SI units [m⁴]

4-Differential $\mathbf{dX} = (cdt, \mathbf{dx})$; $\mathbf{dR} = (cdt, \mathbf{dr})$;
4-UnitTemporal $\mathbf{T} = \gamma(1, \boldsymbol{\beta}) = (\gamma, \gamma\boldsymbol{\beta})$
4-UnitTemporalDifferential $\mathbf{dT} = d[(\gamma, \gamma\boldsymbol{\beta})] = (d[\gamma], d[\gamma\boldsymbol{\beta}])$

$V = \int dV = \int dx\,dy\,dz = \iiint d^3\mathbf{x}$
 $V = V_o/\gamma = 3D\,Spatial\,Volume: SI\,units\,[m^3]$
 $dV = d^3\mathbf{x} = 3D\,Spatial\,Volume\,Element$
 $\gamma = V_o/V$
 $d\gamma = -(V_o/V^2)dV$

$-(V_o)\mathbf{dT}\cdot\mathbf{dX} = Invariant, because (Rest\,Scalar * Lorentz\,Scalar\,Product) = Invariant$
 $= -(V_o)(d[\gamma], d[\gamma\boldsymbol{\beta}]) \cdot (cdt, \mathbf{dx})$
 $= -(V_o)(d[\gamma]cdt - d[\gamma\boldsymbol{\beta}] \cdot \mathbf{dx})$
 $= -(V_o)(-(V_o/V^2)dVcdt - d[\gamma\boldsymbol{\beta}] \cdot \mathbf{dx})$
 $= -(V_o)(-(V_o/V_o^2)dVcdt - d[(1)(\mathbf{0})] \cdot \mathbf{dx})$ by taking the usual rest-case
 $= -(V_o)(-(V_o/V_o^2)dVcdt)$
 $= -(V_o)(-(1/V_o)dVcdt)$
 $= dVcdt$
 $= cdt\,dV$
 $= cdt\,dx\,dy\,dz$
 $= cdt\,d^3\mathbf{x}$
 $= d^4\mathbf{X} = Invariant$

And, this makes sense.

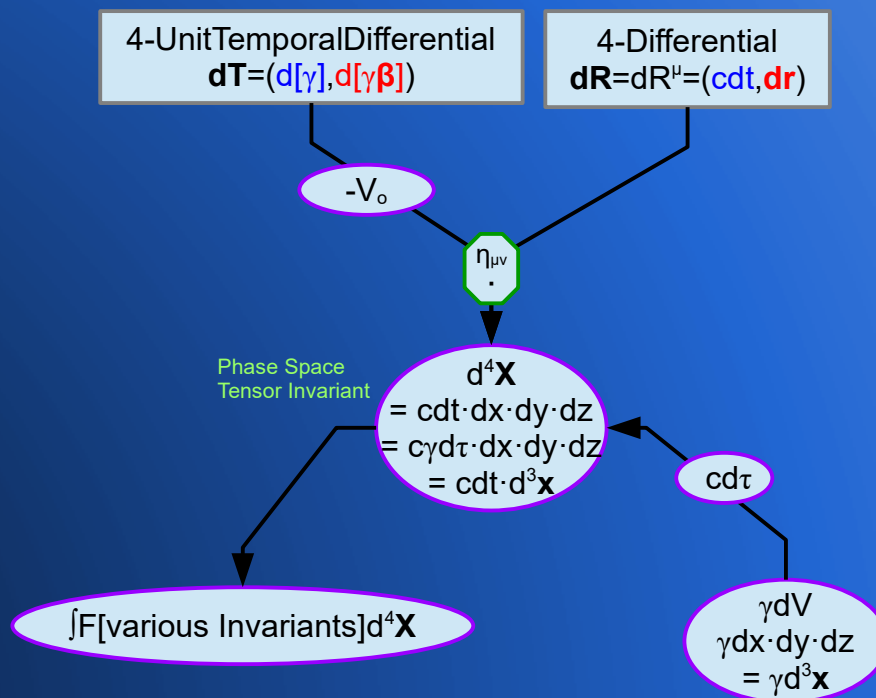
\mathbf{T} is a temporal 4-Vector with fixed magnitude: $\mathbf{T}\cdot\mathbf{T} = 1$

Therefore, \mathbf{dT} must be a spatial 4-Vector

If \mathbf{dX} is also spatial, then the Lorentz scalar product $\{(\mathbf{dT}\cdot\mathbf{dX}) = -magnitude\}$ will be negative with this choice of Minkoski Metric.

Thus, multiplying by $-(V_o)$ gives a positive volume element $\{ cdt\,dx\,dy\,dz = d^4\mathbf{X} \}$

It is sort of quirky though, that the temporal (cdt) comes from the \mathbf{dX} part, and the spatial ($d^3\mathbf{x}$) comes from the \mathbf{dT} part.



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$\rho d^3\mathbf{x} = \rho' d^3\mathbf{x}' = (-V_0/c)d\mathbf{T}\cdot\mathbf{J}$ = Lorentz Scalar Invariant
 $n d^3\mathbf{x} = n' d^3\mathbf{x}' = (-V_0/c)d\mathbf{T}\cdot\mathbf{N}$ = Lorentz Scalar Invariant

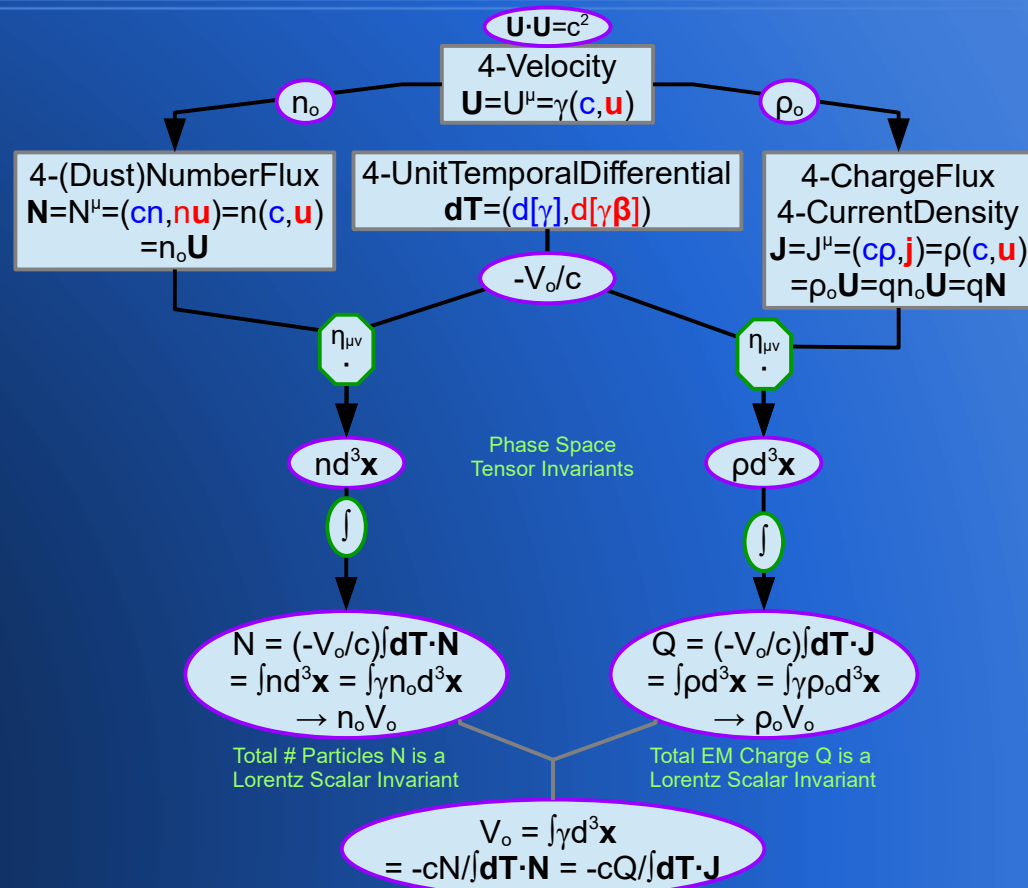
4-Current Density $\mathbf{J} = (\rho c, \mathbf{j})$
 4-Number Flux $\mathbf{N} = (nc, \mathbf{n})$
 4-Unit Temporal $\mathbf{T} = \gamma(1, \boldsymbol{\beta}) = (\gamma, \gamma\boldsymbol{\beta})$
 4-Unit Temporal Differential $d\mathbf{T} = d[(\gamma, \gamma\boldsymbol{\beta})] = (d[\gamma], d[\gamma\boldsymbol{\beta}])$

$V = V_0/\gamma$
 $d\gamma = -(V_0/V^2)dV$

$(-V_0/c)d\mathbf{T}\cdot\mathbf{J}$ = Invariant, because (Rest Scalar * Lorentz Scalar Product) = Invariant
 $= (-V_0/c)(d[\gamma], d[\gamma\boldsymbol{\beta}]) \cdot (\rho c, \mathbf{j})$
 $= (-V_0/c)(d[\gamma]\rho c - d[\gamma\boldsymbol{\beta}] \cdot \mathbf{j})$
 $= (-V_0/c)(- (V_0/V^2)(dV)(\rho c) - d[\gamma\boldsymbol{\beta}] \cdot \mathbf{j})$
 $= (-V_0/c)(- (V_0/V_0^2)(dV)(\rho c) - d[(1)\mathbf{0}] \cdot \mathbf{j})$
 $= (-V_0/c)(- (V_0/V_0^2)(dV)(\rho c))$
 $= (dV/c)(\rho c)$
 $= (\rho c)(dV/c)$
 $= (\rho)(dV)$
 $= \rho d^3\mathbf{x}$

Total Charge $Q = \int \rho d^3\mathbf{x} = \int \rho d^3\mathbf{x}$ = Lorentz Scalar Invariant
 Total Particle # $N = \int n d^3\mathbf{x} = \int n d^3\mathbf{x}$ = Lorentz Scalar Invariant
 Total Rest Volume $V_0 = \int d^3\mathbf{x}$ = Lorentz Scalar Invariant

This also gives an alternate way to define the Rest Volume Invariant V_0 .
 $(-V_0/c)d\mathbf{T}\cdot\mathbf{N} = nd^3\mathbf{x}$
 $N = \int nd^3\mathbf{x} = \int (-V_0/c)d\mathbf{T}\cdot\mathbf{N}$
 $cN/V_0 = -\int d\mathbf{T}\cdot\mathbf{N}$
 $V_0 = -cN/\int d\mathbf{T}\cdot\mathbf{N}$



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 = Lorentz Scalar

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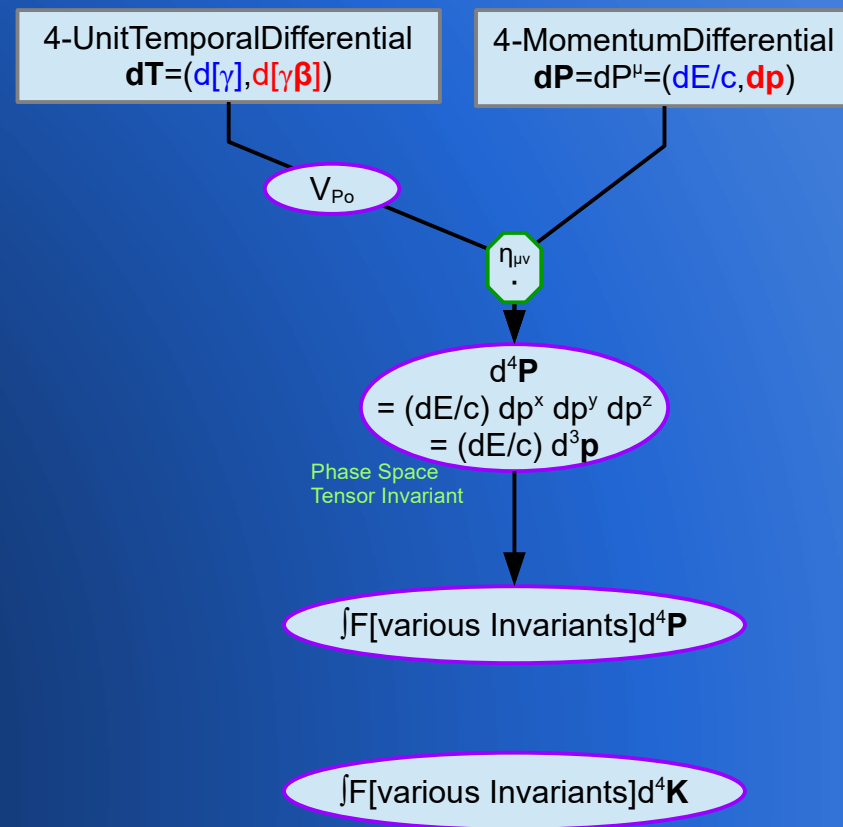
$d^4\mathbf{P} = (V_{P_0})d\mathbf{T} \cdot d\mathbf{P} = (dE/c) d^3\mathbf{p} = (dE/c) dp^x dp^y dp^z$
 $d^4\mathbf{K} = (V_{K_0})d\mathbf{T} \cdot d\mathbf{K} = (d\omega/c) d^3\mathbf{k} = (d\omega/c) dk^x dk^y dk^z$
 The 4D Momentum coords that are integrated to give a 4D Momentum Volume: SI Units [(kg·m/s)⁴]
 The 4D WaveVector coords that are integrated to give a 4D WaveVector Volume: SI Units [(1/m)⁴]

4-DifferentialMomentum $d\mathbf{P} = (dE/c, d\mathbf{p})$
 4-DifferentialWaveVector $d\mathbf{K} = (d\omega/c, d\mathbf{k})$
 4-UnitTemporal $\mathbf{T} = \gamma(1, \boldsymbol{\beta}) = (\gamma, \gamma\boldsymbol{\beta})$
 4-UnitTemporalDifferential $d\mathbf{T} = d[(\gamma, \gamma\boldsymbol{\beta})] = (d[\gamma], d[\gamma\boldsymbol{\beta}])$

$V_P = \int dV_P = \int dp^x \int dp^y \int dp^z = \int d^3\mathbf{p}$
 $V_P = \gamma(V_{P_0}) = 3D \text{ Volume in Momentum Space: SI Units } [(kg \cdot m/s)^3]$
 $dV_P = d\gamma(V_{P_0}) = 3D \text{ Volume Element in Momentum Space}$
 $\gamma = (V_P)/(V_{P_0})$
 $d\gamma = (dV_P)/(V_{P_0})$

$(V_{P_0})d\mathbf{T} \cdot d\mathbf{P} = \text{Invariant, because Rest Scalar} \cdot \text{Lorentz Scalar Product}$
 $= (V_{P_0})(d[\gamma], d[\gamma\boldsymbol{\beta}]) \cdot (dE/c, d\mathbf{p})$
 $= (V_{P_0})(d[\gamma]dE/c - d[\gamma\boldsymbol{\beta}] \cdot d\mathbf{p})$
 $= (V_{P_0})((dV_P/V_{P_0})dE/c - d[\gamma\boldsymbol{\beta}] \cdot d\mathbf{p})$
 $= (V_{P_0})((dV_P/V_{P_0})dE/c - d[(1)(0)] \cdot d\mathbf{p})$ by taking the usual rest-case
 $= (V_{P_0})((dV_P/V_{P_0})dE/c)$
 $= (dV_P) (dE/c)$
 $= d^3\mathbf{p} (dE/c)$
 $= (dE/c) d^3\mathbf{p}$
 $= (dE/c) dp^x dp^y dp^z$
 $= d^4\mathbf{P} = \text{Invariant}$

Likewise, $d^4\mathbf{K} = \text{Invariant}$



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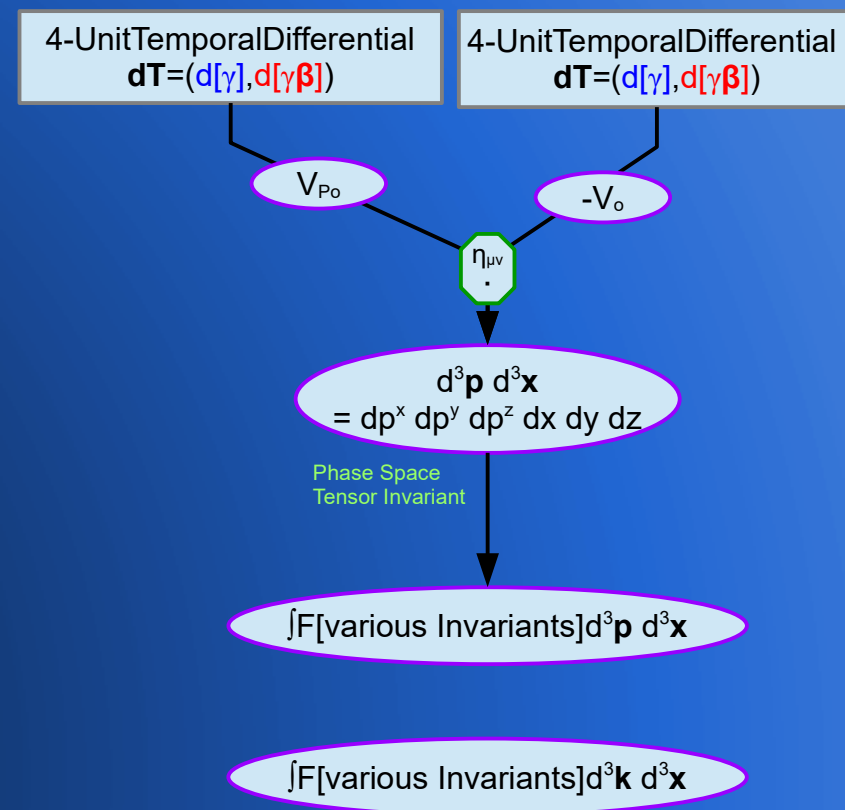
$$d^3\mathbf{p} d^3\mathbf{x} = (V_{Po})d\mathbf{T} \cdot (-V_o)d\mathbf{T} = (-V_o)(V_{Po})d\mathbf{T} \cdot d\mathbf{T}$$

$$d^3\mathbf{k} d^3\mathbf{x} = (V_{Ko})d\mathbf{T} \cdot (-V_o)d\mathbf{T} = (-V_o)(V_{Ko})d\mathbf{T} \cdot d\mathbf{T}$$

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 4-UnitTemporalDifferential $d\mathbf{T} = d[(\gamma, \gamma\boldsymbol{\beta})] = (d[\gamma], d[\gamma\boldsymbol{\beta}])$

$(V_{Po})d\mathbf{T} \cdot (-V_o)d\mathbf{T} = \text{Invariant}$
 $= (V_{Po})(d[\gamma], d[\gamma\boldsymbol{\beta}]) \cdot (-V_o)(d[\gamma], d[\gamma\boldsymbol{\beta}])$
 $= (V_{Po})(-V_o)(d[\gamma]d[\gamma] - d[\gamma\boldsymbol{\beta}] \cdot d[\gamma\boldsymbol{\beta}])$
 $= (V_{Po})(-V_o)(- (V_o/V_o^2)dV(dV_P/(V_{Po})) - d[\gamma\boldsymbol{\beta}] \cdot d[\gamma\boldsymbol{\beta}])$
 $= (V_{Po})(-V_o)(- (V_o/V_o^2)dV(dV_P/(V_{Po})) - d[(1)\mathbf{0}] \cdot d[(1)\mathbf{0}])$
 $= (V_{Po})(-V_o)(- (V_o/V_o^2)dV(dV_P/(V_{Po}))$
 $= (V_{Po})dV(dV_P/(V_{Po}))$
 $= dV dV_P$
 $= dV_P dV$
 $= d^3\mathbf{p} d^3\mathbf{x} = \text{Invariant}$

Likewise, $d^3\mathbf{k} d^3\mathbf{x} = \text{Invariant}$



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SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$

SRQM Study: SR 4-Tensors

General → Symmetric & Anti-Symmetric

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

Any SR Tensor $T^{\mu\nu} = (S^{\mu\nu} + A^{\mu\nu})$ can be decomposed into parts:

Symmetric $S^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2$ with $S^{\mu\nu} = +S^{\nu\mu}$

Anti-Symmetric $A^{\mu\nu} = (T^{\mu\nu} - T^{\nu\mu})/2$ with $A^{\mu\nu} = -A^{\nu\mu}$

$$S^{\mu\nu} + A^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2 + (T^{\mu\nu} - T^{\nu\mu})/2 = T^{\mu\nu}/2 + T^{\nu\mu}/2 + T^{\mu\nu}/2 - T^{\nu\mu}/2 = T^{\mu\nu} + 0 = T^{\mu\nu}$$

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

Note These don't have to be composed from a single general tensor.

$$S^{\mu\nu} A_{\mu\nu} = 0$$

Proof:

$$\begin{aligned} S^{\mu\nu} A_{\mu\nu} &= S^{\nu\mu} A_{\nu\mu}: \text{because we can switch dummy indices} \\ &= (+S^{\mu\nu})A_{\nu\mu}: \text{because of symmetry} \\ &= S^{\mu\nu}(-A_{\mu\nu}): \text{because of anti-symmetry} \\ &= -S^{\mu\nu} A_{\mu\nu} \\ &= 0: \text{because the only solution of } \{c = -c\} \text{ is } 0 \end{aligned}$$

Physically, the anti-symmetric part contains rotational information and the symmetric part contains information about isotropic scaling and anisotropic shear.

Independent components: $\{4^2 = 16 = 10 + 6\}$

Max 16 possible

Max 10 possible

Max 6 possible

General 4-Tensor
 $T^{\mu\nu} =$

$$\begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

Symmetric 4-Tensor
 $S^{\mu\nu} =$

$$\begin{bmatrix} S^{00} & S^{01} & S^{02} & S^{03} \\ S^{10} & S^{11} & S^{12} & S^{13} \\ S^{20} & S^{21} & S^{22} & S^{23} \\ S^{30} & S^{31} & S^{32} & S^{33} \end{bmatrix} = \begin{bmatrix} S^{00} & S^{01} & S^{02} & S^{03} \\ +S^{01} & S^{11} & S^{12} & S^{13} \\ +S^{02} & +S^{12} & S^{22} & S^{23} \\ +S^{03} & +S^{13} & +S^{23} & S^{33} \end{bmatrix}$$

$\text{Tr}[S^{\mu\nu}] = S^{\mu}_{\mu}$

Anti-Symmetric 4-Tensor
 $A^{\mu\nu} =$

$$\begin{bmatrix} A^{00} & A^{01} & A^{02} & A^{03} \\ A^{10} & A^{11} & A^{12} & A^{13} \\ A^{20} & A^{21} & A^{22} & A^{23} \\ A^{30} & A^{31} & A^{32} & A^{33} \end{bmatrix} = \begin{bmatrix} 0 & A^{01} & A^{02} & A^{03} \\ -A^{01} & 0 & A^{12} & A^{13} \\ -A^{02} & -A^{12} & 0 & A^{23} \\ -A^{03} & -A^{13} & -A^{23} & 0 \end{bmatrix}$$

$\text{Tr}[A^{\mu\nu}] = 0$

aka Skew-Symmetric

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$$\begin{aligned} \text{Trace}[T^{\mu\nu}] &= \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T \\ \mathbf{V} \cdot \mathbf{V} &= V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 \\ &= \text{Lorentz Scalar} \end{aligned}$$

SRQM Study: SR 4-Tensors

Symmetric → Isotropic & Anisotropic

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

Any Symmetric SR Tensor $S^{\mu\nu} = (T_{iso}^{\mu\nu} + T_{aniso}^{\mu\nu})$ can be decomposed into parts:

Isotropic $T_{iso}^{\mu\nu} = (1/4)\text{Trace}[S^{\mu\nu}] \eta^{\mu\nu} = (T) \eta^{\mu\nu}$

Anisotropic $T_{aniso}^{\mu\nu} = S^{\mu\nu} - T_{iso}^{\mu\nu}$

The Anisotropic part is Traceless by construction, and the Isotropic part has the same Trace as the original Symmetric Tensor. The Minkowski Metric is a symmetric, isotropic 4-tensor with T=1.

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

Note These don't have to be composed from a single general tensor.

$$S^{\mu\nu} A_{\mu\nu} = 0$$

Proof:
 $S^{\mu\nu} A_{\mu\nu}$
 = $S^{\nu\mu} A_{\nu\mu}$: because we can switch dummy indices
 = $(+S^{\mu\nu})A_{\nu\mu}$: because of symmetry
 = $S^{\mu\nu}(-A_{\mu\nu})$: because of anti-symmetry
 = $-S^{\mu\nu} A_{\mu\nu}$
 = 0: because the only solution of $\{c = -c\}$ is 0

Physically, the isotropic part represents a direction independent transformation (e.g., a uniform scaling or uniform pressure); the deviatoric part represents the distortion

Independent components:

Max 10 possible

Max 1 possible

Max 9 possible

Symmetric 4-Tensor

$$S^{\mu\nu} =$$

$$\begin{bmatrix} S^{00} & S^{01} & S^{02} & S^{03} \\ S^{10} & S^{11} & S^{12} & S^{13} \\ S^{20} & S^{21} & S^{22} & S^{23} \\ S^{30} & S^{31} & S^{32} & S^{33} \end{bmatrix}$$

=

$$\begin{bmatrix} S^{00} & S^{01} & S^{02} & S^{03} \\ +S^{01} & S^{11} & S^{12} & S^{13} \\ +S^{02} & +S^{12} & S^{22} & S^{23} \\ +S^{03} & +S^{13} & +S^{23} & S^{33} \end{bmatrix}$$

$$\text{Tr}[S^{\mu\nu}] = 4T$$

Symmetric Isotropic 4-Tensor

$$T_{iso}^{\mu\nu} =$$

$$\begin{bmatrix} T & 0 & 0 & 0 \\ 0 & -T & 0 & 0 \\ 0 & 0 & -T & 0 \\ 0 & 0 & 0 & -T \end{bmatrix}$$

with T =
(1/4)Trace[S^{μν}]

$$\text{Tr}[T_{iso}^{\mu\nu}] = 4T$$

Symmetric Anisotropic 4-Tensor

$$T_{aniso}^{\mu\nu} =$$

$$\begin{bmatrix} S^{00}-T & S^{01} & S^{02} & S^{03} \\ S^{10} & S^{11}+T & S^{12} & S^{13} \\ S^{20} & S^{21} & S^{22}+T & S^{23} \\ S^{30} & S^{31} & S^{32} & S^{33}+T \end{bmatrix}$$

=

$$\begin{bmatrix} S^{00}-T & S^{01} & S^{02} & S^{03} \\ +S^{01} & S^{11}+T & S^{12} & S^{13} \\ +S^{02} & +S^{12} & S^{22}+T & S^{23} \\ +S^{03} & +S^{13} & +S^{23} & S^{33}+T \end{bmatrix}$$

$$\text{Tr}[T_{aniso}^{\mu\nu}] = 0$$

aka Deviatoric

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (\mathbf{v}_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

$$\begin{aligned} \text{Trace}[T^{\mu\nu}] &= \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T \\ \mathbf{V} \cdot \mathbf{V} &= V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 \\ &= \text{Lorentz Scalar} \end{aligned}$$

SRQM Study: SR 4-Tensors

SR Tensor Invariants

A Tensor Study of Physical 4-Vectors

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(0,0)-Tensor = Lorentz Scalar S: Has either (0) or (1) Tensor Invariant, depending on exact meaning

(S) itself is Invariant

S

(1,0)-Tensor = 4-Vector V^μ: Has (1) Tensor Invariant = The Lorentz Scalar Product

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = \eta_{\mu\nu} V^\mu V^\nu = \text{Tr}[\mathbf{V}^\mu \mathbf{V}^\nu] = V_\nu V^\nu = (v_0 v^0 + v_1 v^1 + v_2 v^2 + v_3 v^3) = (v^0 v^0 - \mathbf{v} \cdot \mathbf{v}) = (v^0_0)^2$$

$$\mathbf{V} = \mathbf{V}^\mu = (v^\mu) = (v^0, \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3)$$

$$\mathbf{V} \cdot \mathbf{V} = (v^0 v^0 - \mathbf{v} \cdot \mathbf{v}) = (v^0_0)^2$$

(2,0)-Tensor = 4-Tensor T^{μν}: Has (4+) Tensor Invariants (though not all independent)

a) T^α_α = Trace = Sum of EigenValues for (1,1)-Tensors (mixed)

b) T^α_{[α T^β_{β]} = Asymm Bi-Product → Inner Product}

c) T^α_{[α T^β_{β T^γ_{γ]} = Asymm Tri-Product → ?Name?}}

d) T^α_{[α T^β_{β T^γ_{γ T^δ_{δ]} = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors}}}

eg. T^α_{[α T^β_{β] = T^α_{α T^β_{β - T^α_{β T^β_{α} = (T^v_{v})² - T^α_{β T^β_{α} = (T^v_{v})² - T^α_{β T^β_{α} (1/4) η_{γδ} η^{γδ}} v^{δ}}}}}}}}}}}}}}}

and, bending tensor rules slightly: = (T^v_{v})² - T^α_{β T^β_{α} (1/4) η_{βδ} η^{βδ}} v^{δ}} = (T^v_{v})² - T^α_{β (η^{βδ}} T^β_{α} (η_{βδ}}) (1/4) = (T^v_{v})² - T^{αδ}} T_{αδ} (1/4)}}}}}}}}}

and, since linear combinations of invariants are invariant:

Examine just the (T^{αδ}} T_{αδ}) part, which for symm|asymm is (±)(T^{αδ}} T_{αδ}) ie. the InnerProduct Invariant}}

a): **Trace** [T^{μν}] = Tr[T^{μν}] = η_{μν} T^{μν} = T_μ^μ = T_v^v = (T₀⁰ + T₁¹ + T₂² + T₃³) = (T⁰⁰ - T¹¹ - T²² - T³³) = (T) for anti-symmetric: = 0}

b): **InnerProduct** T_{μν} T^{μν} = T_{00} T⁰⁰ + T_{i0} Tⁱ⁰ + T_{0j} T^{0j} + T_{ij} T^{ij} = (T⁰⁰)² - Σ_i [Tⁱ⁰]² - Σ_j [T^{0j}]² + Σ_{i,j} [T^{ij}]² for symmetric | anti-symmetric: = (T⁰⁰)² - 2Σ_i [Tⁱ⁰]² + Σ_{i,j} [T^{ij}]² = Σ_{μ=v} [T^{μν}]² - 2Σ_i [Tⁱ⁰]² + 2Σ_{i>j} [T^{ij}]²}}}}}

c): **Antisymmetric Triple Product** T^α_{[α T^β_{β T^γ_{γ]} = Tr[T^{μν}]³ - 3(Tr[T^{μν}])(T^α_{β T^β_{α} + T^α_{β T^β_{γ T^γ_{α} + T^α_{γ T^β_{α T^γ_{β} for anti-symmetric: = 0}}}}}}}}}}

If I got all the math right...

d): **Determinant** Det[T^{μν}] = ? = -(1/2) ε_{αβγδ} T^{αβ} T^{γδ}} for anti-symmetric: Det[T^{μν}] = Pfaffian[T^{μν}] (The Pfaffian is a special polynomial of the matrix entries)}}

Trace Tensor Invariant

$$\text{Tr}[T^{\mu\nu}] = T_{\nu}^{\nu} = (T^{00} - T^{11} - T^{22} - T^{33}) = T$$

Set of 4 EigenValues [T_μ^ν]

Eigenvalues Tensor Invariants

4-Tensor

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

$$T_{\mu\nu} T^{\mu\nu}$$

Inner Product Tensor Invariant

Asymm Tri[T^{μν}]

Asymm Tri-Product Tensor Invariant

Det[T^{μν}]

Determinant Tensor Invariant

Lowered 4-Tensor

$$T_{\mu\nu} = \eta_{\mu\rho} \eta_{\nu\sigma} T^{\rho\sigma}$$

$$= \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix}$$

$$= \begin{bmatrix} +T^{00} & -T^{01} & -T^{02} & -T^{03} \\ -T^{10} & +T^{11} & +T^{12} & +T^{13} \\ -T^{20} & +T^{21} & +T^{22} & +T^{23} \\ -T^{30} & +T^{31} & +T^{32} & +T^{33} \end{bmatrix}$$

The lowered-indices form of a tensor just negativizes the (time-space) and (space-time) sections of the upper-indices tensor

Invariants sometimes seen as

- I₁ = (1/1) Tr[(T^{μν})¹]
- I₂ = (1/2) Tr[(T^{μν})²]
- I₃ = (1/3) Tr[(T^{μν})³]
- I₄ = (1/4) Tr[(T^{μν})⁴]

SR 4-Tensor
(2,0)-Tensor T^{μν}
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor T_{μν}

SR 4-Vector
(1,0)-Tensor V^μ = **V** = (v⁰, **v**)
SR 4-CoVector
(0,1)-Tensor V_μ = (v₀, **-v**)

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Det[T^α_α] = Π_k [λ_k]; with {λ_k} = Eigenvalues
Characteristic Eqns: Det[T^α_α - λ_k I_{(4)}}] = 0

Trace[T^{μν}] = η<sub>μν} T^{μν} = T_μ^μ = T
V · V = V^{μ} η_{μν} V^ν = [(v⁰)² - **v** · **v**] = (v⁰₀)² = Lorentz Scalar}}</sub>

SRQM Study: SR 4-Tensors

SR Tensor Invariants

Tensor Gymnastics

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

Some Tensor Gymnastics:

Matrix **A** = Tensor A^r_c
with rows denoted by "r", columns by "c"

Example with dim=4: r,c={0..3}

Matrix **A** =

$$\begin{bmatrix} A^{r=0}_{c=0} & A^{r=0}_{c=1} & A^{r=0}_{c=2} & A^{r=0}_{c=3} \\ A^{r=1}_{c=0} & A^{r=1}_{c=1} & A^{r=1}_{c=2} & A^{r=1}_{c=3} \\ A^{r=2}_{c=0} & A^{r=2}_{c=1} & A^{r=2}_{c=2} & A^{r=2}_{c=3} \\ A^{r=3}_{c=0} & A^{r=3}_{c=1} & A^{r=3}_{c=2} & A^{r=3}_{c=3} \end{bmatrix}$$

$\mathbf{M} = \mathbf{A} \times \mathbf{B} = A^c_d B^e_c = M^e_d$
,with the rows of **A** multiplied by the columns of **B**
due to the summation over index "c"

If we have sums over both indices:

$$A^c_d B^d_c = M^d_d = \text{Trace}[\mathbf{M}]$$

The sum over "c" gives the matrix multiplication and then the sum over "d" gives the Trace of the resulting matrix M

$$A^c_d A^d_c = (\mathbf{A} \times \mathbf{A})^d_d = (N)^d_d = \text{Trace}[\mathbf{N}] = \text{Trace}[\mathbf{A}^2] = \text{Tr}[\mathbf{A}^2]$$

$$A^c_d A^d_c = (\eta^e_e A^c_e) A^d_c = \eta^e_e (A^c_e A^d_c) = \eta^e_e (N^d_e) = \delta^e_e (N^d_e) = \text{Tr}[\mathbf{N}] = \text{Tr}[\mathbf{A}^2]$$

$$A^c_c A^d_d = A^c_c A^d_d - A^c_d A^d_c = (\text{Tr}[\mathbf{A}])^2 - \text{Tr}[\mathbf{A}^2]$$

,with brackets [...] around the indices indicating anti-symmetric product

$$A^a_a = \text{Tr}[\mathbf{A}]$$

$$A^a_{[a} A^b_{b]} = A^a_a A^b_b - A^a_b A^b_a = (\text{Tr}[\mathbf{A}])^2 - \text{Tr}[\mathbf{A}^2]$$

$$A^a_{[a} A^b_b A^c_c] = +A^a_a A^b_b A^c_c - A^a_a A^b_c A^c_b + A^a_b A^b_c A^c_a - A^a_b A^b_a A^c_c + A^a_c A^b_a A^c_b - A^a_c A^b_b A^c_a$$

$$= +(A^a_a A^b_b A^c_c) - (A^a_a A^b_c A^c_b + A^a_b A^b_a A^c_c + A^a_c A^b_b A^c_a) + (A^a_b A^b_c A^c_a + A^a_c A^b_a A^c_b)$$

$$= +(A^a_a A^b_b A^c_c) - (A^a_a A^b_c A^c_b + A^a_c A^b_a A^c_a + A^a_b A^b_a A^c_c) + (A^a_b A^b_c A^c_a + A^a_c A^b_a A^c_b)$$

$$= +(\text{Tr}[\mathbf{A}])^3 - 3*(\text{Tr}[\mathbf{A}])(\text{Tr}[\mathbf{A}^2]) + 2*(\text{Tr}[\mathbf{A}^3])$$

$$A^a_{[a} A^b_b A^c_c A^d_d] = +A^a_a A^b_b A^c_c A^d_d - A^a_a A^b_b A^c_d A^d_c - A^a_a A^b_c A^c_b A^d_d + A^a_a A^b_c A^c_d A^d_b + A^a_a A^b_d A^c_b A^d_c - A^a_a A^b_d A^c_c A^d_b$$

$$- A^a_b A^b_a A^c_c A^d_d + A^a_b A^b_a A^c_d A^d_c + A^a_b A^b_c A^c_a A^d_d - A^a_b A^b_c A^c_d A^d_a - A^a_b A^b_d A^c_a A^d_c + A^a_b A^b_d A^c_c A^d_a$$

$$+ A^a_c A^b_a A^c_b A^d_d - A^a_c A^b_a A^c_d A^d_b - A^a_c A^b_b A^c_a A^d_d + A^a_c A^b_b A^c_d A^d_a + A^a_c A^b_d A^c_a A^d_b - A^a_c A^b_d A^c_b A^d_a$$

$$- A^a_d A^b_a A^c_c A^d_d + A^a_d A^b_a A^c_d A^d_b + A^a_d A^b_b A^c_a A^d_c - A^a_d A^b_b A^c_c A^d_a - A^a_d A^b_c A^c_a A^d_b + A^a_d A^b_c A^c_b A^d_a$$

$$= +A^a_a A^b_b A^c_c A^d_d$$

$$- A^a_a A^b_b A^c_d A^d_c - A^a_a A^b_c A^c_b A^d_d - A^a_a A^b_c A^c_d A^d_b - A^a_a A^b_d A^c_a A^d_c - A^a_a A^b_d A^c_c A^d_a - A^a_a A^b_d A^c_b A^d_a$$

$$+ A^a_a A^b_b A^c_d A^d_c + A^a_a A^b_d A^c_a A^d_c + A^a_a A^b_d A^c_c A^d_a + A^a_a A^b_d A^c_b A^d_a + A^a_a A^b_a A^c_c A^d_d + A^a_a A^b_a A^c_d A^d_b$$

$$+ A^a_b A^b_a A^c_c A^d_d + A^a_b A^b_a A^c_d A^d_b + A^a_b A^b_c A^c_a A^d_d - A^a_b A^b_c A^c_d A^d_a - A^a_b A^b_d A^c_a A^d_c - A^a_b A^b_d A^c_c A^d_a$$

$$- A^a_b A^b_b A^c_a A^d_c - A^a_b A^b_b A^c_c A^d_a - A^a_b A^b_c A^c_a A^d_b - A^a_b A^b_c A^c_b A^d_a - A^a_b A^b_d A^c_a A^d_c - A^a_b A^b_d A^c_b A^d_a$$

$$= +(\text{Tr}[\mathbf{A}])^4$$

$$- 6*(\text{Tr}[\mathbf{A}])^2(\text{Tr}[\mathbf{A}^2])$$

$$+ 8*(\text{Tr}[\mathbf{A}])(\text{Tr}[\mathbf{A}^3])$$

$$+ 3*(\text{Tr}[\mathbf{A}^2])^2$$

$$- 6*(\text{Tr}[\mathbf{A}^4])$$

$$= +(\text{Tr}[\mathbf{A}])^4 - 6*(\text{Tr}[\mathbf{A}])^2(\text{Tr}[\mathbf{A}^2]) + 8*(\text{Tr}[\mathbf{A}])(\text{Tr}[\mathbf{A}^3]) + 3*(\text{Tr}[\mathbf{A}^2])^2 - 6*(\text{Tr}[\mathbf{A}^4])$$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Det}[T^a_\alpha] = \prod_k[\lambda_k]$; with $\{\lambda_k\} = \text{Eigenvalues}$
Characteristic Eqns: $\text{Det}[T^a_\alpha - \lambda_k I_{(4)}] = 0$

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Study: SR 4-Tensors

SR Tensor Invariants

Cayley-Hamilton Theorem

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General Cayley-Hamilton Theorem

$A^d + c_{d-1}A^{d-1} + \dots + c_0A^0 = 0_{(d)}$, with A = square matrix, d = dimension, A^0 = Identity(d) = $I_{(d)}$

Characteristic Polynomial: $p(\lambda) = \text{Det}[A - \lambda I_{(d)}]$

The following are the Principle Tensor Invariants for dimensions 1..4

$\text{dim} = 1$: $A^1 + c_0A^0 = 0$: $A - I_1 I_{(1)} = 0$

$I_1 = \text{tr}[A] = \text{Det}_{1D}[A] = \lambda_1$

$\text{dim} = 2$: $A^2 + c_1A^1 + c_0A^0 = 0$: $A^2 - I_1 A^1 + I_2 I_{(2)} = 0$

$I_1 = \text{tr}[A] = \Sigma[\text{Eigenvalues}] = \lambda_1 + \lambda_2$

$I_2 = (\text{tr}[A]^2 - \text{tr}[A^2]) / 2 = \text{Det}_{2D}[A] = \Pi[\text{Eigenvalues}] = \lambda_1\lambda_2$

$\text{dim} = 3$: $A^3 + c_2A^2 + c_1A^1 + c_0A^0 = 0$: $A^3 - I_1 A^2 + I_2 A^1 - I_3 I_{(3)} = 0$

$I_1 = \text{tr}[A] = \Sigma[\text{Eigenvalues}] = \lambda_1 + \lambda_2 + \lambda_3$

$I_2 = (\text{tr}[A]^2 - \text{tr}[A^2]) / 2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$

$I_3 = [(\text{tr}[A])^3 - 3 \text{tr}(A^2)(\text{tr}[A]) + 2 \text{tr}(A^3)] / 6 = \text{Det}_{3D}[A] = \Pi[\text{Eigenvalues}] = \lambda_1\lambda_2\lambda_3$

$\text{dim} = 4$: $A^4 + c_3A^3 + c_2A^2 + c_1A^1 + c_0A^0 = 0$: $A^4 - I_1 A^3 + I_2 A^2 - I_3 A^1 + I_4 I_{(4)} = 0$

$I_1 = \text{tr}[A] = \Sigma[\text{Eigenvalues}] = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$

$I_2 = (\text{tr}[A]^2 - \text{tr}[A^2]) / 2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$

$I_3 = [(\text{tr}[A])^3 - 3 \text{tr}(A^2)(\text{tr}[A]) + 2 \text{tr}(A^3)] / 6 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$

$I_4 = ((\text{tr}[A])^4 - 6 \text{tr}(A^2)(\text{tr}[A])^2 + 3(\text{tr}(A^2))^2 + 8 \text{tr}(A^3) \text{tr}[A] - 6 \text{tr}(A^4)) / 24 = \text{Det}_{4D}[A] = \Pi[\text{Eigenvalues}] = \lambda_1\lambda_2\lambda_3\lambda_4$

$I_1 = \Sigma[\text{Unique Eigenvalue Singles}] = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$

$I_2 = \Sigma[\text{Unique Eigenvalue Doubles}] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$

$I_3 = \Sigma[\text{Unique Eigenvalue Triples}] = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$

$I_4 = \Sigma[\text{Unique Eigenvalue Quadruples}] = \lambda_1\lambda_2\lambda_3\lambda_4$

SR 4-Tensor

(2,0)-Tensor $T^{\mu\nu}$

(1,1)-Tensor T^μ_ν or T_{μ}^ν

(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector

(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$

SR 4-CoVector

(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar

(0,0)-Tensor S

Lorentz Scalar

$\text{Det}[T^\alpha_\alpha] = \Pi_k[\lambda_k]$; with $\{\lambda_k\} = \text{Eigenvalues}$

Characteristic Eqns: $\text{Det}[T^\alpha_\alpha - \lambda_k I_{(4)}] = 0$

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2$
 = Lorentz Scalar

SRQM Study: SR 4-Tensors

SR Tensor Invariants

Cayley-Hamilton Theorem

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

General Cayley-Hamilton Theorem	Dim=1	Dim=2	Dim=3	Euclidean 3-Space	Dim=4	Minkowski SpaceTime
$A^d + c_{d-1}A^{d-1} + \dots + c_0A^0 = 0_{(d)}$, with A = square matrix, d = dimension, A^0 = Identity(d) = $I_{(d)}$ $I_0 A^4 - I_1 A^3 + I_2 A^2 - I_3 A^1 + I_4 A^0 = 0$: for 4D Characteristic Polynomial: $p(\lambda) = \text{Det}[A - \lambda I_{(d)}]$	$A = [a]$	$A = [\begin{matrix} a & b \\ c & d \end{matrix}]$	$A = [\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}]$	$A = [\begin{matrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{matrix}]$	$A = [\begin{matrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{matrix}]$	$A = [\begin{matrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{matrix}]$
Tensor Invariants I_n	$= A^i_k : j,k=\{1\}$	$= A^i_k : j,k=\{1,2\}$	$= A^i_k : j,k=\{1,2,3\}$	$= A^i_k : j,k=\{1,2,3\}$	$= A^i_k : j,k=\{1,2,3,4\}$	$= A^i_k : j,k=\{1,2,3,4\}$
$I_0 = 1/0! = 1$	(1) = 1	(1) = 1	(1) = 1	(1) = 1	(1) = 1	(1) = 1
$I_1 = \text{tr}[A]/1!$ $= A^\alpha_\alpha$ $= \Sigma[\text{Unique Eigenvalue Singles}]$	(1) = λ_1 = (a) = $\Sigma[\text{Eigenvalues}]$ = $\text{Det}_{1D}[A]$ = $\Pi[\text{Eigenvalues}]$	(2) = $\lambda_1 + \lambda_2$ = (a + d) = $\Sigma[\text{Eigenvalues}]$	(3) = $\lambda_1 + \lambda_2 + \lambda_3$ = (a + e + i) = $\Sigma[\text{Eigenvalues}]$	(3) = $\lambda_1 + \lambda_2 + \lambda_3$ = (a + e + i) = $\Sigma[\text{Eigenvalues}]$	(4) = $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ = (a + f + k + p) = $\Sigma[\text{Eigenvalues}]$	(4) = $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ = (a + f + k + p) = $\Sigma[\text{Eigenvalues}]$
$I_2 = (\text{tr}[A]^2 - \text{tr}[A^2])/2!$ $= A^\alpha_{[\gamma} A^\beta_{\delta]} / 2$ $= \Sigma[\text{Unique Eigenvalue Doubles}]$	= 0	(1) = $\lambda_1 \lambda_2$ = (ad - bc) = $\text{Det}_{2D}[A]$ = $\Pi[\text{Eigenvalues}]$	(3) = $\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3$ = (ae - bd) + (ai - cg) + (ei - fg)	(3) = $\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3$ = (ae - bd) + (ai - cg) + (ei - fg)	(6) = $\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4$ = (af - be) + (ak - ci) + (ap - dm) + (fk - gi) + (fp - hn) + (kp - lo)	(6) = $\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4$ = (af - be) + (ak - ci) + (ap - dm) + (fk - gi) + (fp - hn) + (kp - lo)
$I_3 = [(\text{tr} A)^3 - 3 \text{tr}(A^2)(\text{tr} A) + 2 \text{tr}(A^3)]/3!$ $= A^\alpha_{[\gamma} A^\beta_{\delta} A^\nu_{\epsilon]} / 6$ $= \Sigma[\text{Unique Eigenvalue Triples}]$	= 0	= 0	(1) = $\lambda_1 \lambda_2 \lambda_3$ = a(ei - fh) - b(di - fg) + c(dh - eg) = $\text{Det}_{3D}[A]$ = $\Pi[\text{Eigenvalues}]$	(1) = $\lambda_1 \lambda_2 \lambda_3$ = a(ei - fh) - b(di - fg) + c(dh - eg) = $\text{Det}_{3D}[A]$ = $\Pi[\text{Eigenvalues}]$	(4) = $\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4$ = ...	(4) = $\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4$ = ...
$I_4 = ((\text{tr} A)^4 - 6 \text{tr}(A^2)(\text{tr} A)^2 + 3(\text{tr}(A^2))^2 + 8 \text{tr}(A^3) \text{tr} A - 6 \text{tr}(A^4))/4!$ $= A^\alpha_{[\gamma} A^\beta_{\delta} A^\nu_{\epsilon} A^\delta_{\zeta]} / 24$ $= \Sigma[\text{Unique Eigenvalue Quadruples}]$	= 0	= 0	= 0	= 0	(1) = $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ = a(f(kp - lo)) + ... = $\text{Det}_{4D}[A]$ = $\Pi[\text{Eigenvalues}]$	(1) = $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ = a(f(kp - lo)) + ... = $\text{Det}_{4D}[A]$ = $\Pi[\text{Eigenvalues}]$

SRQM Study: SR 4-Tensors

SR Tensor Invariants for Faraday EM Tensor

A Tensor Study of Physical 4-Vectors

The Faraday EM Tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge A$ is an anti-symmetric tensor that contains the Electric and Magnetic Fields, defined by the Exterior Product (\wedge). The 3-electric components ($\mathbf{e} = e^i$) are in the temporal-spatial sections. The 3-magnetic components ($\mathbf{b} = b^k$) are in the only-spatial section.

(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$. Has (4+) Tensor Invariants (though not all independent)

- a) $T^\alpha_\alpha = \text{Trace} = \text{Sum of EigenValues for } (1,1)\text{-Tensors (mixed)}$
- b) $T^\alpha_{[\alpha} T^\beta_{\beta]} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product}$
- c) $T^\alpha_{[\alpha} T^\beta_{\beta} T^\gamma_{\gamma]} = \text{Asymm Tri-Product} \rightarrow \text{?Name?}$
- d) $T^\alpha_{[\alpha} T^\beta_{\beta} T^\gamma_{\gamma} T^\delta_{\delta]} = \text{Asymm Quad-Product} \rightarrow \text{4D Determinant} = \text{Product of EigenValues for } (1,1)\text{-Tensors}$

- a): **Faraday Trace** $[F^{\mu\nu}] = F^\nu_\nu = (F^{00} - F^{11} - F^{22} - F^{33}) = (0 - 0 - 0 - 0) = 0$
- b): **Faraday Inner Product** $F_{\mu\nu} F^{\mu\nu} = \sum_{\mu<\nu} [F^{\mu\nu}]^2 - 2\sum_{\mu>\nu} [F^{ij}]^2 = (0) - 2(\mathbf{e}\cdot\mathbf{e}/c^2) + 2(\mathbf{b}\cdot\mathbf{b}) = 2\{(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{e}\cdot\mathbf{e}/c^2)\}$
- c): **Faraday AsymmTri** $[F^{\mu\nu}] = \text{Tr}[F^{\mu\nu}]^3 - 3(\text{Tr}[F^{\mu\nu}])(F^\alpha_\beta F^\beta_\alpha) + F^\alpha_\beta F^\beta_\gamma F^\gamma_\alpha + F^\alpha_\gamma F^\beta_\alpha F^\beta_\gamma = 0 - 3(0) + F^\alpha_\beta F^\beta_\gamma F^\gamma_\alpha + (-F^\beta_\alpha)(-F^\beta_\gamma)(-F^\gamma_\alpha) = 0$
- d): **Faraday Det** $[\text{anti-symmetric } F^{\mu\nu}] = \text{Pfaffian}[F^{\mu\nu}]^2 = [(-e^x/c)(-b^x) - (-e^y/c)(b^y) + (-e^z/c)(-b^z)]^2 = [(e^x b^x/c) + (e^y b^y/c) + (e^z b^z/c)]^2 = \{(\mathbf{e}\cdot\mathbf{b})/c\}^2$

Importantly, the Faraday EM Tensor has only (2) linearly-independent invariants:

- b) $2\{(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{e}\cdot\mathbf{e}/c^2)\}$
 - d) $\{(\mathbf{e}\cdot\mathbf{b})/c\}^2$
- a) & c) give 0=0, and do not provide additional constraints

The 4-Gradient and 4-EMVectorPotential have (4) independent components each, for total of (8). Subtract the (2) invariants which provide constraints to get a total of (6) independent components = (6) independent components of a 4x4 anti-symmetric tensor = (3) 3-electric \mathbf{e} + (3) 3-magnetic \mathbf{b} = (6) independent EM field components

Note: It is possible to have non-zero \mathbf{e} and \mathbf{b} , yet still have zeroes in the Tensor Invariants.

If \mathbf{e} is orthogonal to \mathbf{b} , then $\text{Det}[F^{\alpha\beta}] = \{(\mathbf{b}\cdot\mathbf{e})/c\}^2 = 0$.

If $(\mathbf{b}\cdot\mathbf{b}) = (\mathbf{e}\cdot\mathbf{e}/c^2)$, then $\text{InnerProd}[F^{\alpha\beta}] = 2\{(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{e}\cdot\mathbf{e}/c^2)\} = 0$.

This condition leads to the properties of EM waves = photons = null 4-vectors, which have fields $|\mathbf{b}| = |\mathbf{e}|/c$ and \mathbf{b} orthogonal to \mathbf{e} , travelling at velocity c .

4-Gradient
 $\partial = \partial^\mu = (\partial/c, -\nabla)$

Faraday EM Tensor
 $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge A$

$$\begin{bmatrix} F^{tt} & F^{tx} & F^{ty} & F^{tz} \\ F^{xt} & F^{xx} & F^{xy} & F^{xz} \\ F^{yt} & F^{yx} & F^{yy} & F^{yz} \\ F^{zt} & F^{zx} & F^{zy} & F^{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \partial^0 a^1 - \partial^1 a^0 & \partial^0 a^2 - \partial^2 a^0 & \partial^0 a^3 - \partial^3 a^0 \\ \partial^1 a^0 - \partial^0 a^1 & 0 & \partial^1 a^2 - \partial^2 a^1 & \partial^1 a^3 - \partial^3 a^1 \\ \partial^2 a^0 - \partial^0 a^2 & \partial^2 a^1 - \partial^1 a^2 & 0 & \partial^2 a^3 - \partial^3 a^2 \\ \partial^3 a^0 - \partial^0 a^3 & \partial^3 a^1 - \partial^1 a^3 & \partial^3 a^2 - \partial^2 a^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (\partial^t a^x + \nabla^x \phi)/c & (\partial^t a^y + \nabla^y \phi)/c & (\partial^t a^z + \nabla^z \phi)/c \\ [(-\nabla^x \phi - \partial^t a^x)/c] & 0 & -\nabla^x a^y + \nabla^y a^x & -\nabla^x a^z + \nabla^z a^x \\ [(-\nabla^y \phi - \partial^t a^y)/c] & -\nabla^y a^x + \nabla^x a^y & 0 & -\nabla^y a^z + \nabla^z a^y \\ [(-\nabla^z \phi - \partial^t a^z)/c] & -\nabla^z a^x + \nabla^x a^z & -\nabla^z a^y + \nabla^y a^z & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -e^x/c & -e^y/c & -e^z/c \\ +e^x/c & 0 & -b^z & +b^y \\ +e^y/c & +b^z & 0 & -b^x \\ +e^z/c & -b^y & +b^x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -e/c \\ +e^i/c, & -\epsilon^{ijk} b^k \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -e/c \\ +e^T/c, & -\nabla \wedge \mathbf{a} \end{bmatrix}$$

$\text{Tr}[F^{\mu\nu}] = F^\nu_\nu = 0$

Trace Tensor Invariant

$F_{\mu\nu} F^{\mu\nu} = 2\{(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{e}\cdot\mathbf{e}/c^2)\}$

Inner Product Tensor Invariant

$\text{AsymmTri}[F^{\mu\nu}] = 0$

Asymm Tri-Product Tensor Invariant

$\text{Det}[F^{\mu\nu}] = \{(\mathbf{e}\cdot\mathbf{b})/c\}^2$

Determinant Tensor Invariant

4-(EM)VectorPotential
 $A = A^\mu = (\phi/c, \mathbf{a})$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V}\cdot\mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Study: SR 4-Tensors

SR Tensor Invariants

for 4-AngularMomentum Tensor

A Tensor Study of Physical 4-Vectors

The 4-AngularMomentum Tensor $M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha = \mathbf{X} \wedge \mathbf{P}$ is an anti-symmetric tensor
 The 3-mass-moment components ($\mathbf{n} = \mathbf{n}$) are in the temporal-spatial sections.
 The 3-angular-momentum components ($\mathbf{l} = \mathbf{l}$) are in the only-spatial section.

(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$: Has (4+) Tensor Invariants (though not all independent)

- a) $T^\alpha_\alpha = \text{Trace} = \text{Sum of EigenValues for } (1,1)\text{-Tensors (mixed)}$
- b) $T^\alpha_{[\alpha} T^{\beta]}_{\beta]} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product}$
- c) $T^\alpha_{[\alpha} T^\beta_{\beta} T^\gamma_{\gamma]} = \text{Asymm Tri-Product} \rightarrow \text{?Name?}$
- d) $T^\alpha_{[\alpha} T^\beta_{\beta} T^\gamma_{\gamma} T^\delta_{\delta]} = \text{Asymm Quad-Product} \rightarrow 4\text{D Determinant} = \text{Product of EigenValues for } (1,1)\text{-Tensors}$

- a): 4-AngMom Trace $[M^{\mu\nu}] = M_{\nu}^{\nu} = (M^{00} - M^{11} - M^{22} - M^{33}) = (0 - 0 - 0 - 0) = 0$
- b): 4-AngMom Inner Product $M_{\mu\nu} M^{\mu\nu} = \sum_{\mu=\nu} [M^{\mu\nu}]^2 - 2\sum_i [M^{0i}]^2 + 2\sum_{i>j} [M^{ij}]^2 = (0) - 2(c^2 \mathbf{n} \cdot \mathbf{n}) + 2(\mathbf{l} \cdot \mathbf{l}) = 2\{(\mathbf{l} \cdot \mathbf{l}) - (c^2 \mathbf{n} \cdot \mathbf{n})\}$
- c): 4-AngMom Asymm Tri $[M^{\mu\nu}] = \text{Tr}[M^{\mu\nu}]^3 - 3(\text{Tr}[M^{\mu\nu}]) (M^\alpha_\beta M^\beta_\alpha) + M^\alpha_\beta M^\beta_\gamma M^\gamma_\alpha + M^\alpha_\beta M^\beta_\alpha M^\gamma_\gamma = 0$
- d): 4-AngMom Det $[\text{anti-symmetric } M^{\mu\nu}] = \text{Pfaffian}[M^{\mu\nu}]^2 = [(-cn^x)(+l^x) - (-cn^y)(-l^y) + (-cn^z)(+l^z)]^2 = [-(cn^x l^x) - (cn^y l^y) - (cn^z l^z)]^2 = \{c(\mathbf{n} \cdot \mathbf{l})\}^2$

Importantly, the 4-AngularMomentum Tensor has only (2) linearly-independent invariants:

- b) $2\{(\mathbf{l} \cdot \mathbf{l}) - (c^2 \mathbf{n} \cdot \mathbf{n})\}$: see Wikipedia Laplace-Runge-Lenz_vector, sec. Casimir Invariants
 - d) $\{c(\mathbf{l} \cdot \mathbf{n})\}^2$
- a) & c) give 0=0, and do not provide additional constraints

The 4-Position and 4-Momentum have (4) independent components each, for total of (8).
 Subtract the (2) invariants which provide constraints to get a total of (6) independent components
 = (6) independent components of a 4x4 anti-symmetric tensor
 = (3) 3-mass-moment \mathbf{n} + (3) 3-angular-momentum \mathbf{l} = (6) independent 4-AngularMomentum components

3-massmoment $\mathbf{n} = \mathbf{xm} - \mathbf{tp} = m(\mathbf{x} - \mathbf{tu}) = m(\mathbf{r} - \mathbf{tu}) = m(\mathbf{r} - t(\boldsymbol{\omega} \times \mathbf{r}))$: Tangential velocity $\mathbf{u}_T = (\boldsymbol{\omega} \times \mathbf{r})$

$(-k/r)\mathbf{n} = -mk(\hat{\mathbf{r}} - t(\boldsymbol{\omega} \times \hat{\mathbf{r}})) = mkt(\boldsymbol{\omega} \times \hat{\mathbf{r}}) - mk\hat{\mathbf{r}} = t * d/dt(\mathbf{p}) \times \mathbf{L} - mk\hat{\mathbf{r}} : d/dt(\mathbf{p}) \times \mathbf{L} = mk(\boldsymbol{\omega} \times \hat{\mathbf{r}})$

\mathbf{n} is related to the LRL = Laplace-Runge-Lenz 3-vector: $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$
 which is another classical conserved vector. The invariance is shown here to be relativistic in origin.
 Wikipedia article: Laplace-Runge-Lenz vector shows these as Casimir Invariants.
 See Also: Relativistic Angular Momentum.

4-Position
 $\mathbf{X} = X^\mu = (ct, \mathbf{x})$

$\text{Tr}[M^{\mu\nu}] = M_{\nu}^{\nu} = 0$

Trace
Tensor Invariant

$M_{\mu\nu} M^{\mu\nu} = 2\{(\mathbf{l} \cdot \mathbf{l}) - (c^2 \mathbf{n} \cdot \mathbf{n})\}$

Inner Product
Tensor Invariant

$\text{Asymm Tri}[M^{\mu\nu}] = 0$

Asymm Tri-Product
Tensor Invariant

$\text{Det}[M^{\mu\nu}] = \{c(\mathbf{n} \cdot \mathbf{l})\}^2$

Determinant
Tensor Invariant

4-Momentum
 $\mathbf{P} = P^\mu = (mc, \mathbf{p}) = (E/c, \mathbf{p})$

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$

4-AngularMomentum
Tensor

$M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha = \mathbf{X} \wedge \mathbf{P}$

\rightarrow
 $[M^t M^x M^y M^z]$
 $[M^{xt} M^{xx} M^{xy} M^{xz}]$
 $[M^{yt} M^{yx} M^{yy} M^{yz}]$
 $[M^{zt} M^{zx} M^{zy} M^{zz}]$

$=$
 $[\begin{matrix} 0 & x^0 p^1 - x^1 p^0 & x^0 p^2 - x^2 p^0 & x^0 p^3 - x^3 p^0 \\ x^1 p^0 - x^0 p^1 & 0 & x^1 p^2 - x^2 p^1 & x^1 p^3 - x^3 p^1 \\ x^2 p^0 - x^0 p^2 & x^2 p^1 - x^1 p^2 & 0 & x^2 p^3 - x^3 p^2 \\ x^3 p^0 - x^0 p^3 & x^3 p^1 - x^1 p^3 & x^3 p^2 - x^2 p^3 & 0 \end{matrix}]$

$=$
 $[\begin{matrix} 0 & ctp^x - xE/c & ctp^y - yE/c & ctp^z - zE/c \\ xE/c - ctp^x & 0 & xp^y - yp^x & xp^z - zp^x \\ yE/c - ctp^y & yp^x - xp^y & 0 & yp^z - zp^y \\ zE/c - ctp^z & zp^x - xp^z & zp^y - yp^z & 0 \end{matrix}]$

$=$
 $[\begin{matrix} 0 & c(tp^x - xm) & c(tp^y - ym) & c(tp^z - zm) \\ c(xm - tp^x) & 0 & xp^y - yp^x & xp^z - zp^x \\ c(ym - tp^y) & yp^x - xp^y & 0 & yp^z - zp^y \\ c(zm - tp^z) & zp^x - xp^z & zp^y - yp^z & 0 \end{matrix}]$

$=$
 $[\begin{matrix} 0 & -cn^x & -cn^y & -cn^z \\ +cn^x & 0 & +l^z & -l^y \\ +cn^y & -l^z & 0 & +l^x \\ +cn^z & +l^y & -l^x & 0 \end{matrix}]$

$=$
 $[\begin{matrix} 0 & , & -cn^j \\ +cn^i & , & \epsilon^{ijk} l^k \end{matrix}]$

$=$
 $[\begin{matrix} 0 & , & -c\mathbf{n} \\ +cn^T & , & \mathbf{x} \wedge \mathbf{p} \end{matrix}]$

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or T_{μ}^ν
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

SRQM Study: SR 4-Tensors

SR Tensor Invariants

for Minkowski Metric Tensor

The Minkowski Metric Tensor $\eta^{\mu\nu}$ is the tensor all SR 4-Vectors are measured by.

(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$: Has (4+) Tensor Invariants (though not all independent)

- a) T^{α}_{α} = Trace = Sum of EigenValues for (1,1)-Tensors (mixed)
- b) $T^{\alpha}_{[\alpha} T^{\beta]}_{\beta]}$ = Asymm Bi-Product → Inner Product
- c) $T^{\alpha}_{[\alpha} T^{\beta}_{\beta} T^{\gamma]}_{\gamma]}$ = Asymm Tri-Product → ?Name?
- d) $T^{\alpha}_{[\alpha} T^{\beta}_{\beta} T^{\gamma}_{\gamma} T^{\delta]}_{\delta]}$ = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors

- a) **Minkowski Trace** $[\eta^{\mu\nu}] = 4$
- b) **Minkowski Inner Product** $\eta_{\mu\nu}\eta^{\mu\nu} = 4$
- c) **Minkowski AsymmTri** $[\eta^{\mu\nu}] = 24 = 4!$, if I did the math right...
- d) **Minkowski Det** $[\eta^{\mu\nu}] = -1$

$$\Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} \eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\text{Det}(\text{Exp}[A]) = \text{Exp}(\text{Tr}[A])$$

$$\text{Det}_{4D}(A) = ((\text{tr } A)^4 - 6 \text{tr}(A^2)(\text{tr } A)^2 + 3(\text{tr}(A^2))^2 + 8 \text{tr}(A^3) \text{tr } A - 6 \text{tr}(A^4))/24$$

GR Trace Tensor Invariant
4D SpaceTime

In GR

$$\text{Tr}[g^{\mu\nu}] = g_{\mu\nu}g^{\mu\nu} = g^{\mu}_{\mu} = \delta^{\mu}_{\mu} = 1+1+1+1 = 4$$

4-Gradient
 $\partial = \partial^{\mu} = (\partial/c, -\nabla)$

EigenValues $[\eta^{\mu}_{\nu}] = \text{Set}\{1, 1, 1, 1\}$
Eigenvalues Tensor Invariants

Signature $[\eta^{\mu\nu}] = (+, -, -, -)$
 $= \{1, 3, 0\} = (1-3) = -2$
Signature Tensor Invariant

4-Position
 $R = R^{\mu} = (ct, \mathbf{r})$

Trace Tensor Invariant
 $\text{Tr}[\eta^{\mu\nu}] = (1) - (-1) - (-1) - (-1) = 4$
 $\eta_{\mu\nu}\eta^{\mu\nu} = \eta^{\mu}_{\mu} = \delta^{\mu}_{\mu} = 1+1+1+1$

$\partial[R] = \partial^{\mu}R^{\nu} = \eta^{\mu\nu}$

→

Diag[1, -1, -1, -1]
Diag[1, -I₍₃₎]
Diag[1, -δ^{jk}]
=
[+1 0 0 0]
[0 -1 0 0]
[0 0 -1 0]
[0 0 0 -1]
(in Cartesian form)

$\eta_{\mu\nu}\eta^{\mu\nu} = 4$
Inner Product Tensor Invariant

$[\eta_{\mu\mu}] = 1/[\eta^{\mu\mu}] : \eta_{\mu}^{\nu} = \delta_{\mu}^{\nu}$
SR: Minkowski Metric
"Particle Physics" Convention
 $\text{Det}[\eta^{\mu\nu}] = -1$
 $\text{Det}[\eta^{\nu}_{\mu}] = +1$
Determinant Tensor Invariant

AsymmTri $[\eta^{\mu\nu}] = 24$
Asymm Tri-Product Tensor Invariant

EigenValues not defined for the standard Minkowski Metric Tensor since it is a type (2,0)-Tensor, all upper indices. However, they are defined for the mixed form (1,1)-Tensor
EigenValues are defined for the Lorentz Transforms since they are type (1,1)-Tensors, mixed indices

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (\mathbf{v}_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Det}[T^{\alpha}_{\alpha}] = \prod_k[\lambda_k]$; with $\{\lambda_k\} = \text{Eigenvalues}$
Characteristic Eqns: $\text{Det}[T^{\alpha}_{\alpha} - \lambda_k I_{(4)}] = 0$

Trace $[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Study: SR 4-Tensors

SR Tensor Invariants for Continuous Lorentz Transform Tensors

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The Lorentz Transform Tensor $\{\Lambda^\mu_\nu = \partial x^\mu / \partial x^\nu = \partial_\nu [X^\mu]\}$ is the tensor all SR 4-Vectors must transform by.

- (2,0)-Tensor = 4-Tensor $T^{\mu\nu}$. Has (4+) Tensor Invariants (though not all independent)
- a) $T^\alpha_\alpha = \text{Trace} = \text{Sum of EigenValues for } (1,1)\text{-Tensors (mixed)}$
 - b) $T^\alpha_{[\alpha} T^\beta_{\beta]} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product}$
 - c) $T^\alpha_{[\alpha} T^\beta_\beta T^\gamma_{\gamma]} = \text{Asymm Tri-Product} \rightarrow \text{?Name?}$
 - d) $T^\alpha_{[\alpha} T^\beta_\beta T^\gamma_\gamma T^\delta_{\delta]} = \text{Asymm Quad-Product} \rightarrow \text{4D Determinant} = \text{Product of EigenValues for } (1,1)\text{-Tensors}$

- a): Lorentz Trace $[\Lambda^\mu_\nu] = \{0..4..Infinity\}$ Lorentz Boost meets Rotation at Identity of 4
- b): Lorentz Inner Product $\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$ from $\{\eta_{\mu\nu}\Lambda^\mu_\alpha\Lambda^\nu_\beta = \eta_{\alpha\beta}\}$ and $\{\eta_{\mu\nu}\eta^{\mu\nu} = 4\}$
- c): Lorentz AsymmTri $[\Lambda^{\mu\nu}] =$
- d): Lorentz Det $[\Lambda^{\mu\nu}] = +1$ for Proper Transforms, Continuous Transforms Proper

An even more general version would be with a & b as arbitrary complex values:

could be 2 boosts, 2 rotations, or a boost:rotation combo



SR:Lorentz Transform

$$\partial_\nu [R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu$$

$$\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$$

$$\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$$

Det $[\Lambda^\mu_\nu] = \pm 1$ $\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$

EigenValues $[\Lambda^\mu_\nu]$
=Set $\{e^a, e^{-a}, e^b, e^{-b}\}$

Sum of EigenValues $[\Lambda^\mu_\nu]$
=Tr $[\Lambda^\mu_\nu] = \Lambda^\mu_\mu$
= $\{e^a + e^{-a} + e^b + e^{-b}\}$
= $2(\cosh[a] + \cosh[b])$
= $\{-4..Infinity\}$

Product of EigenValues $[\Lambda^\mu_\nu]$
=Det $[\Lambda^\mu_\nu]$
= $\{e^a \cdot e^{-a} \cdot e^b \cdot e^{-b}\}$
= +1

Inner Product Tensor Invariant

$\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$

Asymm Tri-Product Tensor Invariant

AsymmTri $[\Lambda^\mu_\nu] = ?$
Not yet calc...

Trace Tensor Invariant

Tr $[\text{Cont. } \Lambda^\mu_\nu] = \{0..4..Infinity\}$
Depends on "rotation" amount

Determinant Tensor Invariant

Det $[\text{Proper } \Lambda^\mu_\nu] = +1$
Proper Transform always +1

Rotation(0) = Identity = Boost(0)

Lorentz SR Rotation Tensor $\Lambda^\mu_\nu \rightarrow R^\mu_\nu$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[\theta] & -\sin[\theta] & 0 \\ 0 & \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues $[R^\mu_\nu]$
=Set $\{1, e^{i\theta}, e^{-i\theta}, 1\}$

Sum of EigenValues $[R^\mu_\nu]$
=Tr $[R^\mu_\nu] = R^\mu_\mu$
= $1 + e^{i\theta} + e^{-i\theta} + 1$
= $2 + 2\cos[\theta]$
= $\{0..4\}$

Product of EigenValues $[R^\mu_\nu]$
=Det $[R^\mu_\nu]$
= $1 \cdot e^{i\theta} \cdot e^{-i\theta} \cdot 1$
= +1

Proper

Lorentz SR Identity Tensor $\Lambda^\mu_\nu \rightarrow \eta^\mu_\nu$

$$= R^\mu_\nu [0] = B^\mu_\nu [0] = \delta^\mu_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

= Minkowski Delta

EigenValues $[\eta^\mu_\nu]$
=Set $\{1, 1, 1, 1\}$

Sum of EigenValues $[\eta^\mu_\nu]$
=Tr $[\eta^\mu_\nu] = \eta^\mu_\mu$
= $1 + 1 + 1 + 1$
= 4

Product of EigenValues $[\eta^\mu_\nu]$
=Det $[\eta^\mu_\nu]$
= $1 \cdot 1 \cdot 1 \cdot 1$
= +1

Proper

Lorentz SR Boost Tensor $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$

$$= \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues $[B^\mu_\nu]$
=Set $\{e^\theta, e^{-\theta}, 1, 1\}$

Sum of EigenValues $[B^\mu_\nu]$
=Tr $[B^\mu_\nu] = B^\mu_\mu$
= $e^\theta + e^{-\theta} + 1 + 1$
= $2 + 2\cosh[\theta] = 2 + 2\gamma$
= $\{4..Infinity\}$

Product of EigenValues $[B^\mu_\nu]$
=Det $[B^\mu_\nu]$
= $e^\theta \cdot e^{-\theta} \cdot 1 \cdot 1$
= +1

Proper

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Det $[T^\alpha_\alpha] = \prod_k [\lambda_k]$; with $\{\lambda_k\} = \text{Eigenvalues}$
Characteristic Eqns: Det $[T^\alpha_\alpha - \lambda_k I_{(4)}] = 0$

Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Study: SR 4-Tensors

SR Tensor Invariants for

Discrete Lorentz Transform Tensors

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$$\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$$

$$\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$$

Det $[\Lambda^\mu_\nu] = \pm 1$ **$\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$**

Inner Product Tensor Invariant

$$\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$$

Asymm Tri-Product Tensor Invariant

AsymmTri $[\Lambda^\mu_\nu] = ?$
Not yet calc...



The Trace of various discrete Lorentz transforms varies in steps from $\{-4, -2, 0, 2, 4\}$

This includes Mirror Flips, Time Reversal, and Parity Inverse – essentially taking all combinations of ± 1 on the diagonal of the transform.

Trace Tensor Invariant

$$\text{Tr}[\text{Discrete } \Lambda^\mu_\nu] = \{-4, -2, 0, 2, 4\}$$

Depends on transform

Determinant Tensor Invariant

$$\text{Det}[\Lambda^\mu_\nu] = \pm 1$$

Proper Transform = +1
Improper Transform = -1

Lorentz SR TPcombo
Tensor $\Lambda^\mu_\nu \rightarrow \text{TP}^\mu_\nu$
= $-\eta^\mu_\nu = -\delta^\mu_\nu =$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
 = Negative Identity

EigenValues $[\text{TP}^\mu_\nu]$
= Set $\{-1, -1, -1, -1\}$

Sum of EigenValues $[\text{TP}^\mu_\nu]$
= $\text{Tr}[\text{TP}^\mu_\nu] = \text{TP}^\mu_\mu$
= $-1-1-1-1 = -4$

Product of EigenValues $[\text{TP}^\mu_\nu]$
= $\text{Det}[\text{TP}^\mu_\nu]$
= $-1 \cdot -1 \cdot -1 \cdot -1 = +1$

Proper

Lorentz SR Parity-Inversion
Tensor $\Lambda^\mu_\nu \rightarrow \text{P}^\mu_\nu$
=

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
 = Flip-xyz

EigenValues $[\text{P}^\mu_\nu]$
= Set $\{1, -1, -1, -1\}$

Sum of EigenValues $[\text{P}^\mu_\nu]$
= $\text{Tr}[\text{P}^\mu_\nu] = \text{P}^\mu_\mu$
= $1-1-1-1 = -2$

Product of EigenValues $[\text{P}^\mu_\nu]$
= $\text{Det}[\text{P}^\mu_\nu]$
= $1 \cdot -1 \cdot -1 \cdot -1 = -1$

Improper

Lorentz SR Flip-xy-Combo
Tensor $\Lambda^\mu_\nu \rightarrow \text{Fxy}^\mu_\nu$
= $-\eta^\mu_\nu = -\delta^\mu_\nu =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
 = Rotation-z (π)

EigenValues $[\text{Fxy}^\mu_\nu]$
= Set $\{1, -1, -1, 1\}$

Sum of EigenValues $[\text{Fxy}^\mu_\nu]$
= $\text{Tr}[\text{Fxy}^\mu_\nu] = \text{Fxy}^\mu_\mu$
= $1-1-1+1 = 0$

Product of EigenValues $[\text{Fxy}^\mu_\nu]$
= $\text{Det}[\text{Fxy}^\mu_\nu]$
= $-1 \cdot -1 \cdot -1 \cdot 1 = +1$

Proper

Lorentz SR Time-Reversal
Tensor $\Lambda^\mu_\nu \rightarrow \text{T}^\mu_\nu$
=

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
 = Flip-t

EigenValues $[\text{T}^\mu_\nu]$
= Set $\{-1, 1, 1, 1\}$

Sum of EigenValues $[\text{T}^\mu_\nu]$
= $\text{Tr}[\text{T}^\mu_\nu] = \text{T}^\mu_\mu$
= $-1+1+1+1 = 2$

Product of EigenValues $[\text{T}^\mu_\nu]$
= $\text{Det}[\text{T}^\mu_\nu]$
= $-1 \cdot 1 \cdot 1 \cdot 1 = -1$

Improper

Lorentz SR Identity
Tensor $\Lambda^\mu_\nu \rightarrow \eta^\mu_\nu$
= $\delta^\mu_\nu =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
 = Minkowski Delta

EigenValues $[\eta^\mu_\nu]$
= Set $\{1, 1, 1, 1\}$

Sum of EigenValues $[\eta^\mu_\nu]$
= $\text{Tr}[\eta^\mu_\nu] = \eta^\mu_\mu$
= $1+1+1+1 = 4$

Product of EigenValues $[\eta^\mu_\nu]$
= $\text{Det}[\eta^\mu_\nu]$
= $1 \cdot 1 \cdot 1 \cdot 1 = +1$

Proper

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Det}[T^\alpha_\alpha] = \prod_k [\lambda_k]$; with $\{\lambda_k\} = \text{Eigenvalues}$
Characteristic Eqns: $\text{Det}[T^\alpha_\alpha - \lambda_k I_{(4)}] = 0$

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_\alpha)^2 = \text{Lorentz Scalar}$

SRQM Study: SR 4-Tensors

More SR Tensor Invariants for Discrete Lorentz Transform Tensors

A Tensor Study of Physical 4-Vectors

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John B. Wilson

SR:Lorentz Transform

$$\partial_\nu[R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu$$

$$\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$$

$$\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$$

Det $[\Lambda^\mu_\nu] = \pm 1$ $\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$



Lorentz SR
0-Rotation-z
Tensor $\Lambda^{\mu\nu} \rightarrow R^{\mu\nu}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[0] & -\sin[0] & 0 \\ 0 & \sin[0] & \cos[0] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lorentz SR
Identity
Tensor $\Lambda^{\mu\nu} \rightarrow \eta^{\mu\nu}$

$$= \delta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

= Minkowski Delta

Lorentz SR
Flip-x
Tensor $\Lambda^{\mu\nu} \rightarrow Fx^{\mu\nu}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lorentz SR
Flip-y
Tensor $\Lambda^{\mu\nu} \rightarrow Fy^{\mu\nu}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lorentz SR
Flip-xy-Combo
Tensor $\Lambda^{\mu\nu} \rightarrow Fxy^{\mu\nu}$

$$= -\eta^{\mu\nu} = -\delta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

= Rotation-z (π)

Lorentz SR
 π -Rotation-z
Tensor $\Lambda^{\mu\nu} \rightarrow R^{\mu\nu}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[\pi] & -\sin[\pi] & 0 \\ 0 & \sin[\pi] & \cos[\pi] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EigenValues $[R^{\mu\nu}]$
=Set $\{1, e^{i0}, e^{-i0}, 1\}$

EigenValues $[\eta^{\mu\nu}]$
=Set $\{1, 1, 1, 1\}$

EigenValues $[Fx^{\mu\nu}]$
=Set $\{1, -1, 1, 1\}$

EigenValues $[Fy^{\mu\nu}]$
=Set $\{1, 1, -1, 1\}$

EigenValues $[Fxy^{\mu\nu}]$
=Set $\{1, -1, -1, 1\}$

EigenValues $[R^{\mu\nu}]$
=Set $\{1, e^{i\pi}, e^{-i\pi}, 1\}$

Sum of EigenValues $[R^{\mu\nu}]$
=Tr $[R^{\mu\nu}] = R^{\mu\nu}_{\mu\nu}$
=1+eⁱ⁰+e⁻ⁱ⁰+1
=2+2cos[0]
=4

Sum of EigenValues $[\eta^{\mu\nu}]$
=Tr $[\eta^{\mu\nu}] = \eta^{\mu\nu}_{\mu\nu}$
=1+1+1+1
=2+2cos[0]
=4

Sum of EigenValues $[Fx^{\mu\nu}]$
=Tr $[Fx^{\mu\nu}] = Fx^{\mu\nu}_{\mu\nu}$
=1-1+1+1
=2

Sum of EigenValues $[Fy^{\mu\nu}]$
=Tr $[Fy^{\mu\nu}] = Fy^{\mu\nu}_{\mu\nu}$
=1+1-1+1
=2

Sum of EigenValues $[Fxy^{\mu\nu}]$
=Tr $[Fxy^{\mu\nu}] = Fxy^{\mu\nu}_{\mu\nu}$
=1-1-1+1
=2+2cos[π]
=0

Sum of EigenValues $[R^{\mu\nu}]$
=Tr $[R^{\mu\nu}] = R^{\mu\nu}_{\mu\nu}$
=1+e^{i π} +e^{-i π} +1
=2+2cos[π]
=0

Product of EigenValues $[R^{\mu\nu}]$
=Det $[R^{\mu\nu}]$
=1·eⁱ⁰·e⁻ⁱ⁰·1
=+1

Product of EigenValues $[\eta^{\mu\nu}]$
=Det $[\eta^{\mu\nu}]$
=1·1·1·1
=+1

Product of EigenValues $[Fx^{\mu\nu}]$
=Det $[Fx^{\mu\nu}]$
=1·-1·1·1
=-1

Product of EigenValues $[Fy^{\mu\nu}]$
=Det $[Fy^{\mu\nu}]$
=1·1·-1·1
=-1

Product of EigenValues $[Fxy^{\mu\nu}]$
=Det $[Fxy^{\mu\nu}]$
=-1·-1·-1·1
=+1

Product of EigenValues $[R^{\mu\nu}]$
=Det $[R^{\mu\nu}]$
=1·e^{i π} ·e^{-i π} ·1
=+1

Proper

Proper

Improper

Improper

Proper

Proper

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
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Det $[T^\alpha_\alpha] = \prod_k [\lambda_k]$; with $\{\lambda_k\} =$ Eigenvalues
Characteristic Eqns: Det $[T^\alpha_\alpha - \lambda_k I_{(4)}] = 0$

Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 =$
= Lorentz Scalar

Note:
The Flip-xy-Combo is the equivalent of a π -Rotation-z.
I suspect that this may be related to exchange symmetry and the Spin-Statistics idea that a particle-exchange is the equivalent of a spin-rotation.
A single Flip would not be an exchange because it leaves a mirror-inversion of <right|-left>.
But the extra Flip along an orthogonal axis corrects the mirror-inversion, and would be an overall exchange because the particle is in a different location.

SR 4-Scalars, 4-Vectors, 4-Tensors

Elegantly join many dual physical properties and relations

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

SR 4-Scalars, 4-Vectors, and 4-Tensors beautifully and elegantly display the relations between lots of different physical properties and relations. Their notation makes navigation through the physics very simple.

They also devolve very nicely into the limiting/approximate Newtonian cases of $\{ |v| \ll c \}$ by letting $\{ \gamma \rightarrow 1 \text{ and } \gamma' = d\gamma/dt \rightarrow 0 \}$.

SR tells us that several different physical properties are actually dual aspects of the same thing, with the only real difference being one's point of view, or reference frame.

Examples of 4-Vectors = (1,0)-Tensors include:
(Time , Space), (Energy , Momentum), (Power , Force), (Frequency , WaveNumber),
(Time Differential , Spatial Gradient),
(ChargeDensity , CurrentDensity), (EM-ScalarPotential , EM-VectorPotential), etc.

One can also examine 4-Tensors, which are type (2,0)-Tensors.
The Faraday EM Tensor similarly combines EM fields:
Electric $\{ \mathbf{e} = e^i = (e^x, e^y, e^z) \}$ and Magnetic $\{ \mathbf{b} = b^k = (b^x, b^y, b^z) \}$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -e^j/c \\ +e^i/c & -(\epsilon^{ij}_k b^k) \end{bmatrix}$$

Also, things are even more related than that.
The 4-Momentum is just a constant times 4-Velocity.
The 4-WaveVector is just a constant times 4-Velocity.

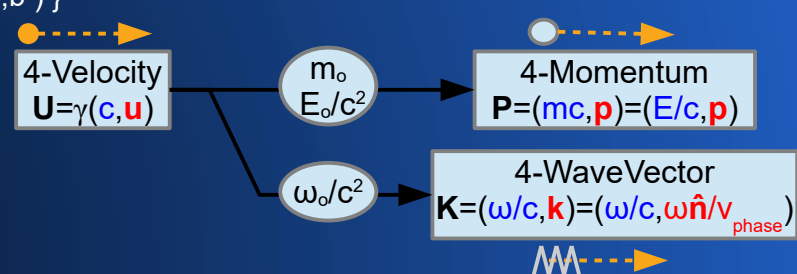
In addition, the very important conservation/continuity equations seem to just fall out of the notation.
The universe apparently has some simple laws which can be easy to write down by using a little math and a super notation.

4-Scalar
S

$$\text{SR 4-Vector } \mathbf{V} = V^\alpha = (v^t, \mathbf{v}) = (v^t, v^x, v^y, v^z) = (\text{temporal} * c^{\pm 1}, \text{spatial})$$

4-Tensor $T^{\alpha\beta}$

$$= \begin{bmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{bmatrix} = \begin{bmatrix} \text{temporal, mixed} \\ \text{mixed, spatial} \end{bmatrix}$$



Faraday EM Tensor $F^{\alpha\beta}$

$$= \begin{bmatrix} 0 & -e^x/c & -e^y/c & -e^z/c \\ +e^x/c & 0 & -b^z & +b^y \\ +e^y/c & +b^z & 0 & -b^x \\ +e^z/c & -b^y & +b^x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -e^j/c \\ +e^i/c & -\epsilon^{ij}_k b^k \end{bmatrix}$$

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(1,1)-Tensor T^μ_ν or T_μ^ν
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(0,0)-Tensor S
Lorentz Scalar

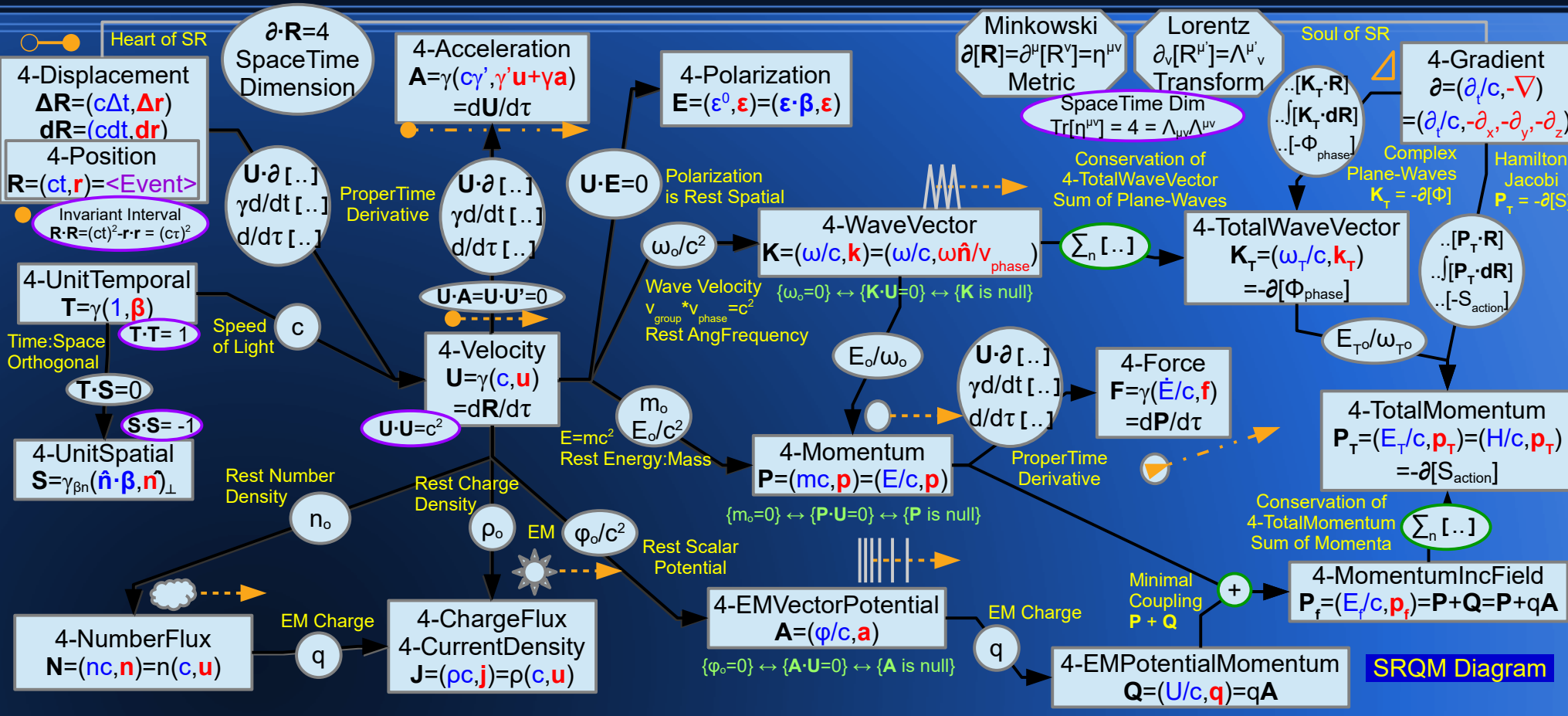
$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$

SRQM Diagram: SR 4-Vectors and Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

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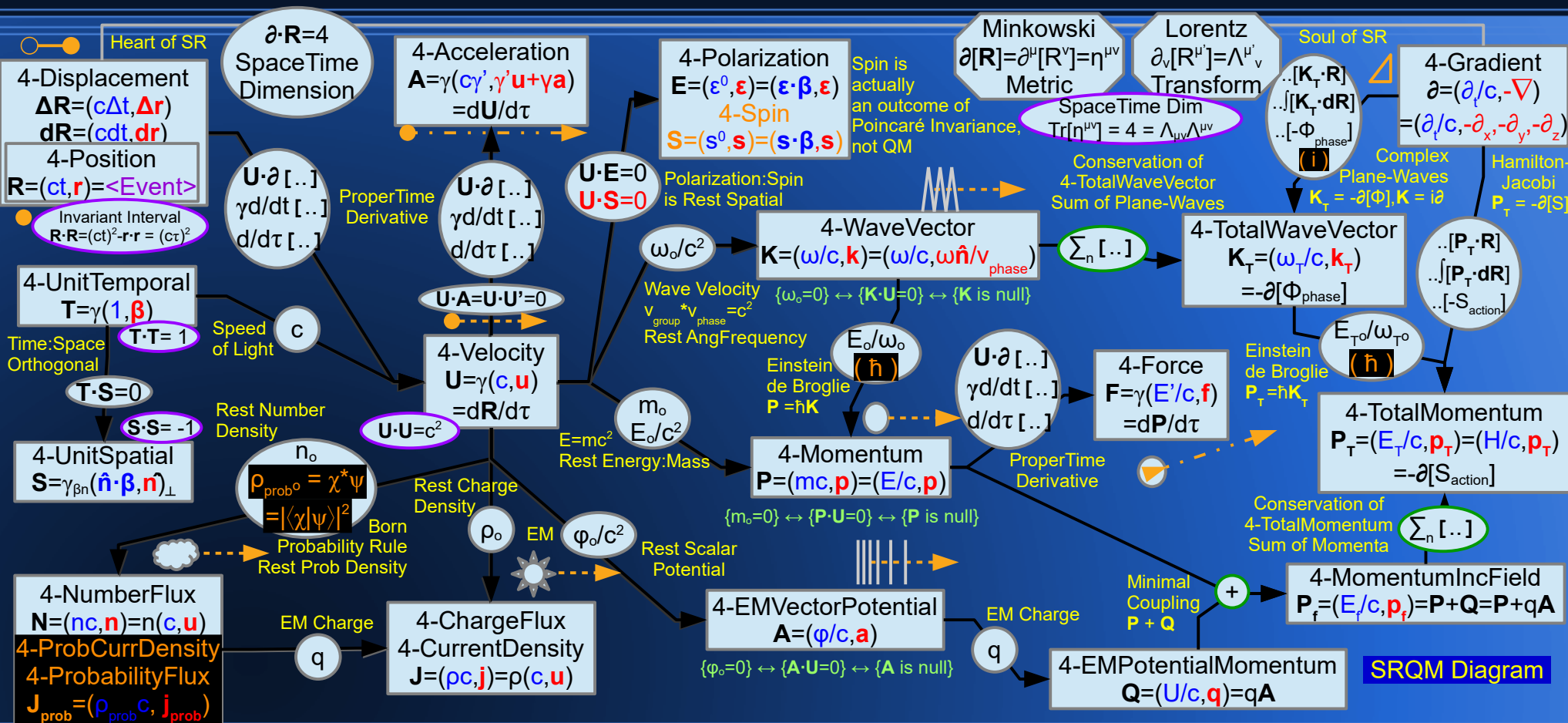


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SRQM Diagram

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu\nu}$ or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = V = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$
 $V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SR Gradient 4-Vectors = (1,0)-Tensors

SR Gradient One-Forms = (0,1)-Tensors

A Tensor Study
of Physical 4-Vectors

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4-Vector = Type (1,0)-Tensor

$$4\text{-Position } \mathbf{R} = R^\mu = (ct, \mathbf{r})$$

$$4\text{-Gradient } \partial_{\mathbf{R}} = \partial = \partial^\mu = \partial/\partial R_\mu = (\partial/c, -\nabla)$$

Standard 4-Vector

$$4\text{-Position } \mathbf{R} = R^\mu = (ct, \mathbf{r})$$

$$4\text{-Velocity } \mathbf{U} = U^\mu = \gamma(c, \mathbf{u})$$

$$4\text{-Momentum } \mathbf{P} = P^\mu = (E/c, \mathbf{p})$$

$$4\text{-WaveVector } \mathbf{K} = K^\mu = (\omega/c, \mathbf{k})$$

[Temporal : Spatial] components

$$[\text{Time (t) : Space (r)}]$$

$$[\text{Time Differential } (\partial_t) : \text{Spatial Gradient}(\nabla)]$$

Related Gradient 4-Vector (from index-raised Gradient One-Form)

$$4\text{-PositionGradient } \partial_{\mathbf{R}} = \partial_{\mathbf{R}}^\mu = \partial/\partial R_\mu = (\partial_{\mathbf{R}^t}/c, -\nabla_{\mathbf{R}}) = \partial = \partial^\mu = 4\text{-Gradient}$$

$$4\text{-VelocityGradient } \partial_{\mathbf{U}} = \partial_{\mathbf{U}}^\mu = \partial/\partial U_\mu = (\partial_{\mathbf{U}^t}/c, -\nabla_{\mathbf{U}})$$

$$4\text{-MomentumGradient } \partial_{\mathbf{P}} = \partial_{\mathbf{P}}^\mu = \partial/\partial P_\mu = (\partial_{\mathbf{P}^t}/c, -\nabla_{\mathbf{P}})$$

$$4\text{-WaveGradient } \partial_{\mathbf{K}} = \partial_{\mathbf{K}}^\mu = \partial/\partial K_\mu = (\partial_{\mathbf{K}^t}/c, -\nabla_{\mathbf{K}})$$

In each case, the (Whichever)Gradient 4-Vector is derived from an SR One-Form or 4-CoVector, which is a type (0,1)-Tensor

$$\text{ex. One-Form PositionGradient } \partial_{\mathbf{R}^v} = \partial/\partial R^v = (\partial_{\mathbf{R}^t}/c, \nabla_{\mathbf{R}})$$

The (Whichever)Gradient 4-Vector is the index-raised version of the SR One-Form (Whichever)Gradient

$$\text{ex. 4-PositionGradient } \partial_{\mathbf{R}}^\mu = \partial/\partial R_\mu = (\partial_{\mathbf{R}^t}/c, -\nabla_{\mathbf{R}}) = \eta^{\mu\nu} \partial_{\mathbf{R}^v} = \eta^{\mu\nu} \partial/\partial R^v = \eta^{\mu\nu} (\partial_{\mathbf{R}^t}/c, \nabla_{\mathbf{R}})_v = \eta^{\mu\nu} (\text{One-Form PositionGradient})_v$$

This is why the 4-Gradient is commonly seen with a minus sign in the spatial component, unlike the other regular 4-Vectors, which have all positive components.

4-Tensors can be constructed from the Tensor Outer Product of 4-Vectors

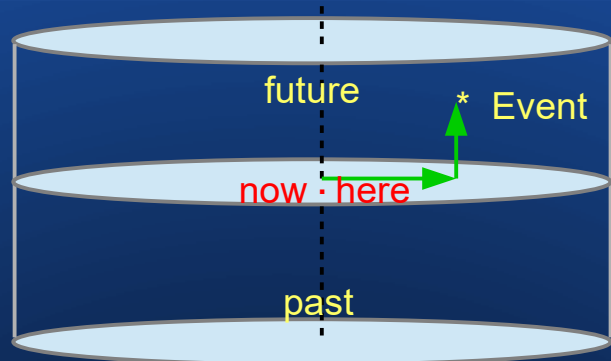
Some Basic 4-Vectors

Minkowski SpaceTime Diagram

Events & Dimensions

A Tensor Study of Physical 4-Vectors

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“Stack of Motion Picture Photos”

Δt time-like interval

$\Delta \mathbf{r}$ space-like interval

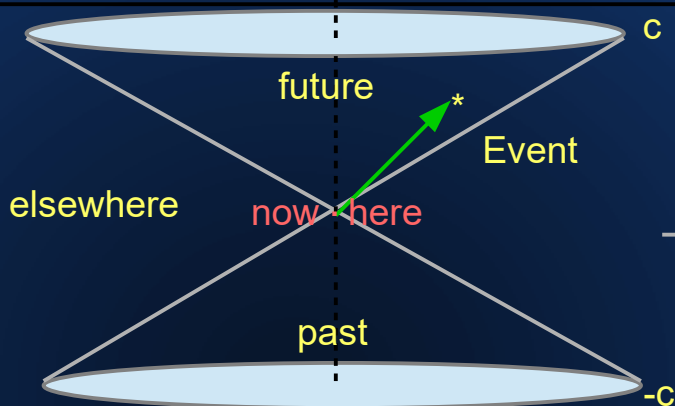
4-Displacement $\Delta \mathbf{R}_{CM} = (c\Delta t, \Delta \mathbf{r})$

$1/c$

Classical Mechanics
time displacement Δt

3-displacement $\Delta \mathbf{r} = \Delta r^i \rightarrow (\Delta x, \Delta y, \Delta z)$

Note the separate dimensional units: (time + 3D space)
 Δt is [time], $|\Delta \mathbf{r}|$ is [length]



LightCone

Δt time-like interval (+)

c light-like interval (0) = null

$\Delta \mathbf{r}$ space-like interval (-)

4-Displacement $\Delta \mathbf{R} = (c\Delta t, \Delta \mathbf{r})$
4-Position $\mathbf{R} = (ct, \mathbf{r})$

$\Delta \mathbf{R} \cdot \Delta \mathbf{R} = [(c\Delta t)^2 - \Delta \mathbf{r} \cdot \Delta \mathbf{r}] = 0$

$(c\Delta \tau)^2$ Time-Like (+)
Light-like: Null (0)
 $-(\Delta r_o)^2$ Space-like (-)

Note the matching dimensional units: (4D SpaceTime)

$(c\Delta t)$ is [length/time]*[time] = [length], $|\Delta \mathbf{r}|$ is [length], $|\Delta \mathbf{R}|$ is [length]

τ is the Proper Time = “rest-time”, time as measured by something not moving spatially

The Minkowski Diagram provides a great visual representation of SpaceTime

Special Relativity

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Classical (scalar) Galilean Invariant
3-vector Not Lorentz Invariant

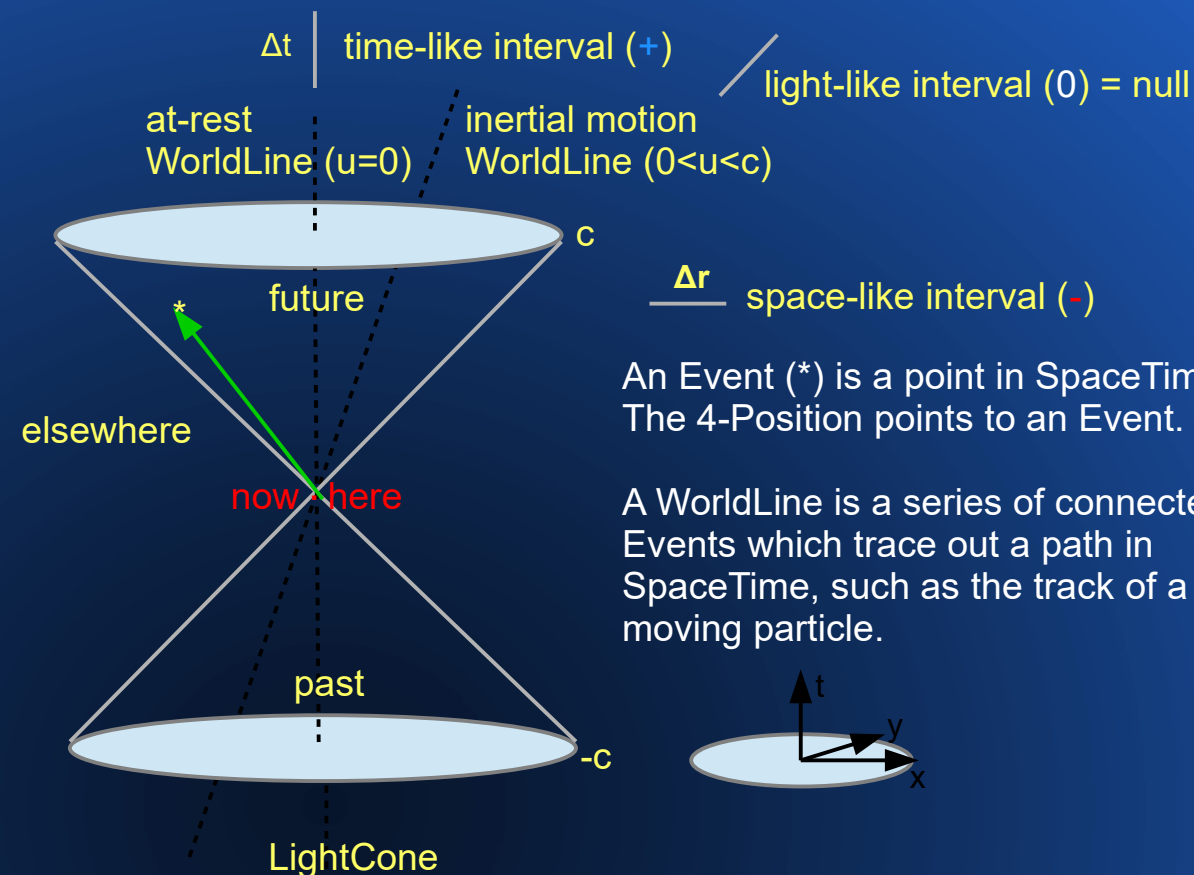
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Some Basic 4-Vectors

Minkowski SpaceTime Diagram, WorldLines, LightSpeed to the Future!

A Tensor Study of Physical 4-Vectors

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4-Displacement
 $\Delta R = (c\Delta t, \Delta r)$

4-Position
 $R = (ct, r) = \langle \text{Event} \rangle$

The 4-Position is a particular type of 4-Displacement, for which the vector base is at the origin (0,0,0,0) = 4-Zero.

4-Position is Lorentz Invariant, but not Poincaré Invariant.
 A standard 4-Displacement is both.

$\Delta R \cdot \Delta R = [(c\Delta t)^2 - \Delta r \cdot \Delta r] = 0$

$(c\Delta \tau)^2$ for time-like (+)
 for light-like (0)
 $-(\Delta r_o)^2$ for space-like (-)

4-Velocity
 $U = \gamma(c, u) = dR/dt$
 $U \cdot U = c^2$

4-Velocity_(rest-frame)
 $U_o = (c, 0)$
 $U_o \cdot U_o = c^2$

4-Velocity_(photonic)
 $U_c = \gamma_c(c, cm)$
 $U_c \cdot U_c = c^2$

$U \cdot U = \gamma(c, u) \cdot \gamma(c, u) = \gamma^2(c^2 - u \cdot u) = (c^2)$
 $\gamma = 1/\sqrt{1-(u/c)^2} = 1/\sqrt{1-(\beta)^2}$

Massive particles move temporally into future at the speed-of-light (c) in their own rest-frame.

Massless particles (photonic) move nullly into the future at the speed-of-light (c), and have no rest-frame.

SR 4-Tensor
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SR Invariant Intervals

Minkowski Diagram: Lorentz Transform

A Tensor Study of Physical 4-Vectors

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SR: Lorentz Transform

$$\partial_\nu [R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu$$

$$\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$$

$$\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$$

$\text{Det}[\Lambda^\mu_\nu] = \pm 1$ $\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$

SR: Minkowski Metric

$$\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$$

$$\text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -I_{(3)}] = \text{Diag}[1, -\delta^{jk}]$$

{in Cartesian form} "Particle Physics" Convention

$$\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu{}^\nu = \delta_\mu{}^\nu$$

$\text{Tr}[\eta^{\mu\nu}] = 4$

Since the SpaceTime magnitude of **U** is a constant (*c*), changes in the components of **U** are like rotating the 4-Vector without changing its length. It keeps the same magnitude.

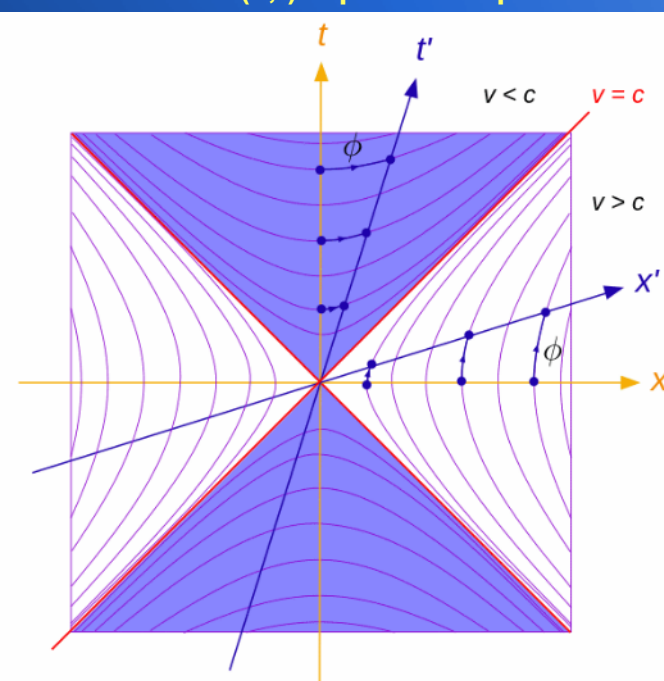
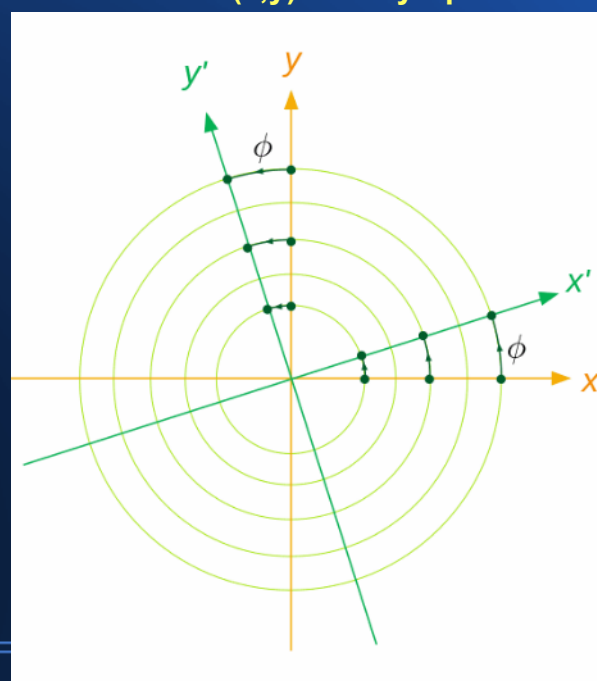
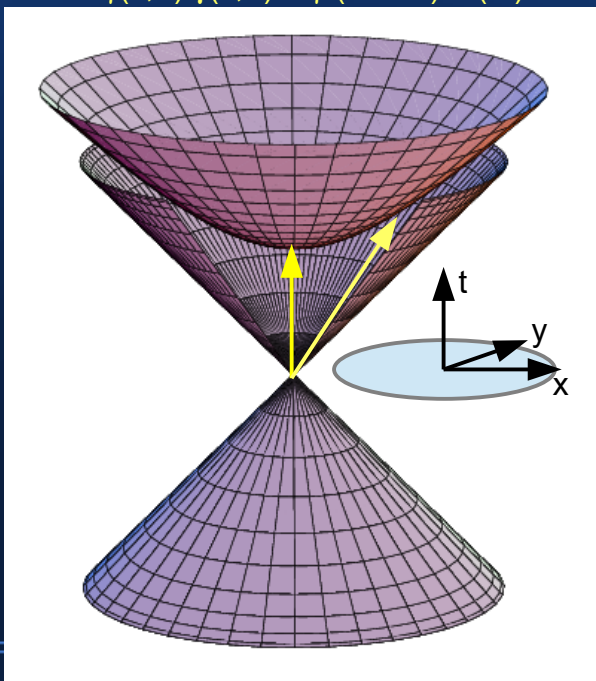
Rotations, purely spatial changes, {eg. along x,y} result in circular displacements. Boosts, or temporal-spatial changes, {eg. along x,t} result in hyperbolic displacements.

The interval between the origin and a given topograph-line is a Lorentz Invariant Constant.

$$\mathbf{U} \cdot \mathbf{U} = \gamma(c, \mathbf{u}) \cdot \gamma(c, \mathbf{u}) = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = (c^2)$$

Rotation (x,y): Purely Spatial

Boost (x,t): Spatial-Temporal



The Minkowski Diagram provides a great visual representation of SpaceTime

SR Invariant Intervals

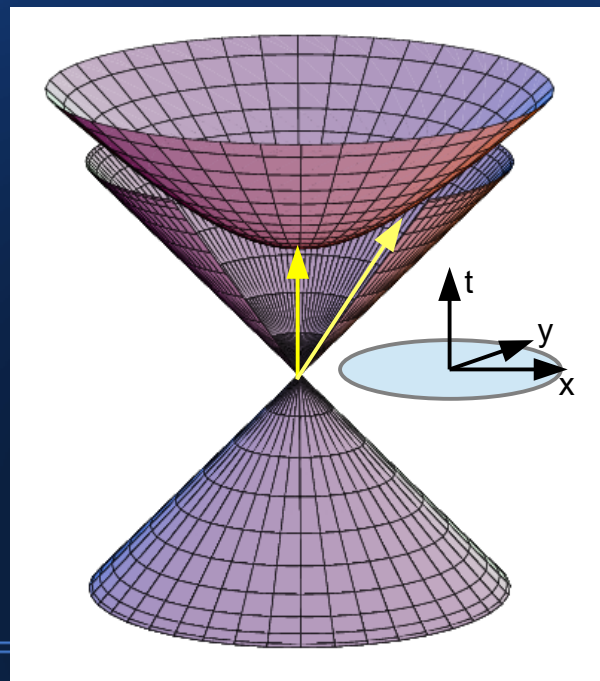
Minkowski Diagram

A Tensor Study of Physical 4-Vectors

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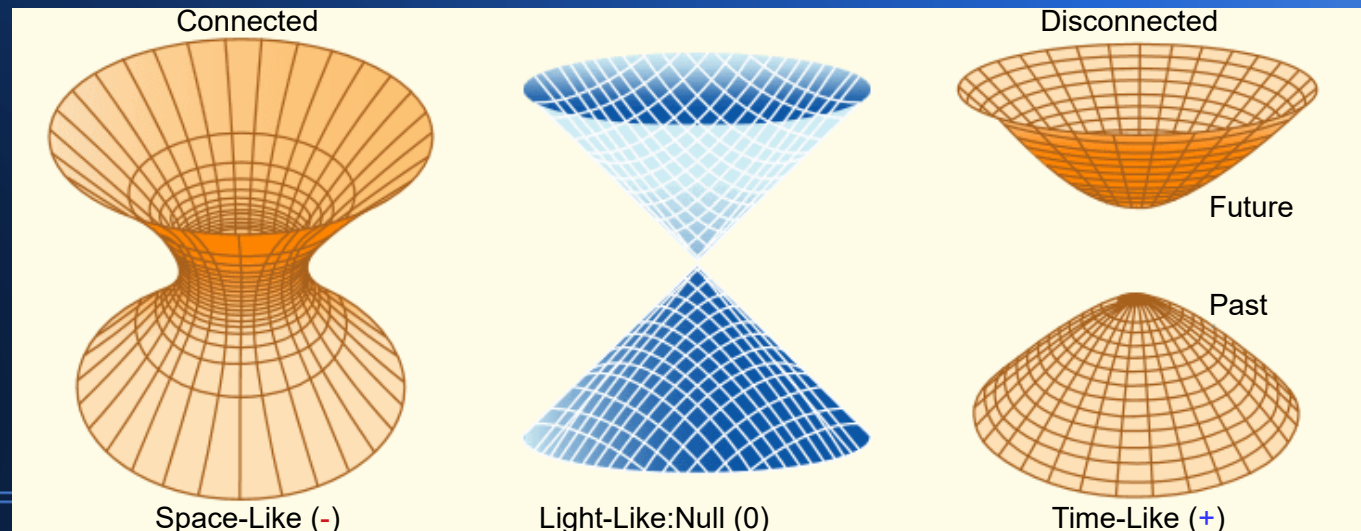
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 $\partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow$
 $\text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -I_{(3)}] = \text{Diag}[1, -\delta^{jk}]$
 {in Cartesian form} "Particle Physics" Convention
 $\{\eta_{\mu\mu}\} = 1/\{\eta^{\mu\mu}\} : \eta_\mu{}^\nu = \delta_\mu{}^\nu$ **Tr $[\eta^{\mu\nu}] = 4$**



$$\Delta R \cdot \Delta R = [(c\Delta t)^2 - \Delta r \cdot \Delta r] = \begin{matrix} (c\Delta t)^2 & \text{Time-like:Temporal} \\ (0) & \text{Light-like:Null:Photonic} \\ -(\Delta r_o)^2 & \text{Space-like:Spatial} \end{matrix}$$

(+) {causal = temporally-ordered}
 (0) {causal, maximum signal speed ($|\Delta r/\Delta t|=c$)}
 (-) {non-causal, spatially-extended}



The Minkowski Diagram provides a great visual representation of SpaceTime

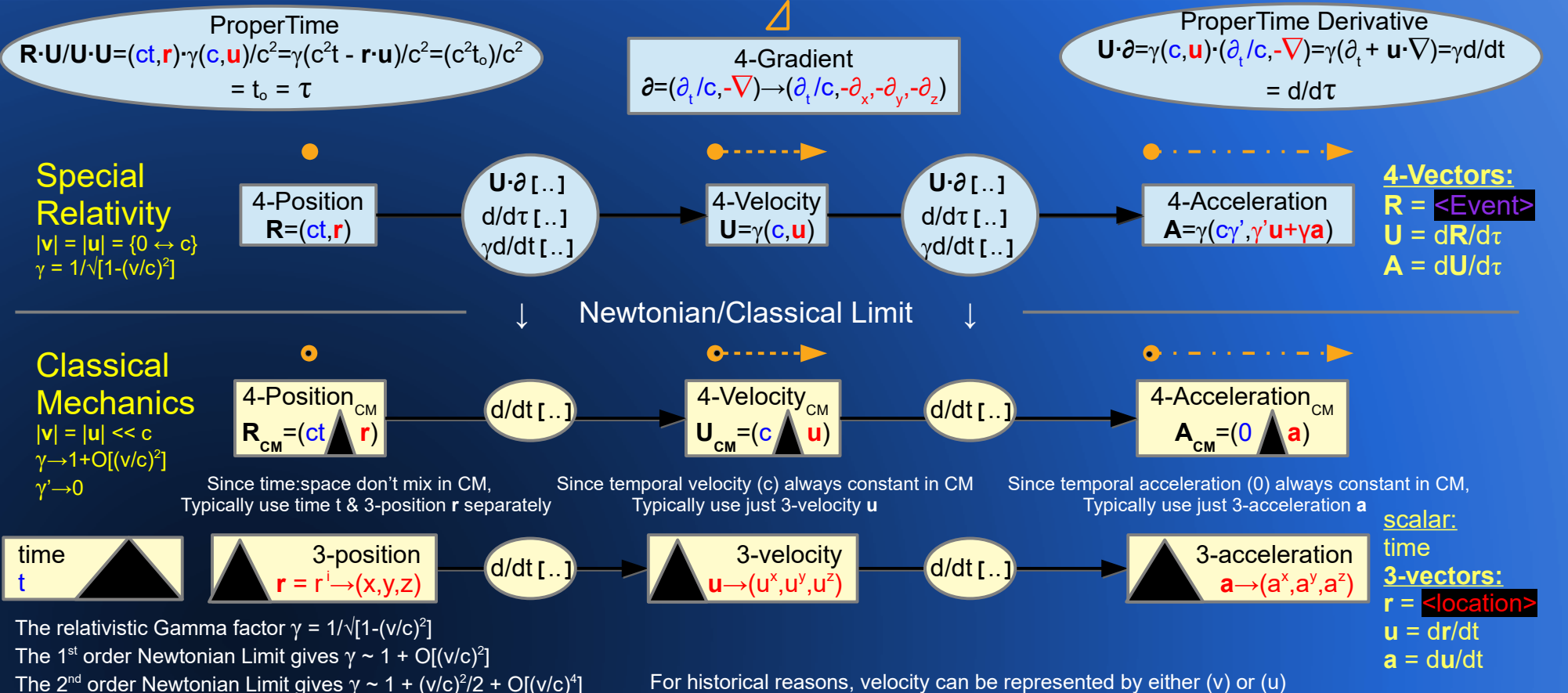
SRQM: Some Basic 4-Vectors

4-Position, 4-Velocity, 4-Acceleration

SpaceTime Kinematics

A Tensor Study of Physical 4-Vectors

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SRQM: Some Basic 4-Vectors

4-Position, 4-Velocity, 4-Acceleration, 4-Momentum, 4-Force

SpaceTime Dynamics

A Tensor Study of Physical 4-Vectors

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ProperTime

$$\mathbf{R} \cdot \mathbf{U} / \mathbf{U} \cdot \mathbf{U} = (\mathbf{ct}, \mathbf{r}) \cdot \gamma(\mathbf{c}, \mathbf{u}) / c^2 = \gamma(c^2 t - \mathbf{r} \cdot \mathbf{u}) / c^2 = (c^2 t_0) / c^2 = t_0 = \tau$$



4-Gradient

$$\partial = (\partial_t / c, -\nabla) \rightarrow (\partial_t / c, -\partial_x, -\partial_y, -\partial_z)$$

ProperTime Derivative

$$\mathbf{U} \cdot \partial = \gamma(\mathbf{c}, \mathbf{u}) \cdot (\partial_t / c, -\nabla) = \gamma(\partial_t + \mathbf{u} \cdot \nabla) = \gamma d/d\tau = d/d\tau$$

Special Relativity

$$|\mathbf{v}| = |\mathbf{u}| = \{0 \leftrightarrow c\}$$

$$\gamma = 1/\sqrt{1-(v/c)^2}$$

4-Position
 $\mathbf{R} = (\mathbf{ct}, \mathbf{r})$

$\mathbf{U} \cdot \partial [\dots]$
 $d/d\tau [\dots]$
 $\gamma d/dt [\dots]$

4-Velocity
 $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$

$\mathbf{U} \cdot \partial [\dots]$
 $d/d\tau [\dots]$
 $\gamma d/dt [\dots]$

4-Acceleration
 $\mathbf{A} = \gamma(\mathbf{c}\gamma', \gamma'\mathbf{u} + \gamma\mathbf{a})$

4-Vectors:
 $\mathbf{R} = \langle \text{Event} \rangle$
 $\mathbf{U} = d\mathbf{R}/d\tau$
 $\mathbf{A} = d\mathbf{U}/d\tau$

$E_0/c^2 = m_0$

4-Momentum
 $\mathbf{P} = (\mathbf{E}/c, \mathbf{p}) = (m\mathbf{c}, \mathbf{p})$

$\mathbf{U} \cdot \partial [\dots]$
 $d/d\tau [\dots]$
 $\gamma d/dt [\dots]$

4-Force
 $\mathbf{F} = \gamma(\dot{\mathbf{E}}/c, \mathbf{f})$

$\mathbf{P} = m_0 \mathbf{U}$
 $\mathbf{F} = d\mathbf{P}/d\tau$

This group of 4-Vectors are the main ones that are connected by the ProperTime Derivative.

$$\mathbf{U} \cdot \partial = d/d\tau = \gamma d/dt = \gamma(c\partial_t + \mathbf{u} \cdot \nabla) = \gamma(\partial_t + \mathbf{u} \cdot \nabla)$$

The classical part of it, the convective derivative, $(\partial_t + \mathbf{u} \cdot \nabla)$, is known by many different names:

The convective derivative is a derivative taken with respect to a moving coordinate system. It is also called the advective derivative, derivative following the motion, hydrodynamic derivative, Lagrangian derivative, material derivative, particle derivative, substantial derivative, substantive derivative, Stokes derivative, or total derivative

SR 4-Tensor

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(0,2)-Tensor $T_{\mu\nu}$

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(0,0)-Tensor S
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SRQM: Some Basic 4-Vectors

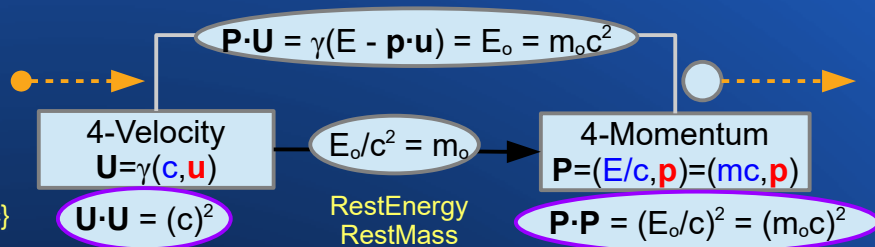
4-Velocity, 4-Momentum, $E=mc^2$

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

Special Relativity

$|v| = |u| = \{0 \leftrightarrow c\}$



$$\mathbf{U} = \gamma(c, \mathbf{u})$$

$$\mathbf{P} = (E/c, \mathbf{p}) = m_0 \mathbf{U} = \gamma m_0 (c, \mathbf{u}) = m(c, \mathbf{u})$$

Temporal part: $E = \gamma m_0 c^2 = mc^2$
{energy}

$$E = m_0 c^2 + (\gamma - 1) m_0 c^2$$

$$E = E_0 + (\gamma - 1) E_0$$

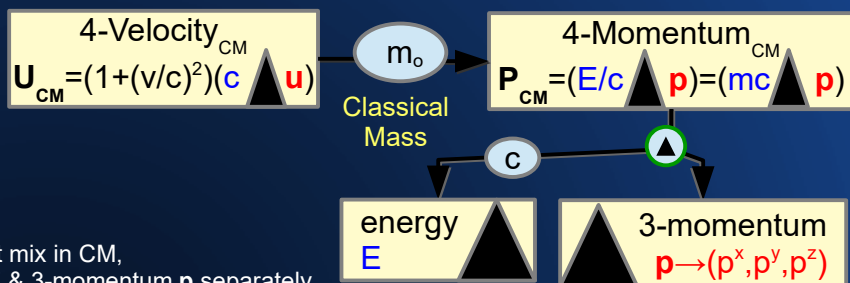
(rest) + (kinetic)

Spatial part: {momentum} $\mathbf{p} = \gamma m_0 \mathbf{u} = m \mathbf{u}$

↓ Newtonian/Classical Limit ↓

Classical Mechanics

$|v| = |u| \ll c$



$$\mathbf{u} \rightarrow (u_x, u_y, u_z)$$

$$\mathbf{P} = (E/c, \mathbf{p}) \sim (1 + (v/c)^2/2) m_0 (c, \mathbf{u})$$

Temporal part: $E \sim (1 + (v/c)^2/2) m_0 c^2 = m_0 c^2 + m_0 v^2/2$
{energy} $E_0 + |\mathbf{p}|^2/2m_0$
(rest) + (kinetic)

Spatial part: {momentum} $\mathbf{p} \sim (1) m_0 \mathbf{u} = m_0 \mathbf{u} \rightarrow m \mathbf{u}$

Since time:space don't mix in CM,
Typically use energy E & 3-momentum \mathbf{p} separately

The relativistic Gamma factor $\gamma = 1/\sqrt{1-(v/c)^2}$
The 1st order Newtonian Limit gives $\gamma \sim 1 + O[(v/c)^2]$
The 2nd order Newtonian Limit gives $\gamma \sim 1 + (v/c)^2/2 + O[(v/c)^4]$

For historical reasons, velocity can be represented by either (v) or (u)

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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Classical (scalar)
Galilean Invariant

3-vector
Not Lorentz Invariant

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

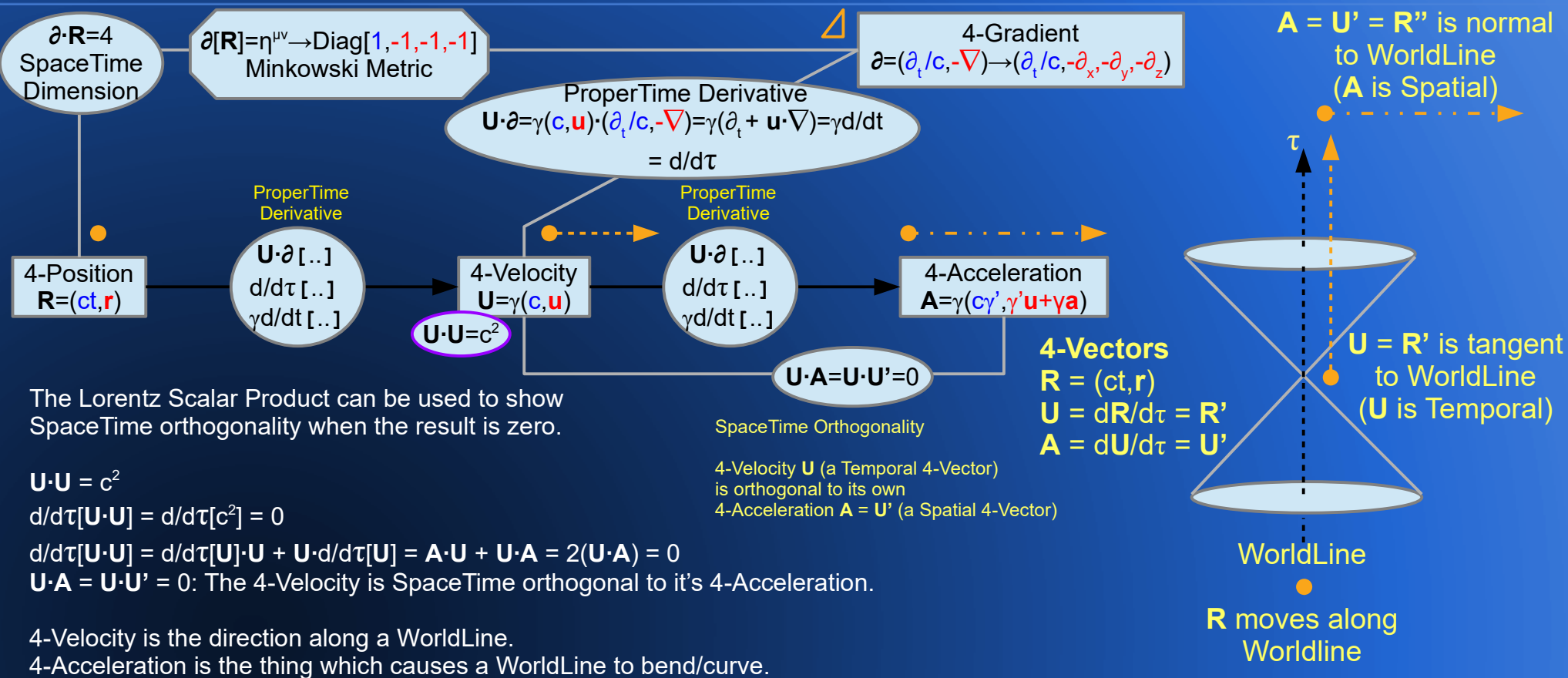
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SRQM: Some Basic 4-Vectors

4-Velocity, 4-Acceleration, SpaceTime Orthogonality

A Tensor Study of Physical 4-Vectors

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SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or T_μ^ν (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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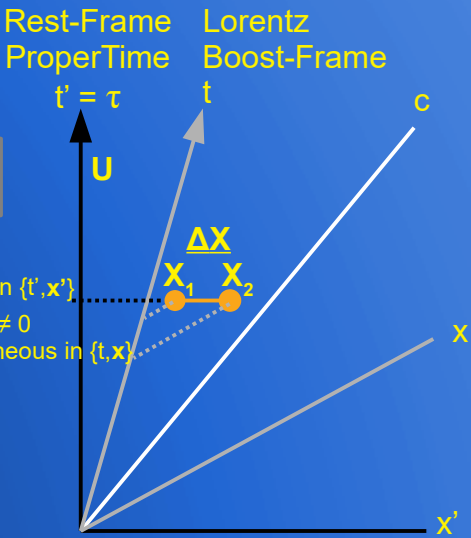
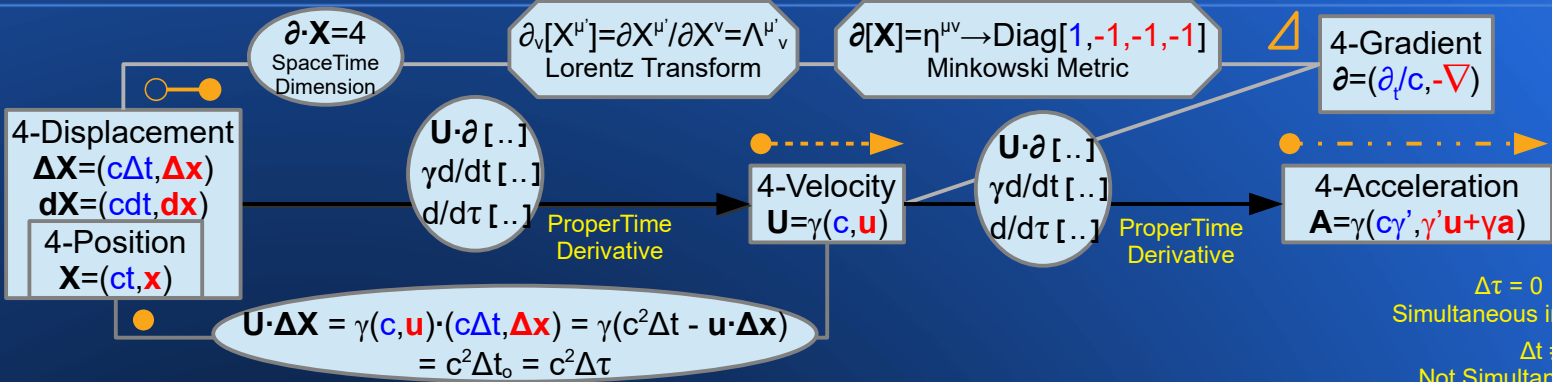
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SRQM: Some Basic 4-Vectors

4-Displacement, 4-Velocity, Relativity of Simultaneity

A Tensor Study of Physical 4-Vectors

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If Lorentz Scalar ($\mathbf{U} \cdot \Delta \mathbf{X} = 0 = c^2\Delta \tau$), then the ProperTime displacement ($\Delta \tau$) is zero, and the event separation ($\Delta \mathbf{X} = \mathbf{X}_2 - \mathbf{X}_1$) is orthogonal to the worldline \mathbf{U} . \mathbf{X}_1 and \mathbf{X}_2 are therefore simultaneous for the observer on this worldline \mathbf{U} .

Examining the equation we get $\gamma(c^2\Delta t - \mathbf{u} \cdot \Delta \mathbf{x}) = 0$. The coordinate time difference between the events is ($\Delta t = \mathbf{u} \cdot \Delta \mathbf{x} / c^2$). The condition for simultaneity in an alternate frame (moving at 3-velocity \mathbf{u} wrt. the worldline \mathbf{U}) is $\Delta t = 0$, which implies $(\mathbf{u} \cdot \Delta \mathbf{x}) = 0$.

- This can be met by:
- $(|\mathbf{u}| = 0)$, the alternate observer is not moving wrt. the events, i.e. is on worldline \mathbf{U} or on a worldline parallel to \mathbf{U} .
 - $(|\Delta \mathbf{x}| = 0)$, the events are at the same spatial location (co-local).
 - $(\mathbf{u} \cdot \Delta \mathbf{x} = 0)$, the alternate observer's motion is perpendicular (orthogonal) to the spatial separation $\Delta \mathbf{x}$ of the events in that frame.

If none of these conditions is met, then the events will not be simultaneous in the alternate reference frame. This is the mathematics behind the concept of Relativity of Simultaneity.

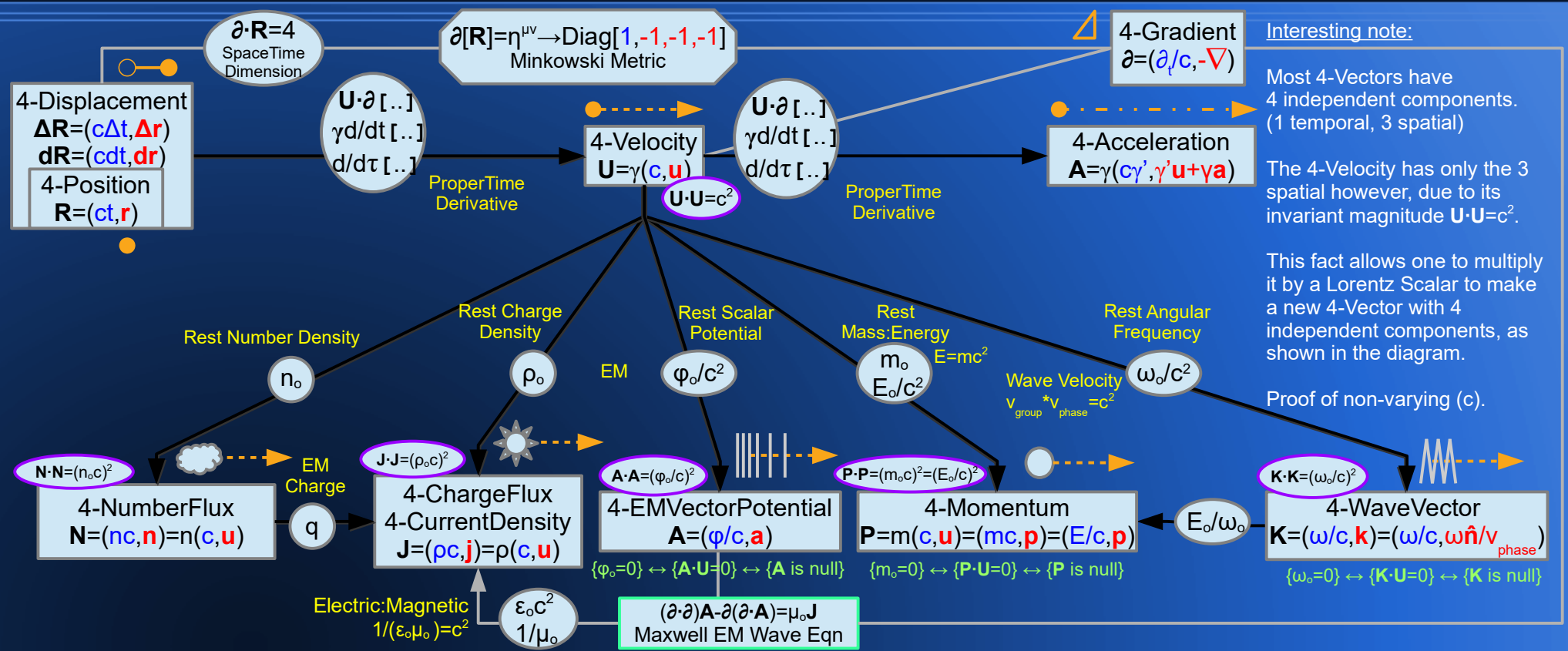
SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or T_μ^ν (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
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SR Diagram: SR Motion * Lorentz Scalar = Interesting Physical 4-Vector

A Tensor Study of Physical 4-Vectors

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SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^{\mu} = V = (v^0, \mathbf{v})$
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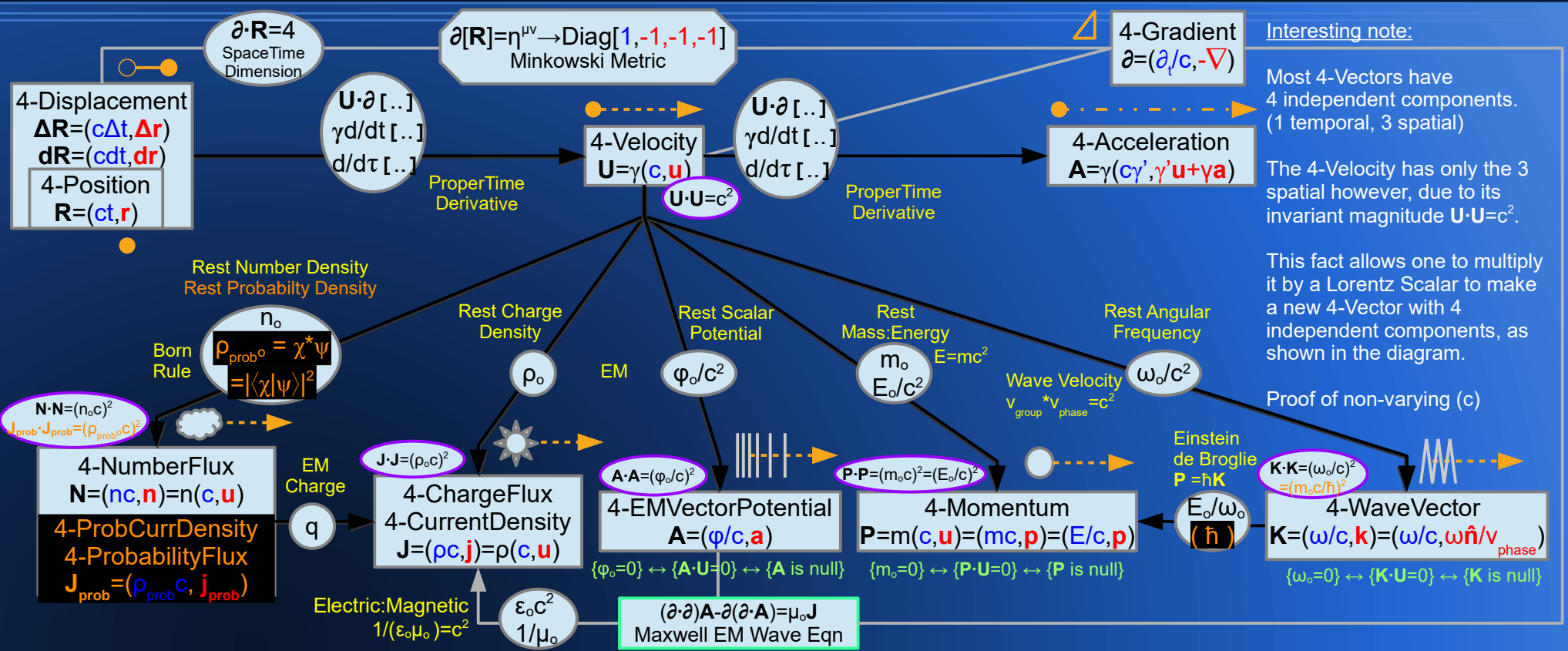
SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
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SRQM Diagram: SRQM Motion * Lorentz Scalar = Interesting Physical 4-Vector

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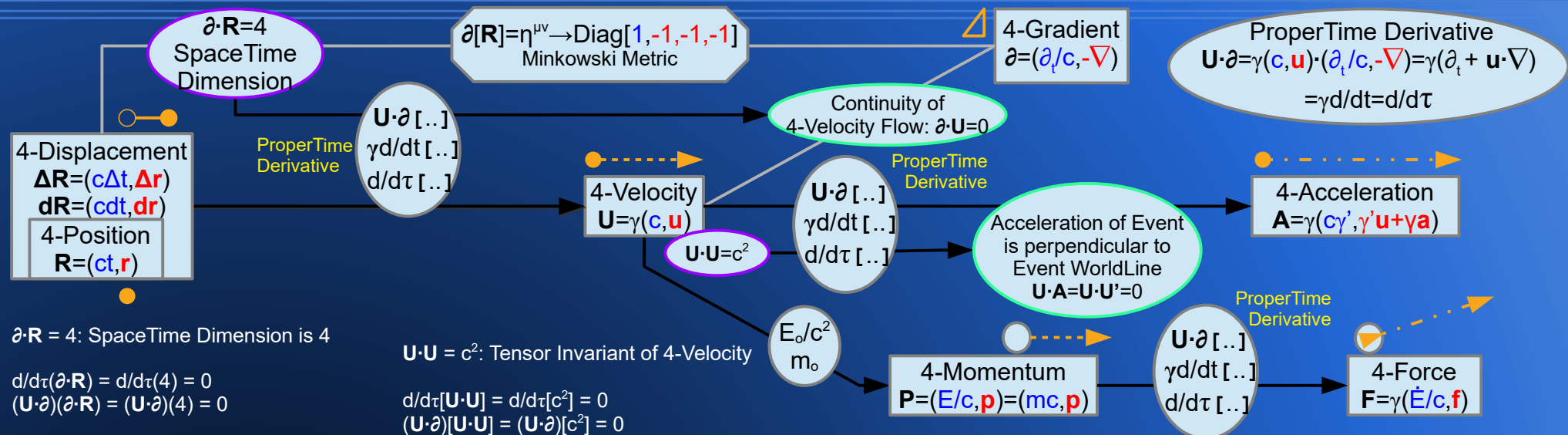
Existing SR Rules
Quantum Principles

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SRQM Diagram: ProperTime Derivative Very Fundamental Results

A Tensor Study of Physical 4-Vectors

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ProperTime Derivative
 $U \cdot \partial = \gamma(c, u) \cdot (\partial_t/c, -\nabla) = \gamma(\partial_t + u \cdot \nabla)$
 $= \gamma d/dt = d/d\tau$

$\partial \cdot R = 4$: SpaceTime Dimension is 4

$d/d\tau(\partial \cdot R) = d/d\tau(4) = 0$
 $(U \cdot \partial)(\partial \cdot R) = (U \cdot \partial)(4) = 0$

$d/d\tau(\partial \cdot R) = d/d\tau(\partial) \cdot R + \partial \cdot d/d\tau(R) = 0$
 $d/d\tau(\partial \cdot R) = d/d\tau[\partial] \cdot R + \partial \cdot U = 0$

$\partial \cdot U = -d/d\tau[\partial] \cdot R$
 $\partial \cdot U = -(U \cdot \partial)[\partial] \cdot R$
 $\partial \cdot U = -(U_\nu \partial^\nu)[\partial_\mu] R^\mu$
 $\partial \cdot U = -U_\nu \partial^\nu \partial_\mu R^\mu$
 $\partial \cdot U = -U_\nu \partial_\mu \partial^\nu R^\mu$
 $\partial \cdot U = -U_\nu \partial_\mu \eta^{\nu\mu}$
 $\partial \cdot U = -U_\nu (0^\nu)$

$\partial \cdot U = 0$: Conservation of the 4-Velocity Flow (4-Velocity Flow-Field)

$U \cdot U = c^2$: Tensor Invariant of 4-Velocity

$d/d\tau[U \cdot U] = d/d\tau[c^2] = 0$
 $(U \cdot \partial)[U \cdot U] = (U \cdot \partial)[c^2] = 0$

$d/d\tau[U \cdot U] = d/d\tau[U] \cdot U + U \cdot d/d\tau[U] = A \cdot U + U \cdot A = 2(U \cdot A) = 0$
 $U \cdot A = U \cdot U' = 0$: The 4-Velocity is SpaceTime orthogonal to its 4-Acceleration.

4-Velocity is the direction of an Event along a WorldLine.
 4-Acceleration of an Event is the thing which causes a WorldLine to bend.

- 4-Vectors:**
 $R = \langle \text{Event} \rangle$
 $U = dR/dt$
 $A = dU/dt$
 $P = m_0 U$
 $F = dP/dt$

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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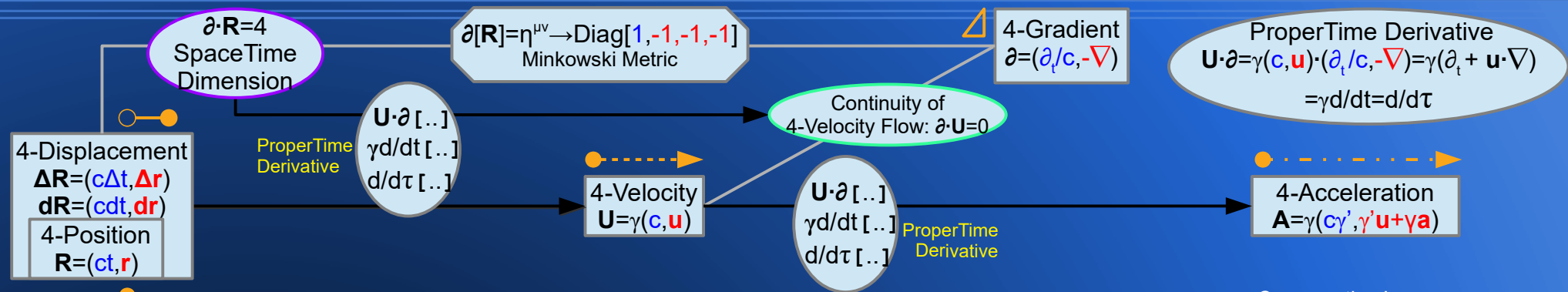
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SRQM Diagram:

Local Continuity of 4-Velocity leads to all the Conservation Laws

A Tensor Study of Physical 4-Vectors

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John B. Wilson



$\partial \cdot \mathbf{R} = 4$
 $d/d\tau(\partial \cdot \mathbf{R}) = d/d\tau(4) = 0$

$d/d\tau(\partial \cdot \mathbf{R}) = d/d\tau(\partial) \cdot \mathbf{R} + \partial \cdot d/d\tau(\mathbf{R}) = 0$
 $d/d\tau(\partial \cdot \mathbf{R}) = d/d\tau[\partial] \cdot \mathbf{R} + \partial \cdot \mathbf{U} = 0$

$\partial \cdot \mathbf{U} = -d/d\tau[\partial] \cdot \mathbf{R}$
 $\partial \cdot \mathbf{U} = -(\mathbf{U} \cdot \partial)[\partial] \cdot \mathbf{R}$
 $\partial \cdot \mathbf{U} = -(\mathbf{U}_\nu \partial^\nu)[\partial_\mu] \mathbf{R}^\mu$
 $\partial \cdot \mathbf{U} = -\mathbf{U}_\nu \partial^\nu \partial_\mu \mathbf{R}^\mu$
 $\partial \cdot \mathbf{U} = -\mathbf{U}_\nu \partial_\mu \partial^\nu \mathbf{R}^\mu$: I believe this is legit, partials commute
 $\partial \cdot \mathbf{U} = -\mathbf{U}_\nu \partial_\mu \eta^{\nu\mu}$
 $\partial \cdot \mathbf{U} = -\mathbf{U}_\nu (0^\nu)$
 $\partial \cdot \mathbf{U} = 0$

$\partial \cdot \mathbf{U} = 0$
 $\partial \cdot (\text{Lorentz Scalar}) \mathbf{U} = 0 (\text{Lorentz Scalar})$
 $\partial \cdot (\text{Lorentz Scalar}) \mathbf{U} = 0$
 $\partial \cdot (\text{Interesting 4-Vector}) = 0$

Example:
 $\partial \cdot (\rho_0) \mathbf{U} = 0$
 $\partial \cdot \mathbf{J} = 0$
 $(\partial_t/c \rho_c + \nabla \cdot \mathbf{j}) = 0$
 $(\partial_t \rho + \nabla \cdot \mathbf{j}) = 0$
 = Conservation of Charge
 = A Continuity Equation

ProperTime Derivative
 $\mathbf{U} \cdot \partial = \gamma(\mathbf{c}, \mathbf{u}) \cdot (\partial_t/c, -\nabla) = \gamma(\partial_t + \mathbf{u} \cdot \nabla)$
 $= \gamma d/dt = d/d\tau$

Conservation Laws:

All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local spatial region.

Conservation of Charge:
 $\partial \cdot \mathbf{J} = (\partial_t \rho + \nabla \cdot \mathbf{j}) = 0$

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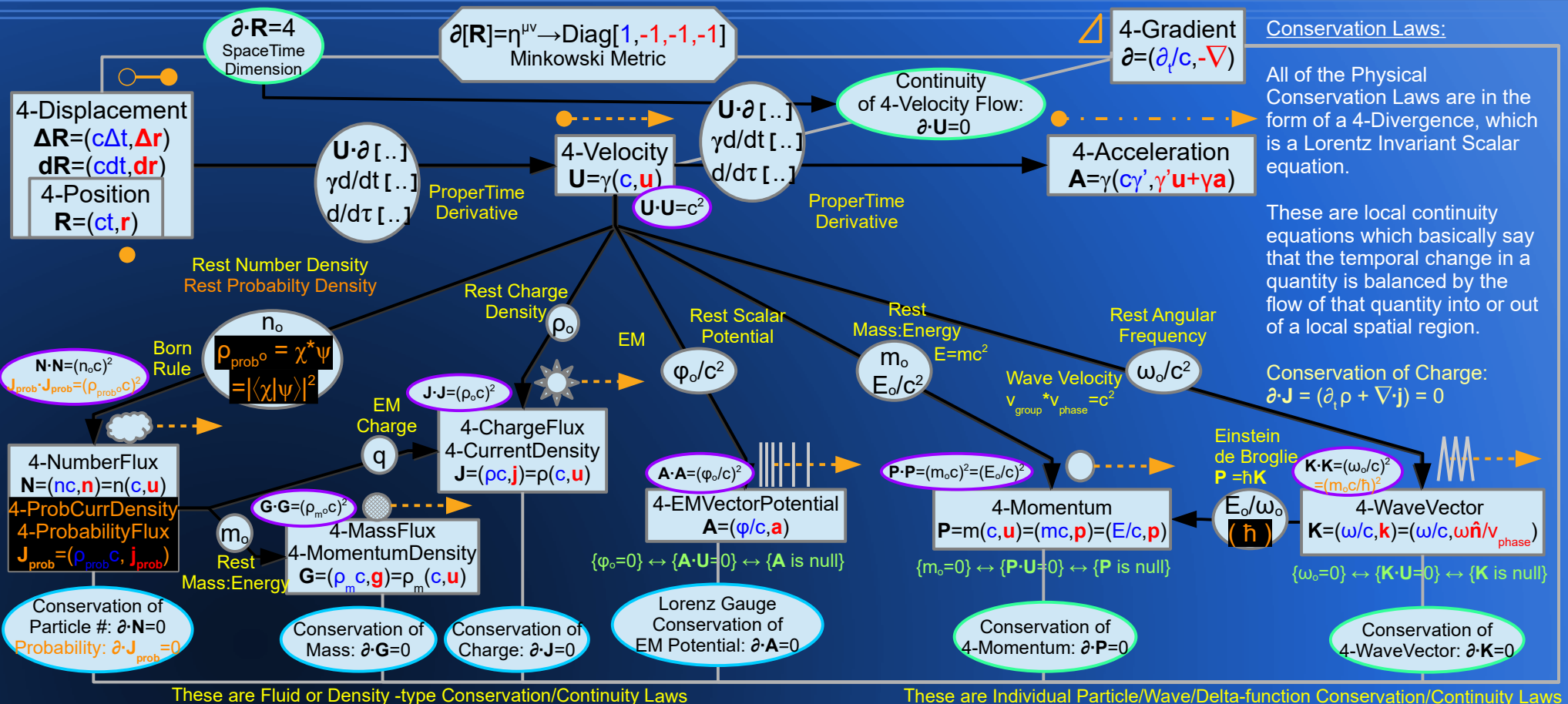
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SRQM Diagram: SRQM Motion * Lorentz Scalar

Conservation Laws, Continuity Eqns

A Tensor Study of Physical 4-Vectors

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SRQM: Some Basic 4-Vectors

4-Velocity, 4-Gradient, Time Dilation

A Tensor Study of Physical 4-Vectors

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at-rest worldline \mathbf{U}_0 ($u=0$) fully temporal

const inertial motion worldline \mathbf{U} ($0 < u < c$) trades some time for space

4-Gradient $\partial = (\partial/\partial c, -\nabla)$

ProperTime Derivative $\mathbf{U} \cdot \partial = d/d\tau = \gamma d/dt$

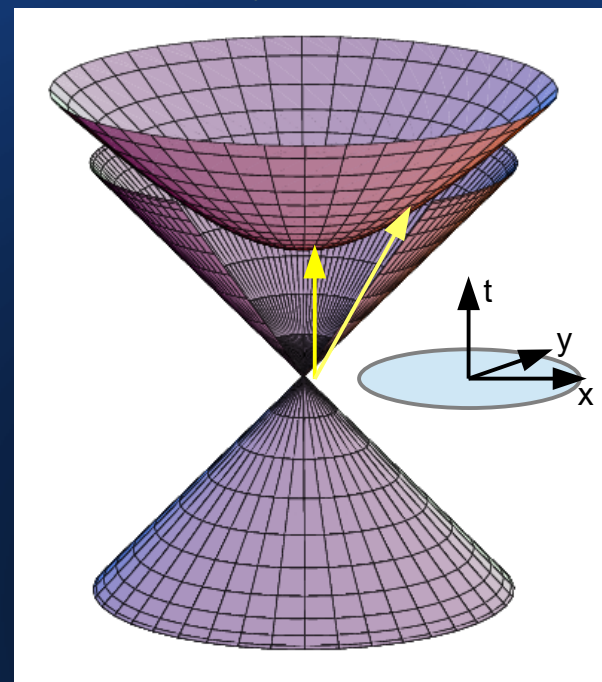
4-Velocity $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$

$\mathbf{U} \cdot \mathbf{U} = \gamma(\mathbf{c}, \mathbf{u}) \cdot \gamma(\mathbf{c}, \mathbf{u}) = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = (c^2)$
 $\gamma = 1/\sqrt{1-(u/c)^2} = 1/\sqrt{1-\beta^2}$

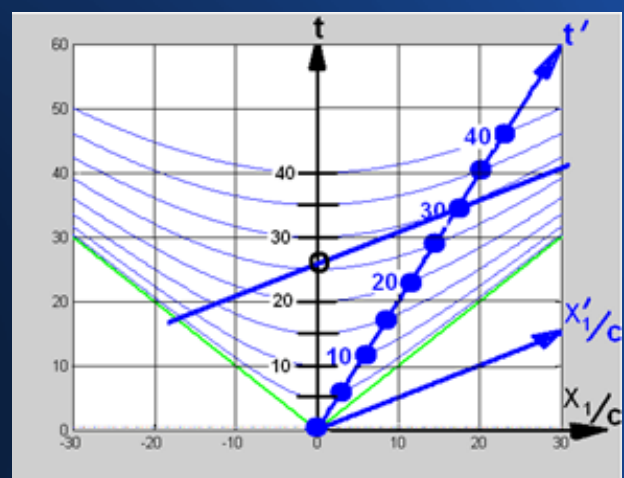
ProperTime Differential $d\tau = (1/\gamma)dt$

4-Velocity (at-rest) $\mathbf{U}_0 = (\mathbf{c}, \mathbf{0})$

Everything moves into future (+t) at the speed-of-light (c) in its own spatial rest-frame



The Minkowski Diagram provides a great visual representation of SpaceTime



Since the SpaceTime magnitude of \mathbf{U} is a constant, changes in the components of \mathbf{U} are like "rotating" the 4-Vector without changing its length. However, as \mathbf{U} gains some spatial velocity, it loses some "relative" temporal velocity. Objects that move in some reference frame "age" more slowly relative to those at rest in the same reference frame.

Time Dilation!
 $\Delta t = \gamma \Delta \tau = \gamma \Delta t_0$
 $dt = \gamma d\tau$
 $d/d\tau = \gamma d/dt$

Each observer will see the other as aging more slowly; similarly to two people moving oppositely along a train track, seeing the other as appearing smaller in the distance.

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SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

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SRQM: Some Basic 4-Vectors

SR 4-WaveVector \mathbf{K}

A Tensor Study of Physical 4-Vectors

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4-WaveVector, aka. Wave 4-Vector, solution of d'Alembertian Wave Eqn.

$$\mathbf{K} = (\omega/c, \mathbf{k}) = (\omega/c, \omega \hat{\mathbf{n}}/v_{\text{phase}}) = (\omega/c, \omega \mathbf{u}/c^2) = (\omega/c^2)(c, \mathbf{u}) = (\omega/c)(1, \boldsymbol{\beta}) = (1/c\mathcal{T}, \hat{\mathbf{n}}/\lambda) = -\partial[\Phi_{\text{phase,plane}}]$$

There are multiple ways of writing out the components of the 4-WaveVector, with each one giving an interesting take on what the 4-WaveVector means.

An SR wave Ψ is actually composed of two tensors:

- (1) 4-Vector propagation part = \mathbf{K}^α , (the engine)
- (2) Variable amplitude part = A (the load), depends on what is waving...

4-Scalar A : $\Psi = A e^{\Lambda(-i\mathbf{K}^\alpha X_\alpha)}$
ex. KG Quantum Wave

4-Vector A^μ : $\Psi^\mu = A^\mu e^{\Lambda(-i\mathbf{K}^\alpha X_\alpha)}$
ex. Maxwell Photon Wave

4-Tensor $A^{\mu\nu}$: $\Psi^{\mu\nu} = A^{\mu\nu} e^{\Lambda(-i\mathbf{K}^\alpha X_\alpha)}$
ex. Gravitational Wave Approx.

The Ψ tensor-type will match the A tensor-type, as the propagation part $e^{\Lambda(-i\mathbf{K}^\alpha X_\alpha)}$ is overall dimensionless.

One comparison I find very interesting is:

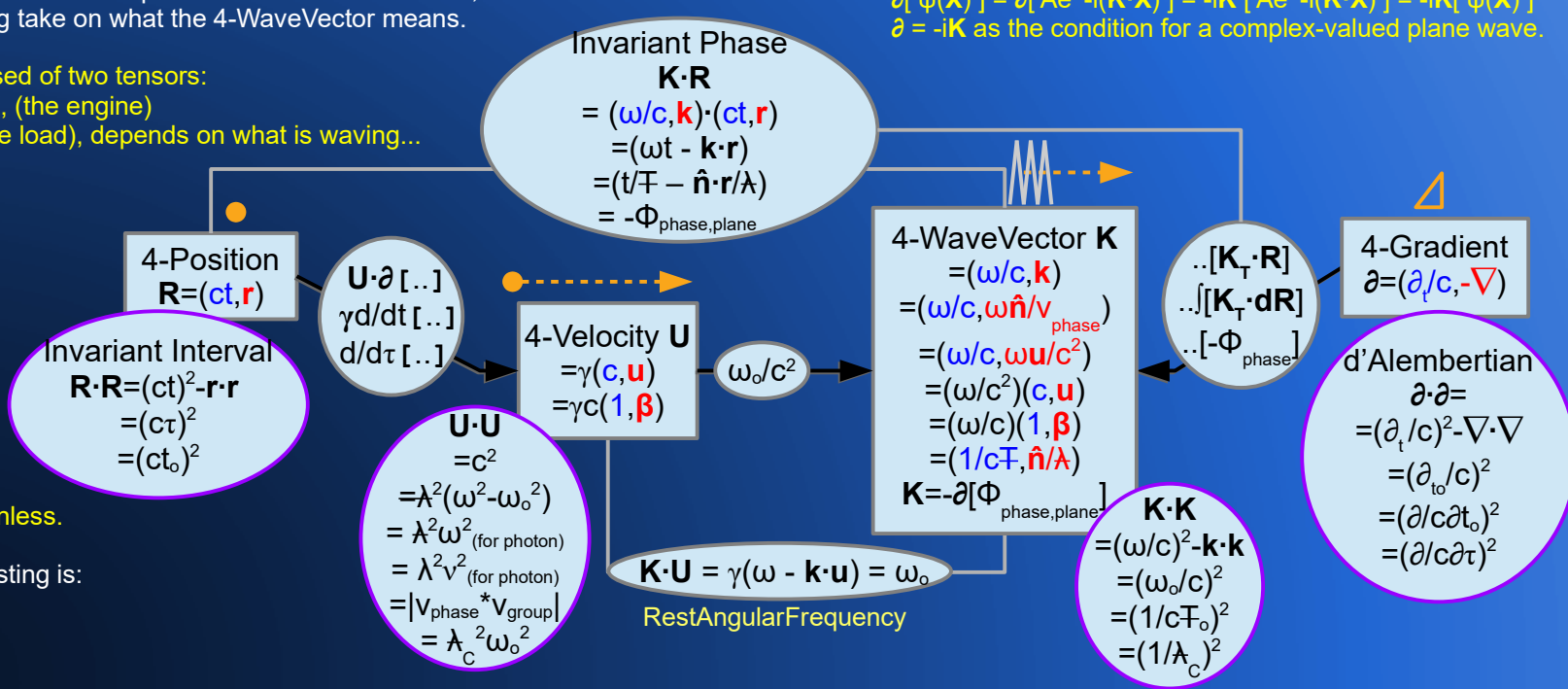
$$\mathbf{R} \cdot \mathbf{R} = (ct_0)^2 = (c\tau)^2$$

$$\mathbf{K} \cdot \mathbf{K} = (1/c\mathcal{T}_0)^2$$

$$\partial \cdot \partial = (\partial/c\partial t_0)^2 = (\partial/c\partial \tau)^2$$

I believe the last one is correct: $(\partial \cdot \partial)[\mathbf{R}] = \mathbf{0} = (\partial/c\partial \tau)^2[\mathbf{R}] = \mathbf{A}_0/c^2 = \mathbf{0}$: The 4-Acceleration seen in the ProperTime Frame = RestFrame = $\mathbf{0}$
Normally $(d/d\tau)^2[\mathbf{R}] = \mathbf{A}$, which could be non-zero. But that is for the total derivative, not the partial derivative.

$\psi_n(\mathbf{X}) = A_n e^{\Lambda(-i(\mathbf{K}_n \cdot \mathbf{X}))}$: Explicit form of an SR plane wave
 $\psi(\mathbf{X}) = \sum_n [\psi_n(\mathbf{X})]$: Complete wave is a superposition of multiple plane waves.
 $\partial[\psi(\mathbf{X})] = \partial[A e^{\Lambda(-i(\mathbf{K} \cdot \mathbf{X}))}] = -i\mathbf{K} [A e^{\Lambda(-i(\mathbf{K} \cdot \mathbf{X}))}] = -i\mathbf{K}[\psi(\mathbf{X})]$
 $\partial = -i\mathbf{K}$ as the condition for a complex-valued plane wave.



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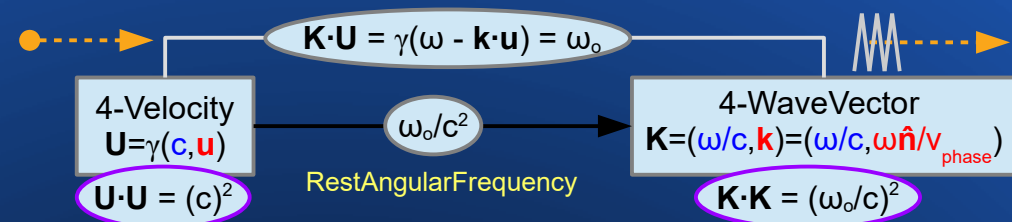
SRQM: Some Basic 4-Vectors

4-Velocity, 4-WaveVector

Wave Properties, Relativistic Doppler Effect

A Tensor Study of Physical 4-Vectors

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$$\mathbf{K} = (\omega/c, \mathbf{k}) = (\omega/c, \omega \hat{\mathbf{n}}/v_{\text{phase}}) = (\omega/c^2) \mathbf{U}$$

$$= (\omega/c^2) \gamma(\mathbf{c}, \mathbf{u}) = (\omega/c^2)(\mathbf{c}, \mathbf{u}) = (\omega/c, (\omega/c^2)\mathbf{u})$$

$$(\omega/c, \omega \hat{\mathbf{n}}/v_{\text{phase}}) = (\omega/c, (\omega/c^2)\mathbf{u})$$

Taking just the spatial components of the 4-WaveVector:

$$\omega \hat{\mathbf{n}}/v_{\text{phase}} = (\omega/c^2)\mathbf{u}$$

$$\hat{\mathbf{n}}/v_{\text{phase}} = (\mathbf{u}/c^2)$$

$$\mathbf{u} * v_{\text{phase}} = c^2$$

$$v_{\text{group}} * v_{\text{phase}} = c^2, \text{ with } \mathbf{u} = v_{\text{group}}$$

Wave Group velocity (v_{group}) is mathematically the same as Particle velocity (\mathbf{u}).

Wave Phase velocity (v_{phase}) is the speed of an individual plane-wave.

The Phase Velocity of a Photon $\{v_{\text{phase}} = c\}$ equals the Particle Velocity of a Photon $\{u = c\}$

The Phase Velocity of a Massive Particle $\{v_{\text{phase}} > c\}$ is greater than the Velocity of a Massive Particle $\{u < c\}$

Relativistic SR Doppler Effect

($\hat{\mathbf{n}}$) here is the unit-directional 3-vector of the photon

Choose an observer frame for which:

$\mathbf{K} = (\omega/c, \mathbf{k})$, with $\mathbf{k}, \hat{\mathbf{n}}$ pointing toward observer

$$\mathbf{U}_{\text{obs}} = (\mathbf{c}, 0) \quad \mathbf{K} \cdot \mathbf{U}_{\text{obs}} = (\omega/c, \mathbf{k}) \cdot (\mathbf{c}, 0) = \omega = \omega_{\text{obs}^0}$$

$$\mathbf{U}_{\text{emit}} = \gamma(\mathbf{c}, \mathbf{u}) \quad \mathbf{K} \cdot \mathbf{U}_{\text{emit}} = (\omega/c, \mathbf{k}) \cdot \gamma(\mathbf{c}, \mathbf{u}) = \gamma(\omega - \mathbf{k} \cdot \mathbf{u}) = \omega_{\text{emit}^0}$$

$$\mathbf{K} \cdot \mathbf{U}_{\text{obs}} / \mathbf{K} \cdot \mathbf{U}_{\text{emit}} = \omega_{\text{obs}^0} / \omega_{\text{emit}^0} = \omega / [\gamma(\omega - \mathbf{k} \cdot \mathbf{u})]$$

For photons, \mathbf{K} is null $\rightarrow \mathbf{K} \cdot \mathbf{K} = 0 \rightarrow \mathbf{k} = (\omega/c)\hat{\mathbf{n}}$

$$\omega_{\text{obs}^0} / \omega_{\text{emit}^0} = \omega / [\gamma(\omega - (\omega/c)\hat{\mathbf{n}} \cdot \mathbf{u})] = 1 / [\gamma(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})] = 1 / [\gamma(1 - |\boldsymbol{\beta}| \cos[\theta_{\text{obs}}])]$$

$$\omega_{\text{obs}} / \omega_{\text{emit}} = \gamma \omega_{\text{obs}^0} / (\gamma \omega_{\text{emit}^0}) = \omega_{\text{obs}^0} / \omega_{\text{emit}^0}$$

$$\omega_{\text{obs}} = \omega_{\text{emit}} / [\gamma(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})] = \omega_{\text{emit}} * \sqrt{[1+|\boldsymbol{\beta}|]} * \sqrt{[1-|\boldsymbol{\beta}|]} / (1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})$$

$$\text{with } \gamma = 1/\sqrt{[1-\beta^2]} = 1/(\sqrt{[1+|\boldsymbol{\beta}|]} * \sqrt{[1-|\boldsymbol{\beta}|]})$$

For motion of emitter $\boldsymbol{\beta}$: (in observer frame of reference)

Away from obs, ($\hat{\mathbf{n}} \cdot \boldsymbol{\beta}) = -\beta$, $\omega_{\text{obs}} = \omega_{\text{emit}} * \sqrt{[1-|\boldsymbol{\beta}|]} / \sqrt{[1+|\boldsymbol{\beta}|]} = \text{Red Shift}$

Toward obs, ($\hat{\mathbf{n}} \cdot \boldsymbol{\beta}) = +\beta$, $\omega_{\text{obs}} = \omega_{\text{emit}} * \sqrt{[1+|\boldsymbol{\beta}|]} / \sqrt{[1-|\boldsymbol{\beta}|]} = \text{Blue Shift}$

Transverse, ($\hat{\mathbf{n}} \cdot \boldsymbol{\beta}) = 0$, $\omega_{\text{obs}} = \omega_{\text{emit}} / \gamma = \text{Transverse Doppler Shift}$

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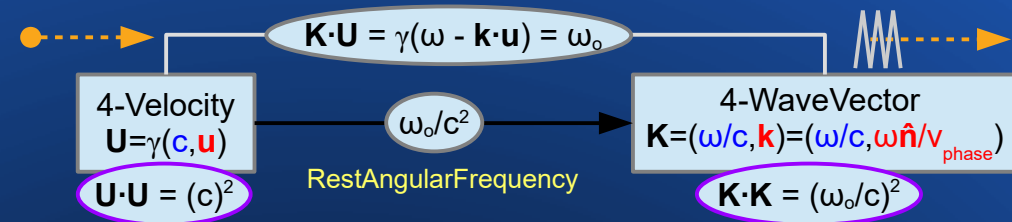
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$$\omega_{\text{obs}} = \omega_{\text{emit}} / [\gamma(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})] = \omega_{\text{emit}} / [\gamma(1 - |\boldsymbol{\beta}| \cos[\theta_{\text{obs}}])]$$

Change reference frames with $\{\text{obs} \rightarrow \text{emit}\}$ & $\{\boldsymbol{\beta} \rightarrow -\boldsymbol{\beta}\}$

$$\omega_{\text{emit}} = \omega_{\text{obs}} / [\gamma(1 + \hat{\mathbf{n}} \cdot \boldsymbol{\beta})] = \omega_{\text{obs}} / [\gamma(1 + |\boldsymbol{\beta}| \cos[\theta_{\text{emit}}])]$$

$$(\omega_{\text{obs}}) * (\omega_{\text{emit}}) = (\omega_{\text{emit}} / [\gamma(1 - |\boldsymbol{\beta}| \cos[\theta_{\text{obs}}])]) * (\omega_{\text{obs}} / [\gamma(1 + |\boldsymbol{\beta}| \cos[\theta_{\text{emit}}])])$$

$$1 = (1 / [\gamma(1 - |\boldsymbol{\beta}| \cos[\theta_{\text{obs}}])]) * (1 / [\gamma(1 + |\boldsymbol{\beta}| \cos[\theta_{\text{emit}}])])$$

$$1 = (\gamma(1 - |\boldsymbol{\beta}| \cos[\theta_{\text{obs}}])) * (\gamma(1 + |\boldsymbol{\beta}| \cos[\theta_{\text{emit}}]))$$

$$1 = \gamma^2(1 - |\boldsymbol{\beta}| \cos[\theta_{\text{obs}}]) * (1 + |\boldsymbol{\beta}| \cos[\theta_{\text{emit}}])$$

Solve for $|\boldsymbol{\beta}| \cos[\theta_{\text{obs}}]$ and use $\{(\gamma^2 - 1) = \beta^2 \gamma^2\}$

Relativistic SR Aberration Effect

$$\cos[\theta_{\text{obs}}] = (\cos[\theta_{\text{emit}}] + |\boldsymbol{\beta}|) / (1 + |\boldsymbol{\beta}| \cos[\theta_{\text{emit}}])$$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$$

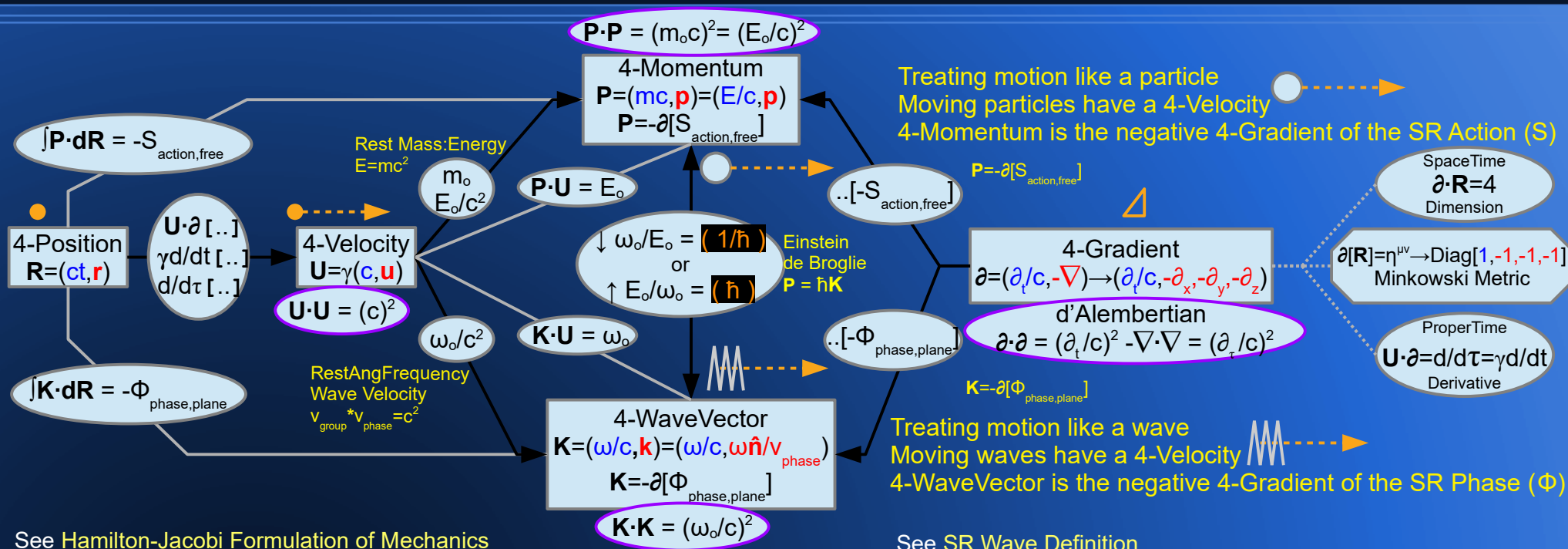
$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$$

SRQM: Some Basic 4-Vectors

4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, **Wave-Particle**

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson



See **Hamilton-Jacobi Formulation of Mechanics** for info on the Lorentz Scalar Invariant SR Action.
 $\{ \mathbf{P} = (E/c, \mathbf{p}) = -\partial[\mathbf{S}] = (-\partial/c \partial t[\mathbf{S}], \nabla[\mathbf{S}]) \}$
 {temporal component} $E = -\partial/\partial t[\mathbf{S}] = -\partial_t[\mathbf{S}]$
 {spatial component} $\mathbf{p} = \nabla[\mathbf{S}]$

****Note**** This is the Action (\mathbf{S}_{action}) for a free particle.
 Generally Action is for the 4-TotalMomentum \mathbf{P}_T of a system.

See **SR Wave Definition** for info on the Lorentz Scalar Invariant SR WavePhase.
 $\{ \mathbf{K} = (\omega/c, \mathbf{k}) = -\partial[\Phi] = (-\partial/c \partial t[\Phi], \nabla[\Phi]) \}$
 {temporal component} $\omega = -\partial/\partial t[\Phi] = -\partial_t[\Phi]$
 {spatial component} $\mathbf{k} = \nabla[\Phi]$

****Note**** This is the Phase (Φ) for a single plane-wave.
 Generally WavePhase is for the 4-TotalWaveVector \mathbf{K}_T of a system.

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 (0,2)-Tensor $T_{\mu\nu}$

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 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Existing SR Rules
Quantum Principles

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

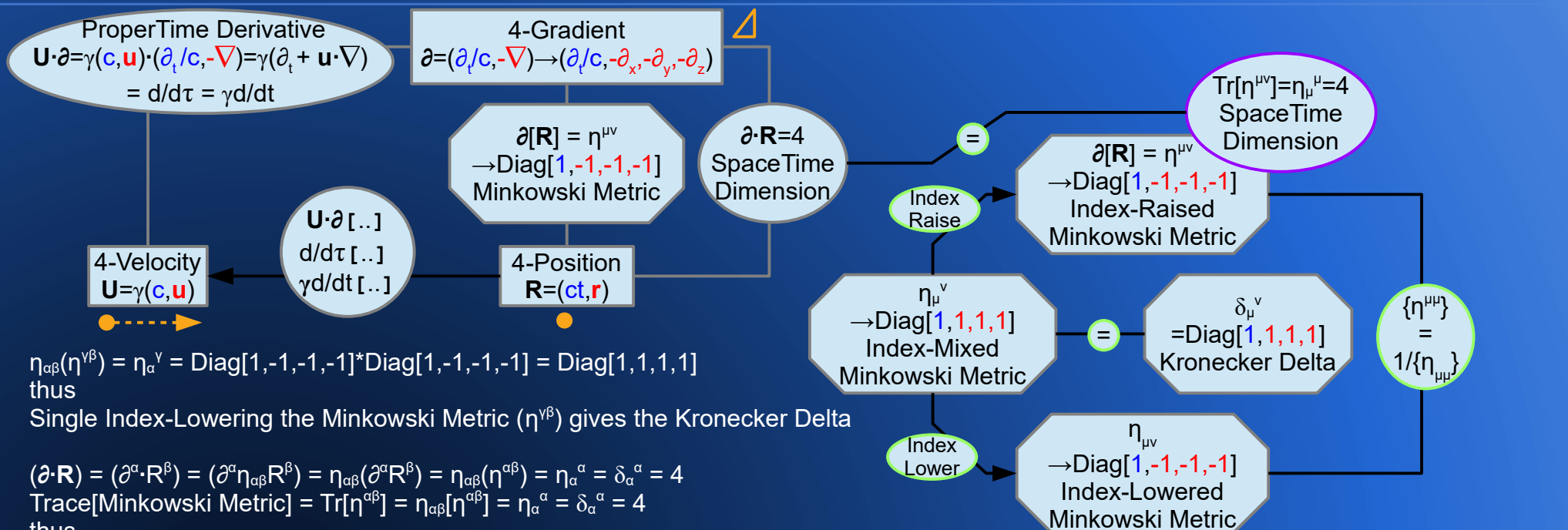
Some Cool Minkowski Metric Tensor Tricks

4-Gradient, 4-Position, 4-Velocity

SpaceTime is 4D

A Tensor Study of Physical 4-Vectors

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$\eta_{\alpha\beta}(\eta^{\gamma\beta}) = \eta_{\alpha}^{\gamma} = \text{Diag}[1, -1, -1, -1] * \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, 1, 1, 1]$
 thus

Single Index-Lowering the Minkowski Metric ($\eta^{\gamma\beta}$) gives the Kronecker Delta

$(\partial \cdot \mathbf{R}) = (\partial^{\alpha} \cdot \mathbf{R}^{\beta}) = (\partial^{\alpha} \eta_{\alpha\beta} \mathbf{R}^{\beta}) = \eta_{\alpha\beta} (\partial^{\alpha} \mathbf{R}^{\beta}) = \eta_{\alpha\beta} (\eta^{\alpha\beta}) = \eta_{\alpha}^{\alpha} = \delta_{\alpha}^{\alpha} = 4$

Trace[Minkowski Metric] = $\text{Tr}[\eta^{\alpha\beta}] = \eta_{\alpha\beta}[\eta^{\alpha\beta}] = \eta_{\alpha}^{\alpha} = \delta_{\alpha}^{\alpha} = 4$

thus

The Divergence of 4-Position ($\partial \cdot \mathbf{R}$) = "Magnitude" of the Minkowski Metric $\text{Tr}[\eta^{\alpha\beta}]$ = the Dimension of SpaceTime (4)

$(\mathbf{U} \cdot \partial)[\mathbf{R}] = (\mathbf{U}^{\alpha} \cdot \partial^{\beta})[\mathbf{R}^{\gamma}] = (\mathbf{U}^{\alpha} \eta_{\alpha\beta} \partial^{\beta})[\mathbf{R}^{\gamma}] = (\mathbf{U}_{\beta} \partial^{\beta})[\mathbf{R}^{\gamma}] = (\mathbf{U}_{\beta}) \partial^{\beta}[\mathbf{R}^{\gamma}] = (\mathbf{U}_{\beta}) \eta^{\beta\gamma} = \mathbf{U}^{\gamma} = \mathbf{U} = (d/d\tau)[\mathbf{R}]$

thus

Lorentz Scalar Product ($\mathbf{U} \cdot \partial$) = Derivative wrt. ProperTime ($d/d\tau$) = Relativistic Factor * Derivative wrt. CoordinateTime $\gamma(d/dt)$:

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 (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
 (0,2)-Tensor $T_{\mu\nu}$

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 (1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM+EM Diagram: 4-Vectors

A Tensor Study of Physical 4-Vectors

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(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
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SRQM+EM Diagram: 4-Vectors, 4-Tensors

A Tensor Study of Physical 4-Vectors

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$\partial[\mathbf{R}] = \eta^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1]$
Minkowski Metric

SR Perfect Fluid
 $T^{\mu\nu} = ((\rho_{eo} + p_o)/c^2)U^\mu U^\nu - (p_o)\eta^{\mu\nu}$
 $T^{\mu\nu} = (\rho_{eo})V^{\mu\nu} - (p_o)H^{\mu\nu}$
 StressEnergy 4-Tensor

Einstein GR
 $G^{\mu\nu} = R^{\mu\nu} - g^{\mu\nu}R/2$
 4-Tensor

4-Gradient
 $\partial = (\partial/c, -\nabla)$

4-Displacement
 $\Delta\mathbf{R} = (c\Delta t, \Delta\mathbf{r})$
 $d\mathbf{R} = (cdt, d\mathbf{r})$
 4-Position
 $\mathbf{R} = (ct, \mathbf{r})$

4-Acceleration
 $\mathbf{A} = \gamma(c\gamma', \gamma'\mathbf{u} + \gamma\mathbf{a})$

4-Polarization: Spin
 $\mathbf{E} = (\epsilon^0, \boldsymbol{\epsilon}) = (\boldsymbol{\epsilon} \cdot \boldsymbol{\beta}, \boldsymbol{\epsilon})$
 $\mathbf{S} = (\mathbf{s}^0, \mathbf{s}) = (\mathbf{s} \cdot \boldsymbol{\beta}, \mathbf{s})$
 Complex

4-WaveVector
 $\mathbf{K} = (\omega/c, \mathbf{k})$

4-TotalWaveVector
 $\mathbf{K}_T = (\omega_T/c, \mathbf{k}_T)$

4-UnitTemporal
 $\mathbf{T} = \gamma(1, \boldsymbol{\beta})$

4-Velocity
 $\mathbf{U} = \gamma(c, \mathbf{u})$

$\eta_{\mu\nu}$

EM Faraday
 $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$
 $= [\begin{matrix} 0 & -e/c \\ +e/c & -\epsilon^i_k b^k \end{matrix}]$
 4-Tensor

$\eta_{\mu\nu}$

4-UnitSpatial
 $\mathbf{S} = \gamma\beta n(\hat{\mathbf{n}} \cdot \boldsymbol{\beta}, \hat{\mathbf{n}})_\perp$

4-Momentum
 $\mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p})$

4-Force
 $\mathbf{F} = \gamma(E'/c, \mathbf{f})$

4-TotalMomentum
 $\mathbf{P}_T = (E_T/c, \mathbf{p}_T) = (H/c, \mathbf{p}_T)$

4-NumberFlux
 $\mathbf{N} = (nc, \mathbf{n}) = n(c, \mathbf{u})$
 4-ProbCurrDensity
 4-ProbabilityFlux
 $\mathbf{J}_{prob} = (p_{prob}c, \mathbf{j}_{prob})$
 $= (i\hbar/2m_0)(\psi^* \partial[\psi] - \partial[\psi^*]\psi) + (q/m_0)(\psi^* \psi)\mathbf{A}$
 Complex

4-ChargeFlux
 4-CurrentDensity
 $\mathbf{J} = (\rho c, \mathbf{j}) = \rho(c, \mathbf{u})$

4-EMVectorPotential
 $\mathbf{A} = (\phi/c, \mathbf{a})$

4-MassFlux
 4-MomentumDensity
 $\mathbf{G} = (\rho_m c, \mathbf{g}) = (\rho_e/c, \mathbf{g})$

4-ForceDensity
 $\mathbf{F}_{den} = \gamma(E'_{den}/c, \mathbf{f}_{den})$

4-MomentumField
 $\mathbf{P}_f = (E_f/c, \mathbf{p}_f)$
 $= \mathbf{P} + \mathbf{Q} = \mathbf{P} + q\mathbf{A}$

4-EMPotentialMomentum
 $\mathbf{Q} = (U/c, \mathbf{q}) = q\mathbf{A}$

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SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Existing SR Rules
Quantum Principles

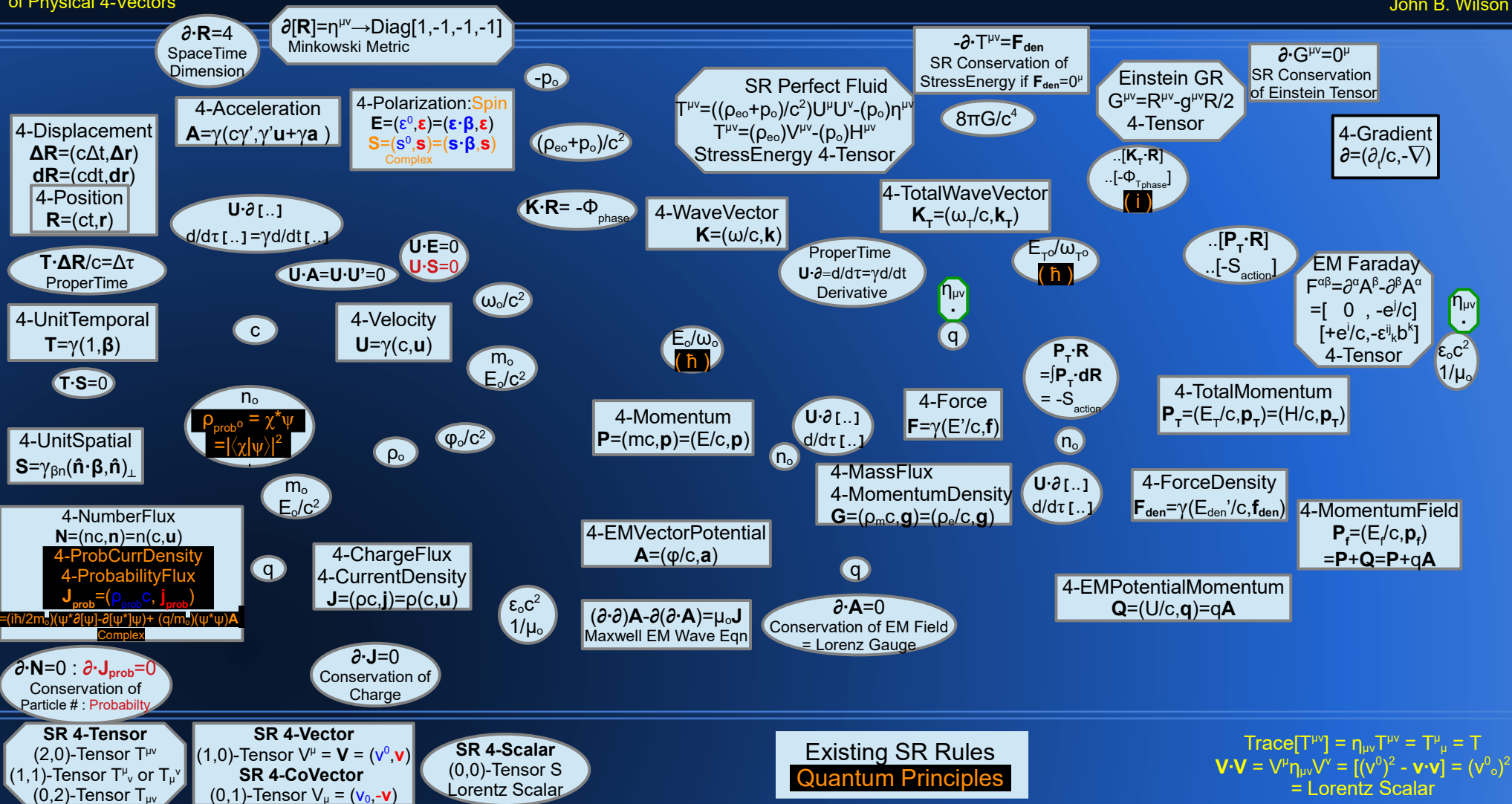
Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu} = T^\mu_\mu = T$
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SRQM+EM Diagram: 4-Vectors, 4-Tensors

Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

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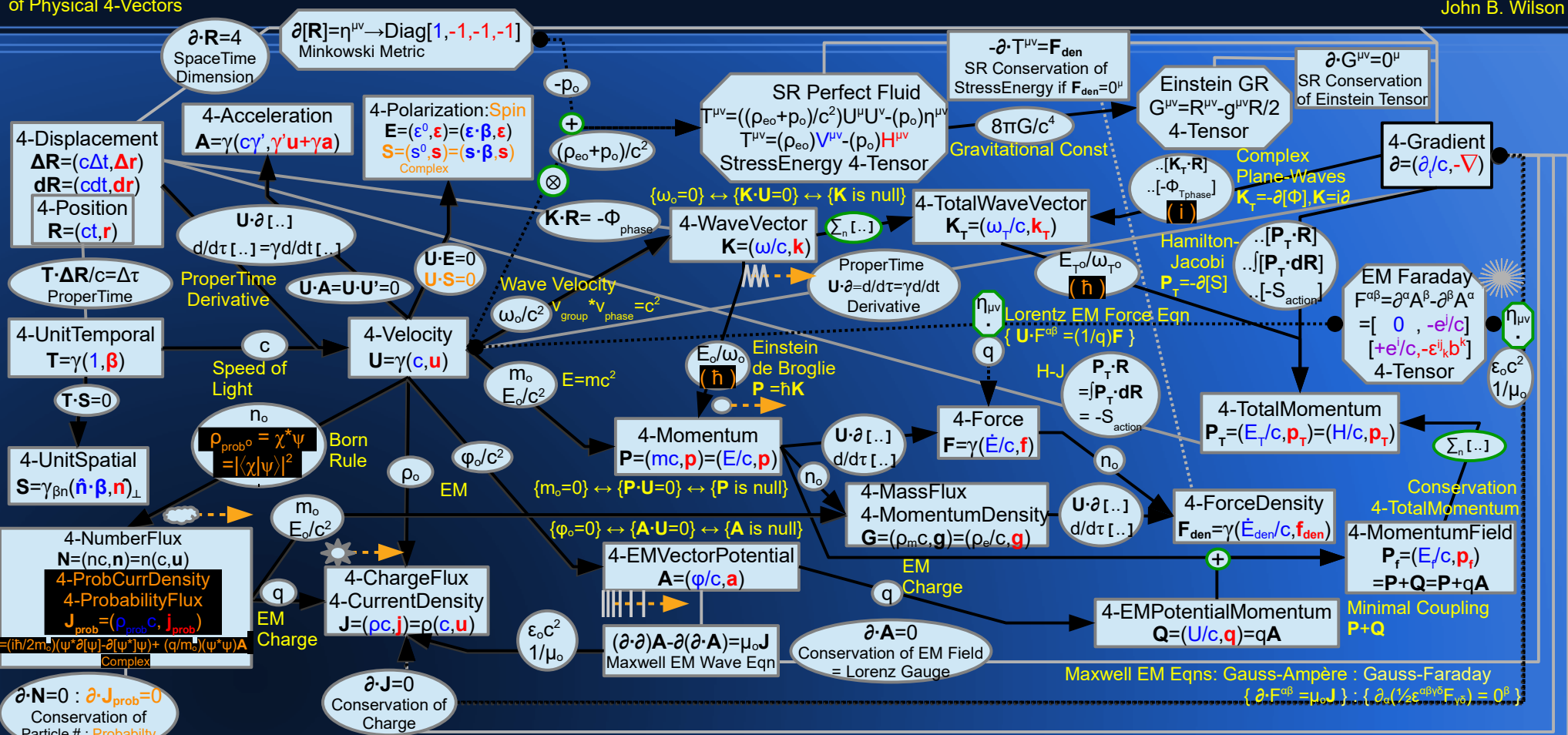


SRQM+EM Diagram: 4-Vectors, 4-Tensors

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SR 4-Tensor
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Existing SR Rules
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SRQM+EM Diagram: 4-Vectors, 4-Tensors Lorentz Scalars / Physical Constants with Tensor Invariants

A Tensor Study of Physical 4-Vectors



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or T_{ν}^μ
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 (0,1)-Tensor $V_\mu = (\mathbf{v}_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

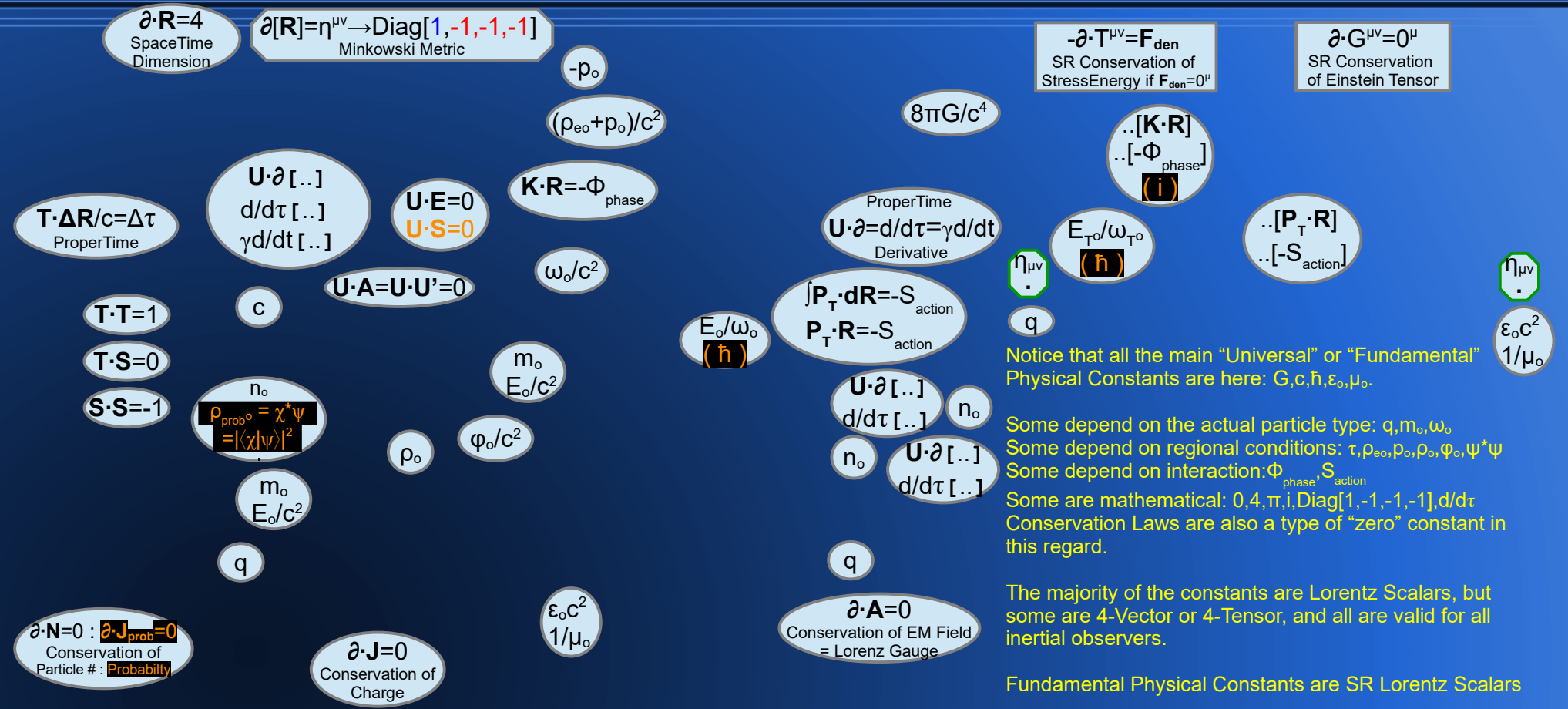
Existing SR Rules
Quantum Principles

Trace $T^{\mu\nu} = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram: Physical Constants Emphasized

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Notice that all the main "Universal" or "Fundamental" Physical Constants are here: $G, c, \hbar, \epsilon_0, \mu_0$.

Some depend on the actual particle type: q, m_0, ω_0
 Some depend on regional conditions: $\tau, \rho_{e0}, \rho_0, \Phi_0, \Psi^* \Psi$
 Some depend on interaction: $\Phi_{\text{phase}}, S_{\text{action}}$
 Some are mathematical: $0, 4, \pi, i, \text{Diag}[1, -1, -1, -1], d/d\tau$
 Conservation Laws are also a type of "zero" constant in this regard.

The majority of the constants are Lorentz Scalars, but some are 4-Vector or 4-Tensor, and all are valid for all inertial observers.

Fundamental Physical Constants are SR Lorentz Scalars

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor $T^{\mu\nu}$ or $T_{\mu\nu}$
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Existing SR Rules
Quantum Principles

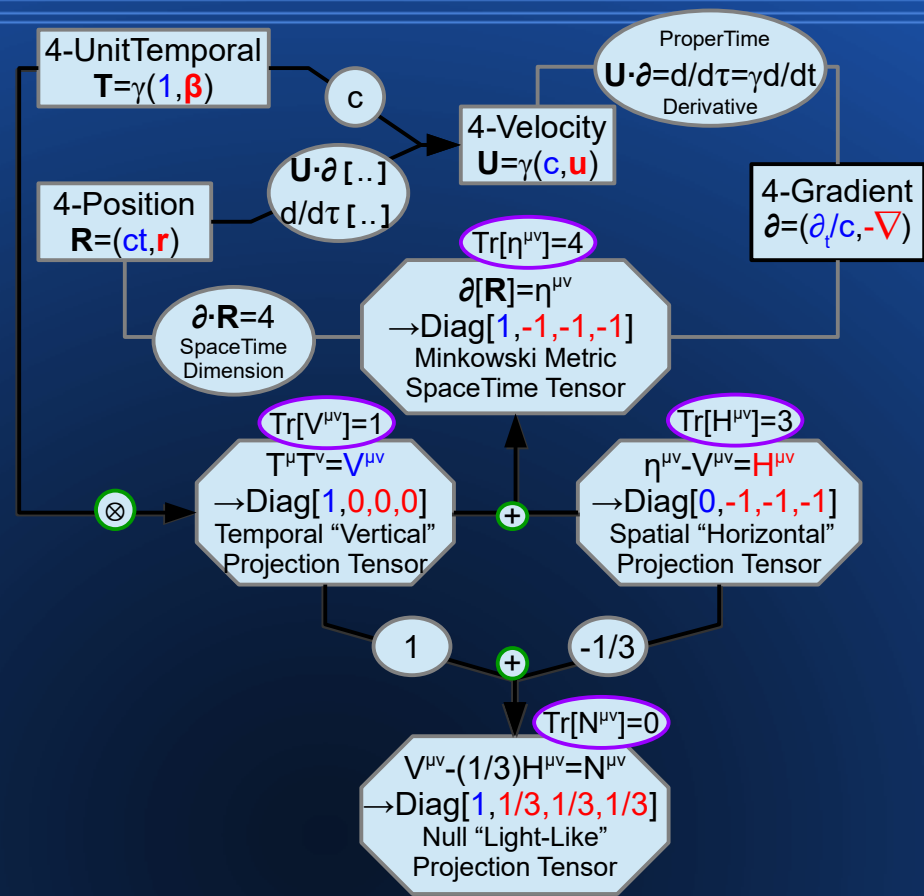
The fact that these "tie together" a network of 4-Vectors is a good argument for why their values are constant. Changing even one would change the relationship properties among all of the 4-Vectors.

SRQM Diagram: Projection Tensors

Temporal, Spatial, Null, SpaceTime

A Tensor Study of Physical 4-Vectors

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Projection Tensors act as follows:

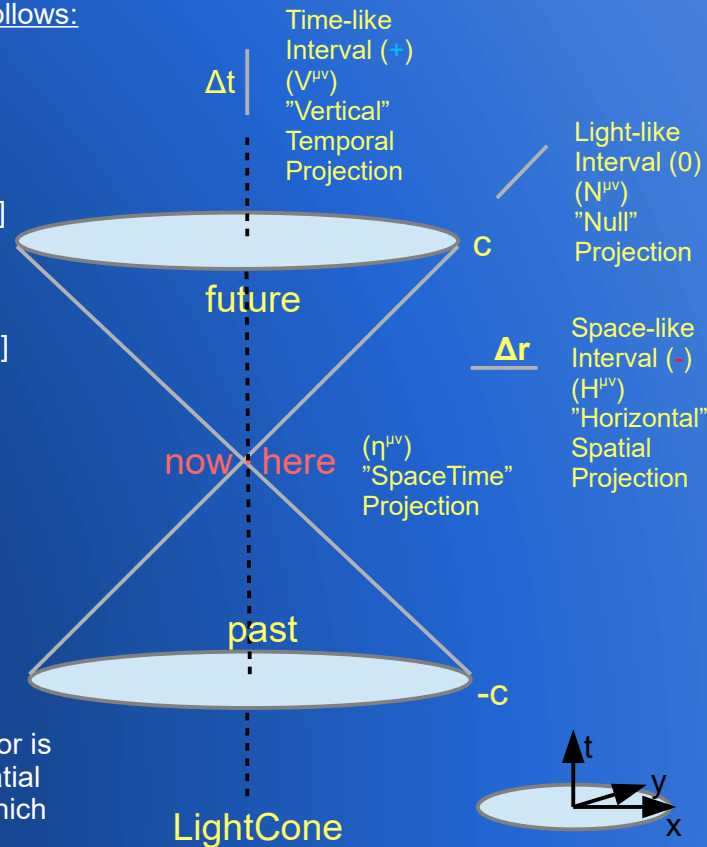
Generic 4-Vector:
 $A^\nu = (a^0, \mathbf{a}) = (a^0, a^1, a^2, a^3)$

Temporal Projection:
 $V^\nu_\nu = \eta_{\omega\nu} V^{\mu\omega} \rightarrow \text{Diag}[1, 0, 0, 0]$
 $V^\nu_\nu A^\nu = (a^0, 0, 0, 0) = (a^0, \mathbf{0})$

Spatial Projection:
 $H^\nu_\nu = \eta_{\omega\nu} H^{\mu\omega} \rightarrow \text{Diag}[0, 1, 1, 1]$
 $H^\nu_\nu A^\nu = (0, a^1, a^2, a^3) = (0, \mathbf{a})$

SpaceTime Projection:
 $V^\nu_\nu A^\nu + H^\nu_\nu A^\nu = \eta^\mu_\nu A^\nu$
 $= \delta^\mu_\nu A^\nu = A^\mu = (a^0, \mathbf{a})$

$V^\mu_\nu + H^\mu_\nu = \eta^\mu_\nu = \delta^\mu_\nu$
 $V^{\mu\nu} + H^{\mu\nu} = \eta^{\mu\nu}$
The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

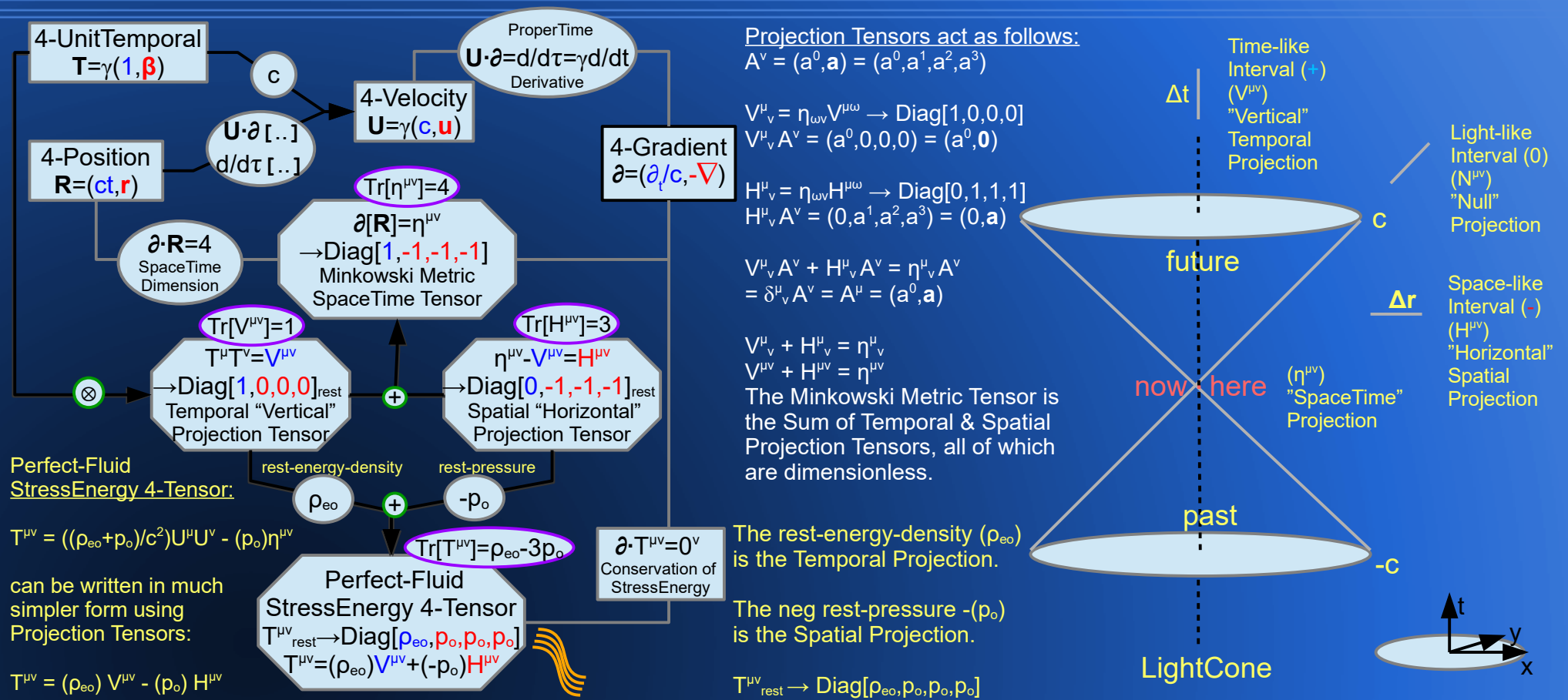
SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram: Projection Tensors & Perfect-Fluid Stress-Energy Tensor

A Tensor Study of Physical 4-Vectors

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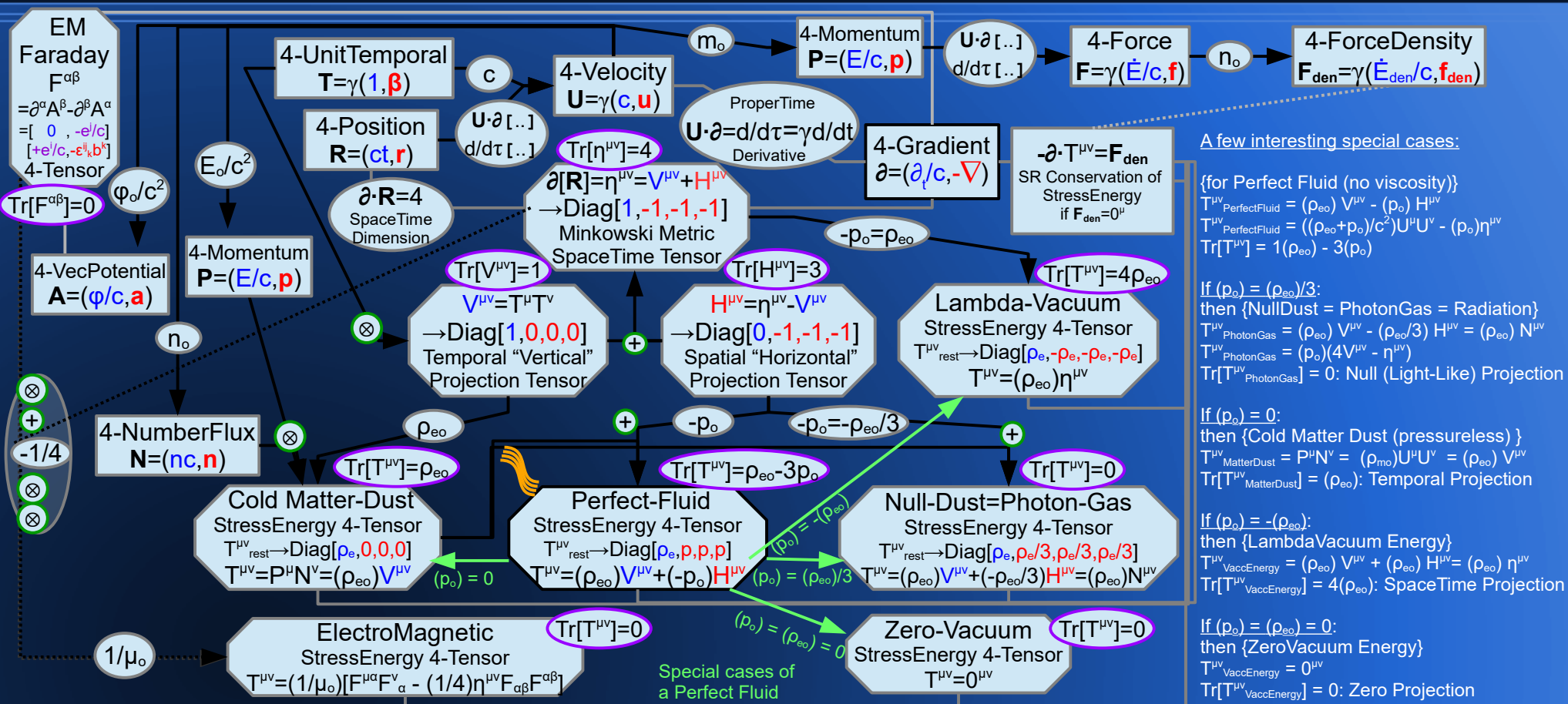
SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_{ν} or T_{μ}^{ν} (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (\mathbf{v}_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
-------------------------------------------------------------------------------------------------------------------------------	-----------------------------------------------------------------------------------------------------------------------------------------------------------	----------------------------------------------------------

Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$

SRQM+EM Diagram: Projection Tensors & Stress-Energy Tensors: Special Cases

A Tensor Study of Physical 4-Vectors

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SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
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SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

$\text{Tr}[\] = \text{Trace Function} = \eta_{\mu\nu}$
 $N^{\mu\nu} = V^{\mu\nu} - (1/3) H^{\mu\nu} = \text{Null Projection Tensor}$
 $N^{\mu\nu} \rightarrow \text{Diag}[1, 1/3, 1/3, 1/3]$ with $\text{Tr}[N^{\mu\nu}] = 0$

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_o)^2 = \text{Lorentz Scalar}$

SRQM Study: 4D Gauss' Theorem

A Tensor Study
of Physical 4-Vectors

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Gauss' Theorem in SR:

$$\int_{\Omega} d^4\mathbf{X} (\partial_{\mu} V^{\mu}) = \oint_{\partial\Omega} dS (V^{\mu} N_{\mu})$$

$$\int_{\Omega} d^4\mathbf{X} (\partial \cdot \mathbf{V}) = \oint_{\partial\Omega} dS (\mathbf{V} \cdot \mathbf{N})$$

where:

$\mathbf{V} = V^{\mu}$ is a 4-Vector field defined in Ω

$(\partial \cdot \mathbf{V}) = (\partial_{\mu} V^{\mu})$ is the 4-Divergence of \mathbf{V}

$(\mathbf{V} \cdot \mathbf{N}) = (V^{\mu} N_{\mu})$ is the component of \mathbf{V} along the \mathbf{N} -direction

Ω is a 4D simply-connected region of Minkowski SpaceTime

$\partial\Omega = S$ is its 3D boundary with its own 3D Volume element dS and outward pointing normal \mathbf{N} .

$\mathbf{N} = N^{\mu}$ is the outward-pointing normal

$d^4\mathbf{X} = (c dt)(d^3\mathbf{x}) = (c dt)(dx dy dz)$ is the 4D differential volume element

In vector calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem, is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface.

More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface.

Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.

In vector calculus, and more generally in differential geometry,

the generalized Stokes' theorem is a statement about the integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus.

SRQM Diagram:

Minimal Coupling = Potential Interaction Conservation of 4-TotalMomentum

A Tensor Study of Physical 4-Vectors

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- $\mathbf{P} = (E/c, \mathbf{p})$: 4-Momentum
- $\mathbf{Q} = (V/c, \mathbf{q})$: 4-PotentialMomentum
- $\mathbf{A} = (\phi/c, \mathbf{a})$: 4-VectorPotential
- $\mathbf{P}_f = (E_f/c, \mathbf{p}_f)$: 4-MomentumIncPotentialField
- $\mathbf{P}_T = (E_T/c, \mathbf{p}_T) = (H/c, \mathbf{p}_T)$: 4-TotalMomentum

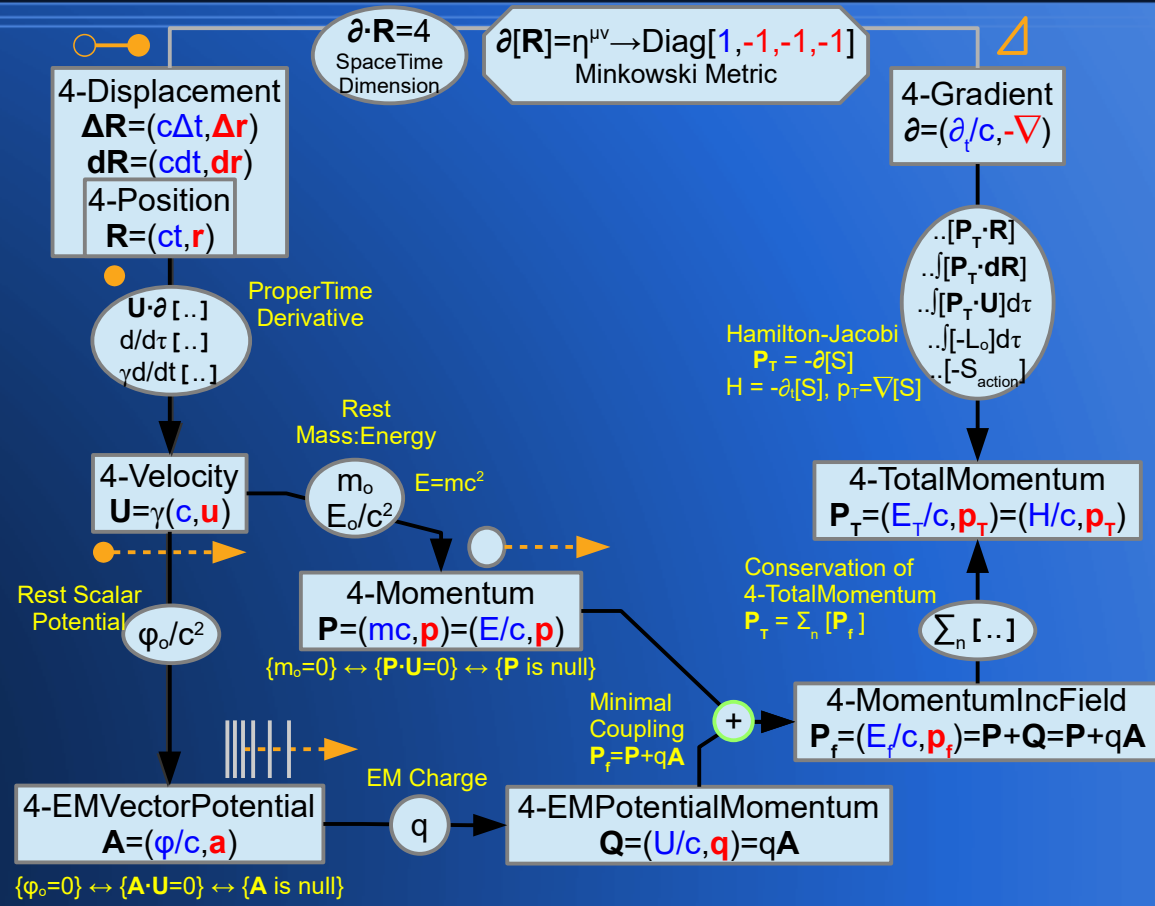
$\mathbf{P} = \mathbf{P}_f - q\mathbf{A} = (E_f/c - q\phi/c, \mathbf{p}_f - q\mathbf{a})$: Minimal Coupling Relation

$\mathbf{P}_f = \mathbf{P} + \mathbf{Q} = \mathbf{P} + q\mathbf{A}$: Conservation of 4-MomentumIncPotentialField

- $\mathbf{P}_f = \mathbf{P} + \mathbf{Q}$
- $\mathbf{P}_f = \mathbf{P} + q\mathbf{A}$
- $\mathbf{P}_f = (m_0)\mathbf{U} + (q\phi_0/c^2)\mathbf{U}$
- $\mathbf{P}_f = (E_0/c^2)\mathbf{U} + (q\phi_0/c^2)\mathbf{U}$
- $\mathbf{P}_f = ((E_0 + q\phi_0)/c^2)\mathbf{U}$
- $\mathbf{P}_f = ((E + q\phi)/c^2)(c, \mathbf{u})$
- $\mathbf{P}_f = ((E + q\phi)/c, \mathbf{p} + q\mathbf{a})$

4-MomentumIncPotentialField has a contribution from a Mass "charge" (m_0) an EM charge (q) interacting with a potential (ϕ_0)

$\mathbf{P}_T = \sum_n [\mathbf{P}_f]$: Conservation of 4-TotalMomentum
4-TotalMomentum is the Sum over all such 4-Momenta



SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν} (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
---------------------------------------------------------------------------------------------------------------------------------	------------------------------------------------------------------------------------------------------------------------------------------------------	----------------------------------------------------------

Trace[T^{μ}_{μ}] = $\eta_{\mu\nu}T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Hamiltonian:Lagrangian Connection

$$H + L = (\mathbf{p}_T \cdot \mathbf{u}) = \gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma$$

A Tensor Study
of Physical 4-Vectors

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4-Momentum $\mathbf{P} = m_0\mathbf{U} = (E_0/c^2)\mathbf{U}$; 4-Vector Potential $\mathbf{A} = (\phi_0/c^2)\mathbf{U}$

4-Total Momentum $\mathbf{P}_T = (\mathbf{P} + q\mathbf{A}) = (H/c, \mathbf{p}_T)$

$\mathbf{P} \cdot \mathbf{U} = \gamma(E - \mathbf{p} \cdot \mathbf{u}) = E_0 = m_0c^2$; $\mathbf{A} \cdot \mathbf{U} = \gamma(\phi - \mathbf{a} \cdot \mathbf{u}) = \phi_0$

$\mathbf{P}_T \cdot \mathbf{U} = (\mathbf{P} \cdot \mathbf{U} + q\mathbf{A} \cdot \mathbf{U}) = E_0 + q\phi_0 = m_0c^2 + q\phi_0$

$\gamma = 1/\text{Sqrt}[1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}]$: Relativistic Gamma Identity

$(\gamma - 1/\gamma) = (\gamma \boldsymbol{\beta} \cdot \boldsymbol{\beta})$: Manipulate into this form... still an identity

$(\gamma - 1/\gamma)(\mathbf{P}_T \cdot \mathbf{U}) = (\gamma \boldsymbol{\beta} \cdot \boldsymbol{\beta})(\mathbf{P}_T \cdot \mathbf{U})$: Still covariant with Lorentz Scalar

$\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma \boldsymbol{\beta} \cdot \boldsymbol{\beta})(\mathbf{P}_T \cdot \mathbf{U})$

$\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma \boldsymbol{\beta} \cdot \boldsymbol{\beta})(E_0 + q\phi_0)$

$\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma \mathbf{u} \cdot \mathbf{u})(E_0 + q\phi_0)/c^2$

$\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma(E_0/c^2 + q\phi_0/c^2)\mathbf{u} \cdot \mathbf{u})$

$\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = ((\gamma E_0/c^2 + \gamma q\phi_0/c^2) \cdot \mathbf{u})$

$\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = ((E\mathbf{u}/c^2 + q\phi\mathbf{u}/c^2) \cdot \mathbf{u})$

$\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = ((\mathbf{p} + q\mathbf{a}) \cdot \mathbf{u})$

$\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\mathbf{p}_T \cdot \mathbf{u})$

$\{ H \} + \{ L \} = (\mathbf{p}_T \cdot \mathbf{u})$: The Hamiltonian/Lagrangian connection

$H = \gamma(\mathbf{P}_T \cdot \mathbf{U}) = \gamma((\mathbf{P} + q\mathbf{A}) \cdot \mathbf{U}) =$ The Hamiltonian with minimal coupling

$L = -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = -((\mathbf{P} + q\mathbf{A}) \cdot \mathbf{U})/\gamma =$ The Lagrangian with minimal coupling

H:L Connection in Density Format

$H + L = (\mathbf{p}_T \cdot \mathbf{u})$

$nH + nL = n(\mathbf{p}_T \cdot \mathbf{u})$, with number density $n = \gamma n_0$

$\mathcal{H} + \mathcal{L} = (\mathbf{g}_T \cdot \mathbf{u})$, with

momentum density $\{\mathbf{g}_T = n\mathbf{p}_T\}$

Hamiltonian density $\{\mathcal{H} = nH\}$

Lagrangian Density $\{\mathcal{L} = nL = (\gamma n_0)(L_0/\gamma) = n_0 L_0\}$

Lagrangian Density is Lorentz Scalar

for an EM field (photonic):

$\mathcal{H} = (1/2)\{\epsilon_0 \mathbf{e} \cdot \mathbf{e} + \mathbf{b} \cdot \mathbf{b}/\mu_0\}$

$\mathcal{L} = (1/2)\{\epsilon_0 \mathbf{e} \cdot \mathbf{e} - \mathbf{b} \cdot \mathbf{b}/\mu_0\} = (-1/4\mu_0)F_{\mu\nu}F^{\mu\nu}$

$\mathcal{H} + \mathcal{L} = \epsilon_0 \mathbf{e} \cdot \mathbf{e} = (\mathbf{g}_T \cdot \mathbf{u})$

$|\mathbf{u}| = c$

$|\mathbf{g}_T| = \epsilon_0 \mathbf{e} \cdot \mathbf{e}/c$

Poynting Vector $|\mathbf{s}| = |\mathbf{g}|c^2 \rightarrow c\epsilon_0 \mathbf{e} \cdot \mathbf{e}$

$H_0 + L_0 = 0$ Calculating the Rest Values

$H_0 = (\mathbf{P}_T \cdot \mathbf{U})$

$H = \gamma H_0$

$L_0 = -(\mathbf{P}_T \cdot \mathbf{U})$

$L = L_0/\gamma$

4-Vector notation gives a very nice way to find the Hamiltonian/Lagrangian connection:

$(H) + (L) = (\mathbf{p}_T \cdot \mathbf{u})$, where $H = \gamma(\mathbf{P}_T \cdot \mathbf{U})$ & $L = -(\mathbf{P}_T \cdot \mathbf{U})/\gamma$

SRQM Study:

SR Lagrangian, Lagrangian Density, and Relativistic Action (S)

A Tensor Study
of Physical 4-Vectors

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Relativistic Action (S) is Lorentz Scalar Invariant

$$S = \int L dt = \int (L_o/\gamma)(\gamma d\tau) = \int (L_o)(d\tau)$$

$$S = \int L dt = \int (\mathcal{L}/n) dt = \int \mathcal{L}/(n) dt = \int \mathcal{L}(d^3x) dt = \int (\mathcal{L}/c)(d^3x)(cdt) = \int (\mathcal{L}/c)(d^4x)$$

Explicitly-Covariant Relativistic Action (S)

Particle Form	Density Form {= n _o *Particle}
$S = \int L_o d\tau = -\int H_o d\tau$	$S = (1/c) \int (n_o L_o)(d^4x) = -(1/c) \int (n_o H_o)(d^4x)$
$S = -\int (\mathbf{P}_T \cdot \mathbf{U}) d\tau$	$S = (1/c) \int (\mathcal{L})(d^4x)$
$S = -\int (\mathbf{P}_T \cdot d\mathbf{R}/d\tau) d\tau$	
$S = -\int (\mathbf{P}_T \cdot d\mathbf{R})$	$S = \int (\mathcal{L}/c)(d^4x)$

$S = -\int (\mathbf{P}_T \cdot \mathbf{U}) d\tau$	$S = -(1/c) \int n_o (\mathbf{P}_T \cdot \mathbf{U})(d^4x)$
$S = -\int ((\mathbf{P} + q\mathbf{A}) \cdot \mathbf{U}) d\tau$	$S = -(1/c) \int n_o ((\mathbf{P} + q\mathbf{A}) \cdot \mathbf{U})(d^4x)$
$S = -\int (\mathbf{P} \cdot \mathbf{U} + q\mathbf{A} \cdot \mathbf{U}) d\tau$	$S = -(1/c) \int (n_o \mathbf{P} \cdot \mathbf{U} + n_o q \mathbf{A} \cdot \mathbf{U})(d^4x)$
$S = -\int (E_o + q\mathbf{U} \cdot \mathbf{A}) d\tau$	$S = -(1/c) \int (n_o E_o + n_o q \mathbf{U} \cdot \mathbf{A})(d^4x)$
$S = -\int (E_o + q\phi_o) d\tau$	$S = -(1/c) \int (\rho_{E_o} + \mathbf{J} \cdot \mathbf{A})(d^4x)$
$S = -\int (E_o + V) d\tau$	
$S = -\int (m_o c^2 + V) d\tau$	$S = (1/c) \int (\mathcal{L})(d^4x)$

with $V = q\phi_o$

$$S = (1/c) \int ((1/2) \{ \epsilon_o \mathbf{e} \cdot \mathbf{e} - \mathbf{b} \cdot \mathbf{b} / \mu_o \}) (d^4x)$$

$$S = (1/c) \int ((-1/4 \mu_o) F_{\mu\nu} F^{\mu\nu}) (d^4x)$$

for an EM field = no rest frame

Lagrangian $\{L = (\mathbf{p}_T \cdot \mathbf{u}) - H\}$ is *not* Lorentz Scalar Invariant

Rest Lagrangian $\{L_o = \gamma L = -(\mathbf{P}_T \cdot \mathbf{U})\}$ is Lorentz Scalar Invariant

Lagrangian Density $\{\mathcal{L} = nL = (\gamma n_o)(L_o/\gamma) = n_o L_o\}$ is Lorentz Scalar Invariant

$$n = \gamma n_o = \# / d^3x = \# / (dx)(dy)(dz) = \text{number density}$$

$$dt = \gamma d\tau$$

$$cd\tau = n_o (cdt)(dx)(dy)(dz) = n_o (d^4x)$$

$$d\tau = (n_o/c)(d^4x)$$

H:L Connection in Density Format for Photonic System (no rest-frame)

$$H + L = (\mathbf{p}_T \cdot \mathbf{u})$$

$$nH + nL = n(\mathbf{p}_T \cdot \mathbf{u}), \text{ with number density } n = \gamma n_o$$

$$\mathcal{H} + \mathcal{L} = (\mathbf{g}_T \cdot \mathbf{u}), \text{ with}$$

momentum density $\{\mathbf{g}_T = n\mathbf{p}_T\}$
 Hamiltonian density $\{\mathcal{H} = nH\}$
 Lagrangian Density $\{\mathcal{L} = nL = (\gamma n_o)(L_o/\gamma) = n_o L_o\}$
 Lagrangian Density is Lorentz Scalar

for an EM field (photonic):

$$\mathcal{H} = (1/2) \{ \epsilon_o \mathbf{e} \cdot \mathbf{e} + \mathbf{b} \cdot \mathbf{b} / \mu_o \} = n_o E_o = \rho_{E_o} = \text{EM Field Energy Density}$$

$$\mathcal{L} = (1/2) \{ \epsilon_o \mathbf{e} \cdot \mathbf{e} - \mathbf{b} \cdot \mathbf{b} / \mu_o \} = (-1/4 \mu_o) F_{\mu\nu} F^{\mu\nu} = (-1/4 \mu_o) * \text{Faraday EM Tensor Inner Product}$$

$$\mathcal{H} + \mathcal{L} = \epsilon_o \mathbf{e} \cdot \mathbf{e} = (\mathbf{g}_T \cdot \mathbf{u})$$

$$|\mathbf{u}| = c$$

$$|\mathbf{g}_T| = \epsilon_o \mathbf{e} \cdot \mathbf{e} / c$$

Poynting Vector $|\mathbf{s}| = |\mathbf{g}|c^2 \rightarrow c\epsilon_o \mathbf{e} \cdot \mathbf{e}$

$\epsilon_o \mu_o = 1/c^2$:Electric:Magnetic Constant Eqn

The Relativistic Action Equation is seen in many different formats

SRQM Study:

SR Hamilton-Jacobi Equation and Relativistic Action (S)

Lagrangian $\{L = (\mathbf{p}_T \cdot \mathbf{u}) - H\}$ is *not* a Lorentz Scalar
 Rest Lagrangian $\{L_o = \gamma L = -(\mathbf{P}_T \cdot \mathbf{U})\}$ is a Lorentz Scalar

Relativistic Action (S) is Lorentz Scalar

$$S = \int L dt$$

$$S = \int (L_o/\gamma)(\gamma d\tau)$$

$$S = \int (L_o)(d\tau)$$

Explicitly Covariant Relativistic Action (S)

$$S = \int L_o d\tau = -\int H_o d\tau$$

$$S = -\int (\mathbf{P}_T \cdot \mathbf{U}) d\tau$$

$$S = -\int (\mathbf{P}_T \cdot d\mathbf{R}/d\tau) d\tau$$

$$S = -\int (\mathbf{P}_T \cdot d\mathbf{R})$$

$$S = -\int (\mathbf{P}_T \cdot \mathbf{U}) d\tau$$

$$S = -\int ((\mathbf{P} + q\mathbf{A}) \cdot \mathbf{U}) d\tau$$

$$S = -\int (\mathbf{P} \cdot \mathbf{U} + q\mathbf{A} \cdot \mathbf{U}) d\tau$$

$$S = -\int (E_o + q\phi_o) d\tau$$

$$S = -\int (E_o + V) d\tau \quad \text{with } V = q\phi_o$$

$$S = -\int (m_o c^2 + V) d\tau$$

$$S = -\int (H_o) d\tau$$

4-Scalars
Relativistic Action Eqn
Integral Format

$$S_{\text{action}} = -\int [\mathbf{P}_T \cdot d\mathbf{R}]$$

$$= -\int [\mathbf{P}_T \cdot \mathbf{U}] d\tau$$

$$= -\int [(\mathbf{H}/c, \mathbf{p}_T) \cdot \gamma(c, \mathbf{u})] d\tau$$

$$= -\int [\gamma(H - \mathbf{p}_T \cdot \mathbf{u})] d\tau$$

Inverse

4-Vectors
Relativistic Hamilton-Jacobi Eqn
Differential Format

4-TotalMomentum

$$\mathbf{P}_T = (E_T/c, \mathbf{p}_T) = (H/c, \mathbf{p}_T)$$

$$\mathbf{P}_T = -\partial[S_{\text{action}}]$$

$$(H/c, \mathbf{p}_T) = (-\partial_t[S_{\text{action}}], \nabla[S_{\text{action}}])$$

Hamilton-Jacobi Equation
 $\partial[-S] = -\partial[S] = \mathbf{P}_T$

$$S = -\int (E_o + q\phi_o) d\tau$$

$$S = -(E_o + q\phi_o) \int d\tau$$

$$S = -(E_o + q\phi_o)(\tau + \text{const})$$

$$-S = (E_o + q\phi_o)(\tau + \text{const})$$

$$\partial[-S] = (E_o + q\phi_o)\partial[(\tau + \text{const})]$$

$$\partial[-S] = (E_o + q\phi_o)\partial[\tau]$$

$$\partial[-S] = (E_o + q\phi_o)\partial[\mathbf{R} \cdot \mathbf{U}/c^2]$$

$$\partial[-S] = ((E_o + q\phi_o)/c^2)\partial[\mathbf{R} \cdot \mathbf{U}]$$

$$\partial[-S] = (E_o/c^2 + q\phi_o/c^2)\mathbf{U}$$

$$\partial[-S] = (m_o + q\phi_o/c^2)\mathbf{U}$$

$$\partial[-S] = m_o\mathbf{U} + q(\phi_o/c^2)\mathbf{U}$$

$$\partial[-S] = \mathbf{P} + q\mathbf{A}$$

$$\partial[-S] = \mathbf{P}_T$$

Verified!

$$\mathbf{R} \cdot \mathbf{U} = c^2\tau : \tau = \mathbf{R} \cdot \mathbf{U}/c^2$$

The Hamilton-Jacobi Equation is incredibly simple in 4-Vector form

SR 4-Tensor

(2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector

(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
 SR 4-CoVector
 (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar

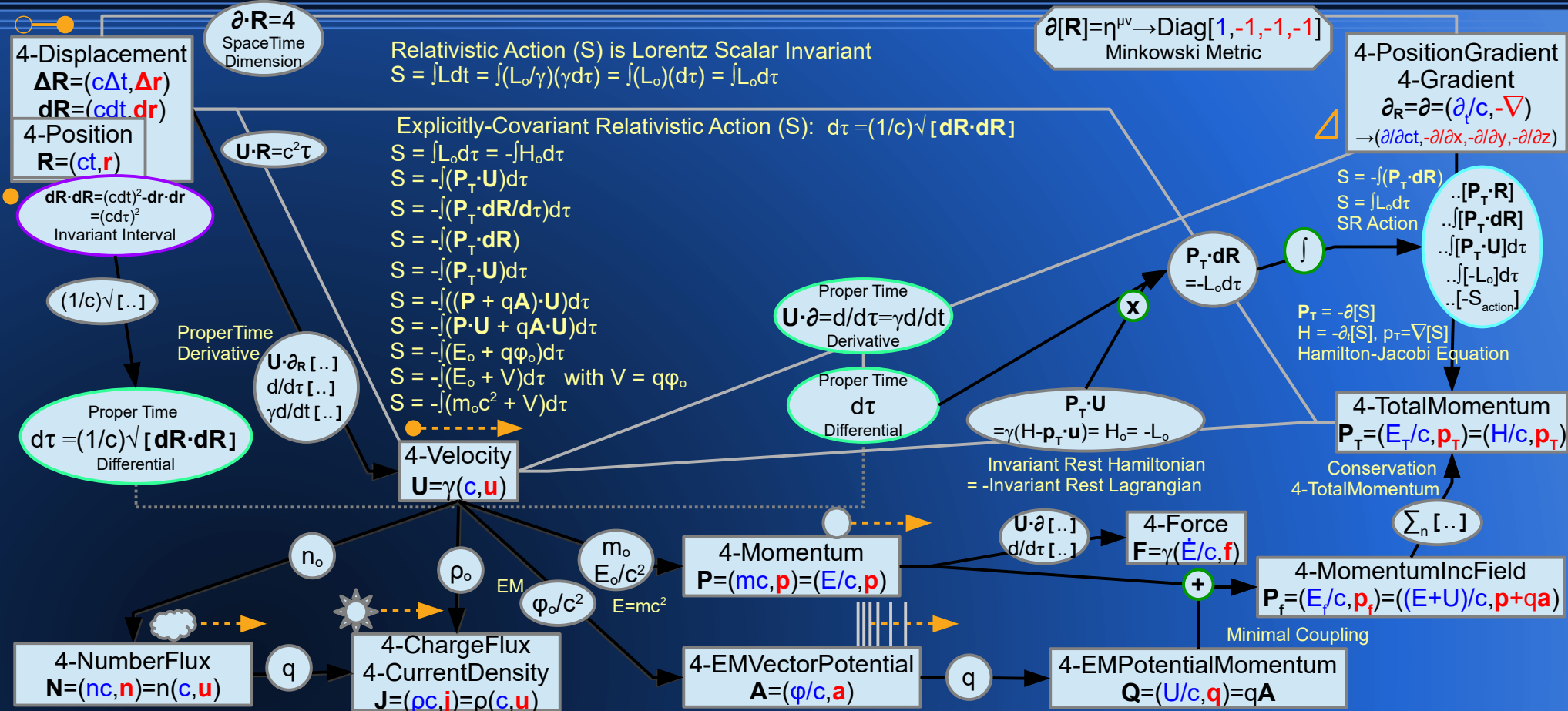
(0,0)-Tensor S
Lorentz Scalar

SRQM Diagram:

Relativistic Hamilton-Jacobi Equation ($P_T = -\partial[S]$) Differential Format : 4-Vectors

A Tensor Study of Physical 4-Vectors

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SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = V = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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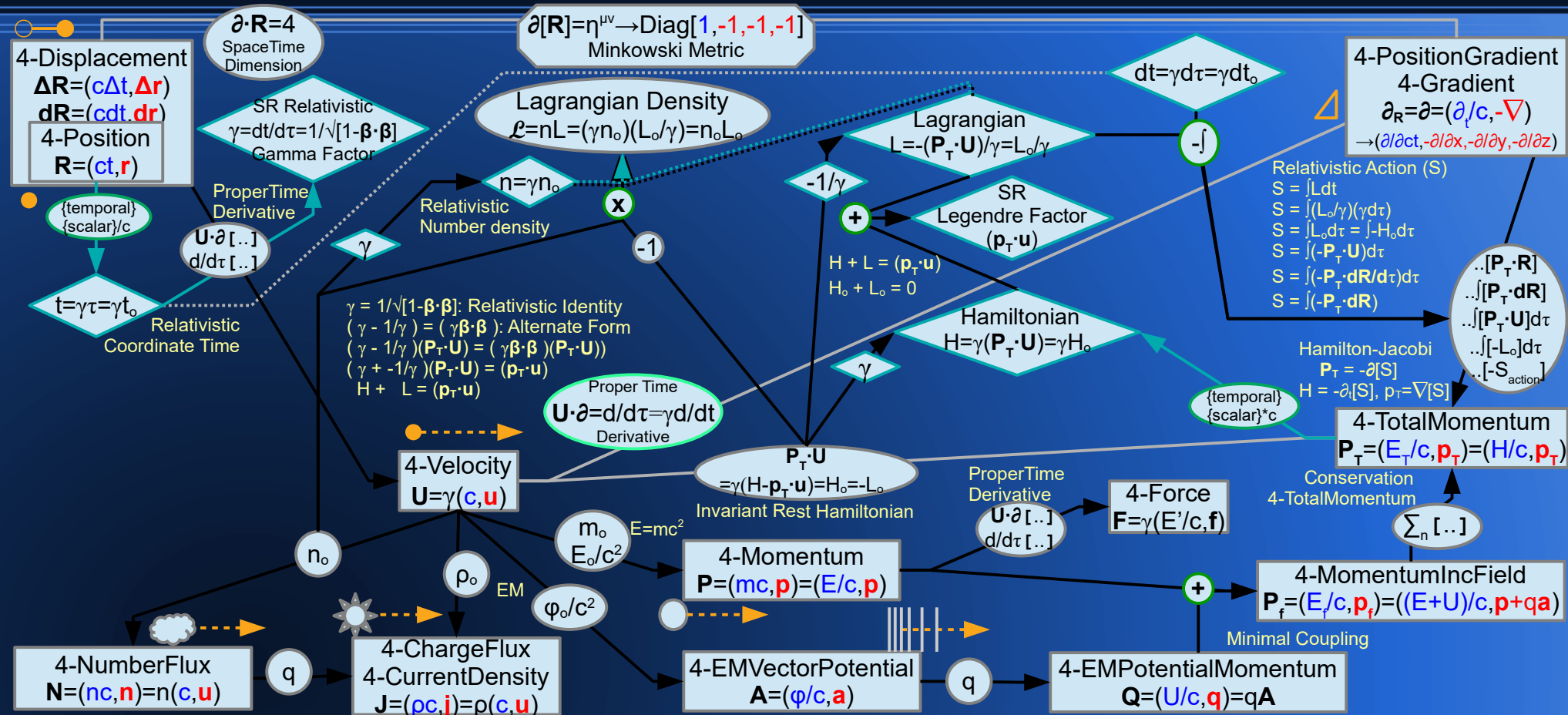
Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram: Relativistic Action Equation

(S = -∫(P_T·dR)) Integral Format : 4-Scalars

A Tensor Study of Physical 4-Vectors

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SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = V = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

SR Relativistic Scalar
(not Lorentz Invariant)

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $V \cdot V = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2$
 = Lorentz Scalar

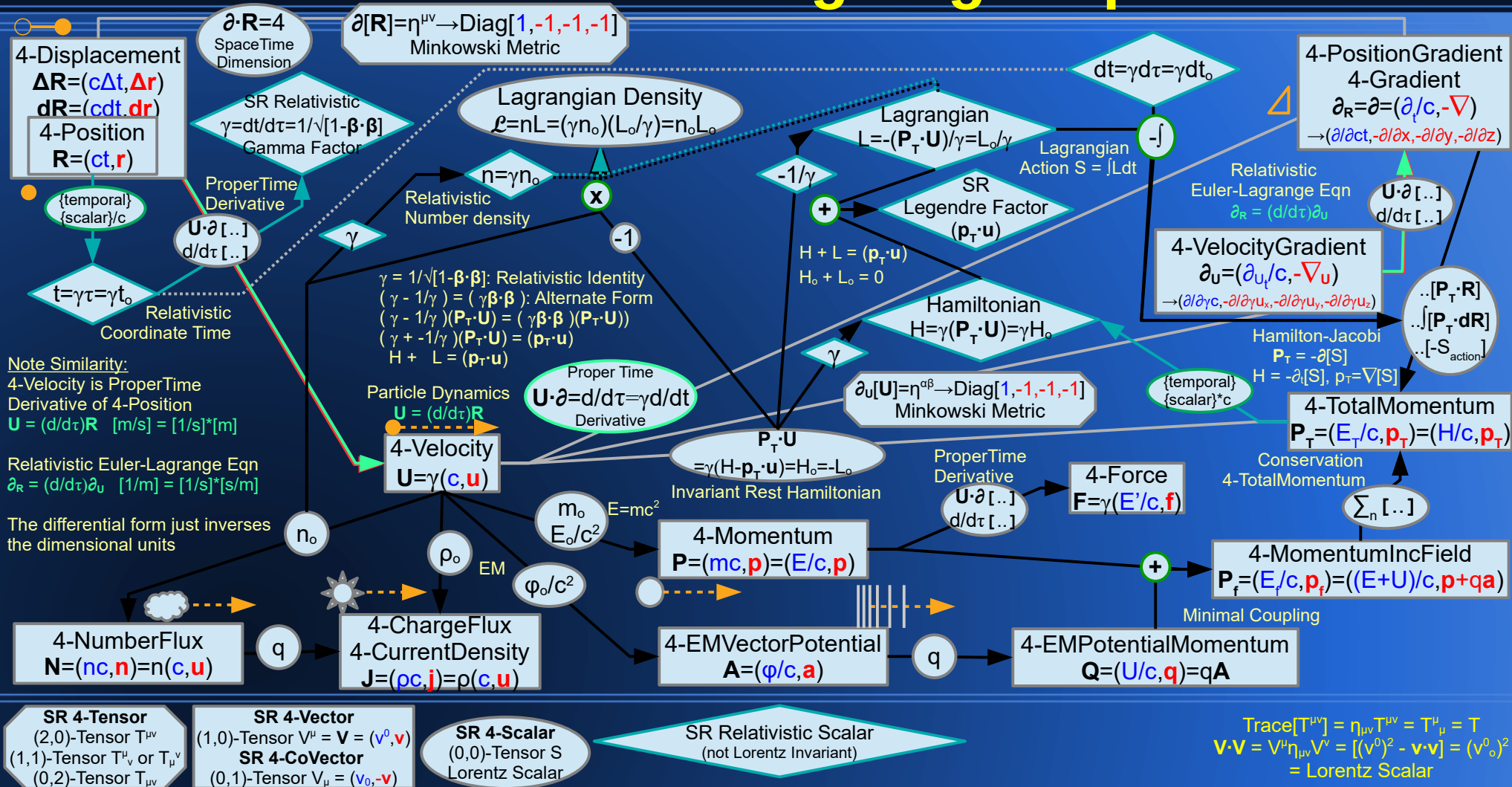
SRQM Diagram: Relativistic Factors

Hamiltonian & Lagrangian

Relativistic Euler-Lagrange Equation

A Tensor Study of Physical 4-Vectors

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SRQM Diagram:

Relativistic Euler-Lagrange Equation

The Easy Derivation $(\mathbf{U}=(d/d\tau)\mathbf{R}) \rightarrow (\partial_{\mathbf{R}}=(d/d\tau)\partial_{\mathbf{U}})$

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Note Similarity:

4-Velocity is ProperTime Derivative of 4-Position
 $\mathbf{U} = (d/d\tau)\mathbf{R}$ [m/s] = [1/s]*[m]

Relativistic Euler-Lagrange Eqn
 $\partial_{\mathbf{R}} = (d/d\tau)\partial_{\mathbf{U}}$ [1/m] = [1/s]*[s/m]

The differential form just inverts the dimensional units, so the placement of the \mathbf{R} and \mathbf{U} switch.

That is it: so simple!
Much, much easier than how I was taught in Grad School.

To complete the process and create the Equations of Motion, one just applies the base form to a Lagrangian.

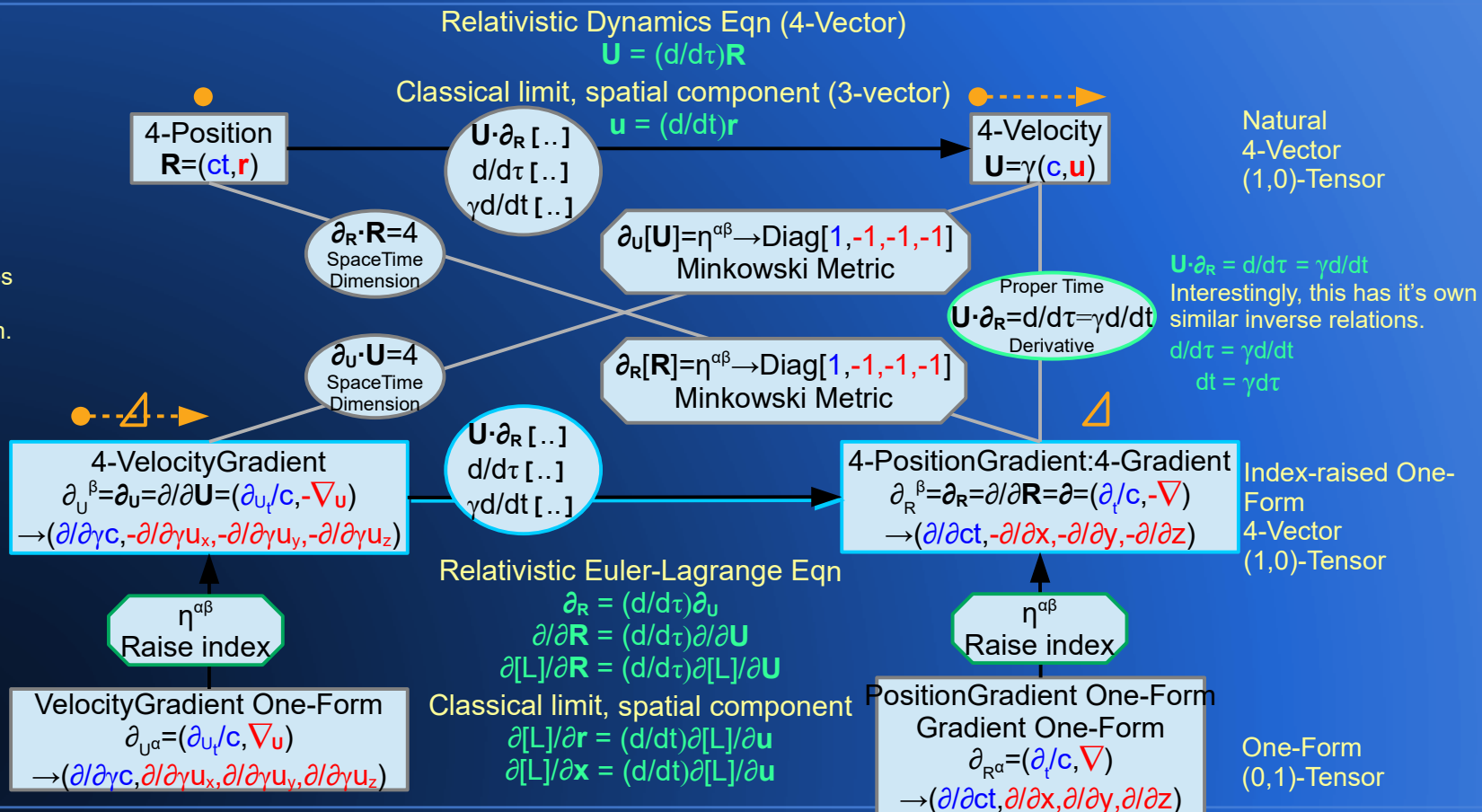
This can be:
 a classical Lagrangian
 a relativistic Lagrangian
 a Lorentz scalar Lagrangian
 a quantum Lagrangian

Relativistic Dynamics Eqn (4-Vector)

$$\mathbf{U} = (d/d\tau)\mathbf{R}$$

Classical limit, spatial component (3-vector)

$$\mathbf{u} = (d/dt)\mathbf{r}$$



Natural 4-Vector (1,0)-Tensor

$\mathbf{U}\cdot\partial_{\mathbf{R}} = d/d\tau = \gamma d/dt$
 Interestingly, this has it's own similar inverse relations.
 $d/d\tau = \gamma d/dt$
 $dt = \gamma d\tau$

Index-raised One-Form 4-Vector (1,0)-Tensor

One-Form (0,1)-Tensor

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu}T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

Relativistic Euler-Lagrange Equation Alternate Forms: Particle vs. Density

A Tensor Study of Physical 4-Vectors

SciRealm.org
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4-Velocity \mathbf{U} is ProperTime Derivative of 4-Position \mathbf{R} . The Euler-Lagrange Eqn can be generated by taking the differential form of the same equation.

Relativistic 4-Vector Kinematical Eqn

$$\mathbf{U} = (d/d\tau)\mathbf{R}$$

$$\mathbf{U} \cdot \mathbf{K} = (d/d\tau)\mathbf{R} \cdot \mathbf{K}$$

Relativistic Euler-Lagrange Eqns

{uses gradient-type 4-Vectors}

$$\partial_{\mathbf{R}} = (d/d\tau)\partial_{\mathbf{U}}: \{\text{particle format}\}$$

$$\partial_{\mathbf{R} \cdot \mathbf{K}} = (d/d\tau) \partial_{\mathbf{U} \cdot \mathbf{K}}$$

$$\partial_{(-\Phi)} = (d/d\tau) \partial_{\mathbf{U} \cdot \mathbf{K}}$$

$$\partial_{(-\Phi)} = (\mathbf{U} \cdot \partial_{\mathbf{R}}) \partial_{\mathbf{U} \cdot \mathbf{K}}$$

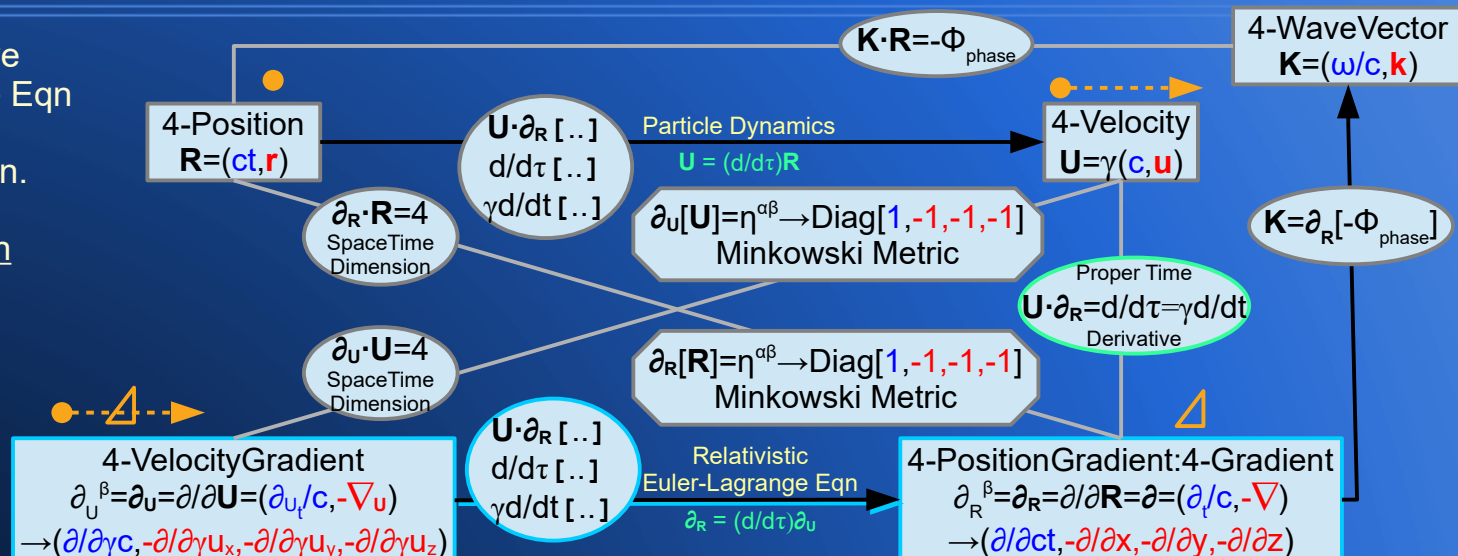
$$\partial \partial (-\Phi) = (\mathbf{U} \cdot \partial_{\mathbf{R}}) \partial \partial [\mathbf{U} \cdot \mathbf{K}]$$

$$\partial \partial (-\Phi) = (\partial_{\mathbf{R}}) \partial \partial [\mathbf{K}]$$

$$\partial \partial (-\Phi) = (\partial_{\mathbf{R}}) \partial \partial [\partial_{\mathbf{R}}(-\Phi)]$$

$$\partial \partial (\Phi) = (\partial_{\mathbf{R}}) \partial \partial [\partial_{\mathbf{R}}(\Phi)]$$

$$\partial_{[\Phi]} = (\partial_{\mathbf{R}}) \partial_{[\partial_{\mathbf{R}}(\Phi)]}: \{\text{density format}\}$$



$$\mathcal{L} = (1/2)\{ \partial_{\mathbf{R}}[\Phi] \cdot \partial_{\mathbf{R}}[\Phi] - (m_0 c/\hbar)^2 \Phi^2 \}: \text{KG Lagrangian Density}$$

$$\partial_{[\Phi]} \mathcal{L} = (\partial_{\mathbf{R}}) \partial_{[\partial_{\mathbf{R}}(\Phi)]} \mathcal{L}: \text{Euler-Lagrange Eqn \{density format\}}$$

$$-(m_0 c/\hbar)^2 \Phi = (\partial_{\mathbf{R}}) \cdot \partial_{\mathbf{R}}[\Phi]$$

$$(\partial_{\mathbf{R}} \cdot \partial_{\mathbf{R}})[\Phi] = - (m_0 c/\hbar)^2 \Phi$$

$$(\partial \cdot \partial) = - (m_0 c/\hbar)^2: \text{KG Eqn of Motion}$$

Klein-Gordon Relativistic Quantum Wave Eqn

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

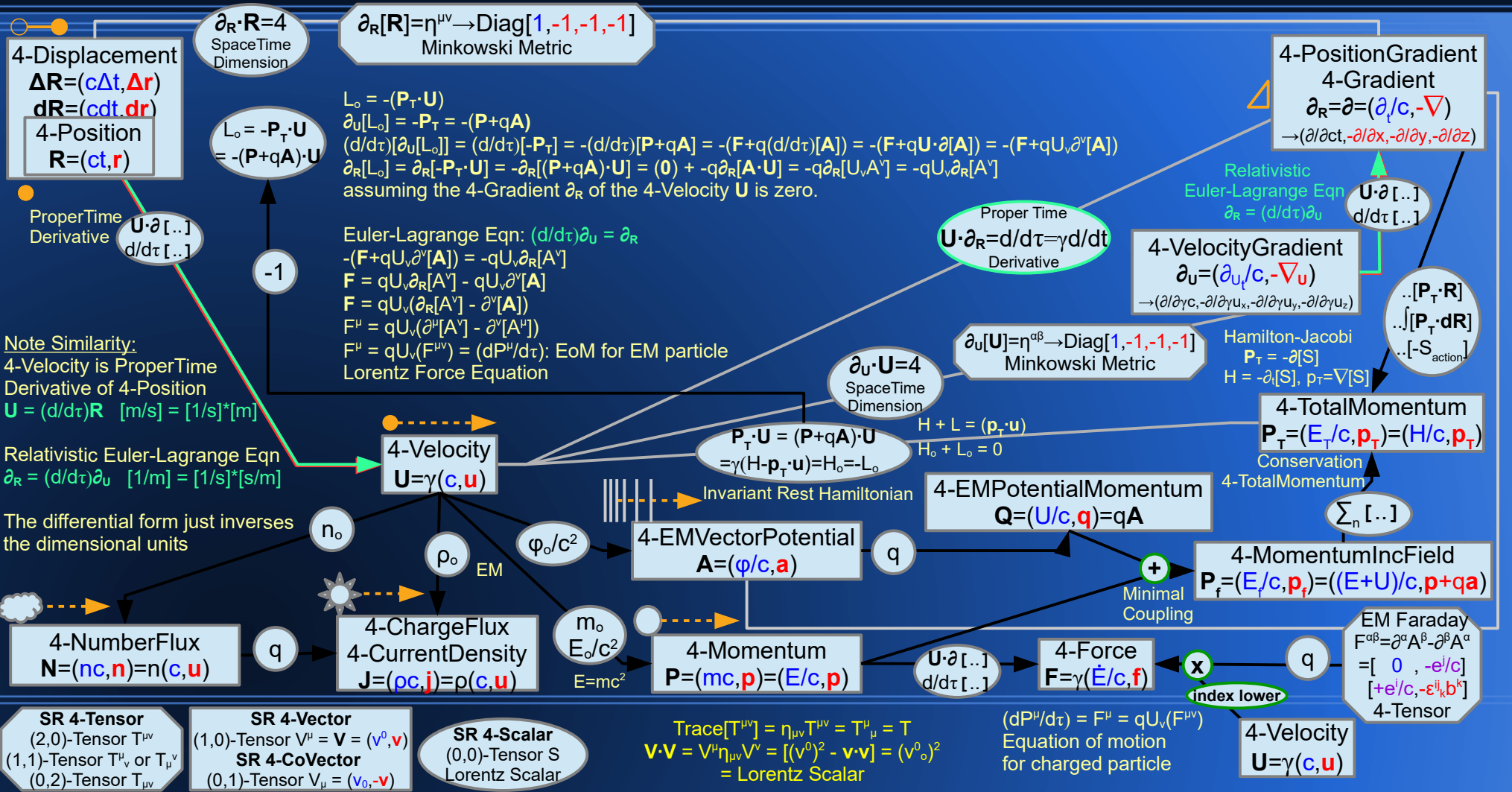
$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$$

SRQM Diagram: Relativistic Euler-Lagrange Equation Equation of Motion (EoM) for EM particle

A Tensor Study of Physical 4-Vectors

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John B. Wilson



SRQM Diagram: Relativistic Euler-Lagrange Equation Equation of Motion (EoM) for EM particle

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

$\gamma = 1/\text{Sqrt}[1-\beta\cdot\beta]$: Relativistic Gamma Identity
 $(\gamma - 1/\gamma) = (\gamma\beta\cdot\beta)$: Manipulate into this form... still an identity
 $\gamma(\mathbf{P}_T\cdot\mathbf{U}) + -(\mathbf{P}_T\cdot\mathbf{U})/\gamma = (\gamma\beta\cdot\beta)(\mathbf{P}_T\cdot\mathbf{U})$
 $\gamma(\mathbf{P}_T\cdot\mathbf{U}) + -(\mathbf{P}_T\cdot\mathbf{U})/\gamma = (\mathbf{p}_T\cdot\mathbf{u})$
 $\{ H \} + \{ L \} = (\mathbf{p}_T\cdot\mathbf{u})$: The Hamiltonian/Lagrangian connection

$H = \gamma H_0 = \gamma(\mathbf{P}_T\cdot\mathbf{U}) = \gamma((\mathbf{P}+q\mathbf{A})\cdot\mathbf{U})$ = The Hamiltonian with minimal coupling
 $L = L_0/\gamma = -(\mathbf{P}_T\cdot\mathbf{U})/\gamma = -((\mathbf{P}+q\mathbf{A})\cdot\mathbf{U})/\gamma$ = The Lagrangian with minimal coupling

$H_0 = (\mathbf{P}_T\cdot\mathbf{U}) = -L_0 = (\mathbf{U}\cdot\mathbf{P}_T)$: Rest Hamiltonian = Total RestEnergy
 $L_0 = -(\mathbf{P}_T\cdot\mathbf{U}) = -H_0$

$(d/d\tau)\partial_u[L_0] = \partial_R[L_0]$

4-Velocity is ProperTime
Derivative of 4-Position
 $\mathbf{U} = (d/d\tau)\mathbf{R}$ [m/s] = [1/s]*[m]

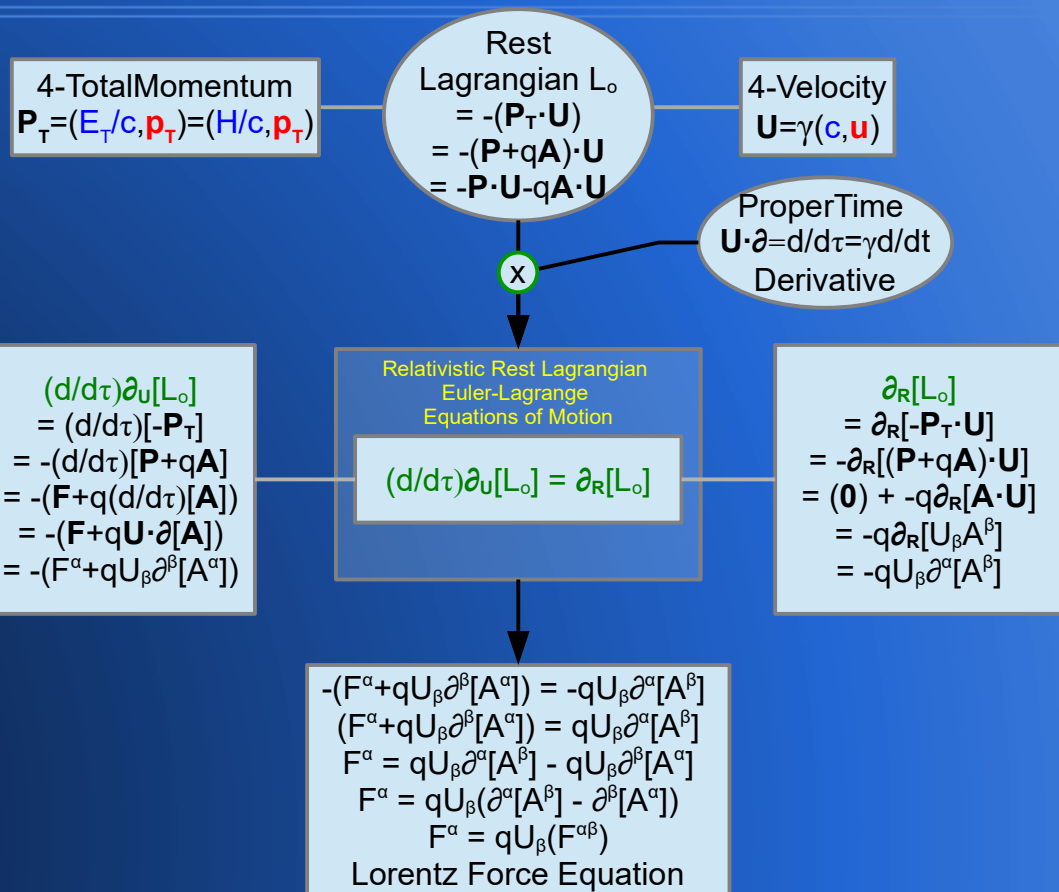
Relativistic Euler-Lagrange Eqn
 $\partial_R = (d/d\tau)\partial_u$ [1/m] = [1/s]*[s/m]

$\partial/\partial\mathbf{R} = (d/d\tau)\partial/\partial\mathbf{U}$
 $\partial[L]/\partial\mathbf{R} = (d/d\tau)\partial[L]/\partial\mathbf{U}$

Classical limit, spatial component
 $\partial[L]/\partial\mathbf{r} = (d/dt)\partial[L]/\partial\mathbf{u}$
 $\partial[L]/\partial\mathbf{x} = (d/dt)\partial[L]/\partial\mathbf{u}$

$\mathbf{F}_{EM} = \gamma q\{ (\mathbf{u}\cdot\mathbf{e})/c, (\mathbf{e}) + (\mathbf{u}\times\mathbf{b}) \}$
 $\mathbf{e} = (-\nabla\phi - \partial_t\mathbf{a})$ and $\mathbf{b} = [\nabla\times\mathbf{a}]$

If $\mathbf{a}\sim 0$, then $\mathbf{f} = -q\nabla\phi = -\nabla U$, the force is neg grad of a potential



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 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or T_μ^ν
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM Diagram: Relativistic Hamilton's Equations Equation of Motion (EoM) for EM particle

A Tensor Study of Physical 4-Vectors

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$\gamma = 1/\text{sqrt}[1-\beta \cdot \beta]$: Relativistic Gamma Identity
 $(\gamma - 1/\gamma) = (\gamma \beta \cdot \beta)$: Manipulate into this form... still an identity
 $\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma \beta \cdot \beta)(\mathbf{P}_T \cdot \mathbf{U})$
 $\gamma(\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\mathbf{p}_T \cdot \mathbf{u})$
 $\{ H \} + \{ L \} = (\mathbf{p}_T \cdot \mathbf{u})$: The Hamiltonian/Lagrangian connection

$H = \gamma H_0 = \gamma(\mathbf{P}_T \cdot \mathbf{U}) = \gamma((\mathbf{P} + q\mathbf{A}) \cdot \mathbf{U})$ = The Hamiltonian with minimal coupling
 $L = L_0/\gamma = -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = -((\mathbf{P} + q\mathbf{A}) \cdot \mathbf{U})/\gamma$ = The Lagrangian with minimal coupling

$H_0 = (\mathbf{P}_T \cdot \mathbf{U}) = -L_0 = (\mathbf{U} \cdot \mathbf{P}_T)$: Rest Hamiltonian = Total RestEnergy
 $L_0 = -(\mathbf{P}_T \cdot \mathbf{U}) = -H_0$

$\partial_{\mathbf{P}_T}[H_0] = \partial_{\mathbf{P}_T}[\mathbf{U} \cdot \mathbf{P}_T] = \partial_{\mathbf{P}_T}[\mathbf{U}] \cdot \mathbf{P}_T + \mathbf{U} \cdot \partial_{\mathbf{P}_T}[\mathbf{P}_T] = 0 + \mathbf{U} \cdot \partial_{\mathbf{P}_T}[\mathbf{P}_T] = \mathbf{U} = d/d\tau[\mathbf{X}]$
 Thus: $(d/d\tau)[\mathbf{X}] = (\partial/\partial \mathbf{P}_T)[H_0]$
 $\partial_{\mathbf{X}}[H_0] = \partial_{\mathbf{X}}[\mathbf{U} \cdot \mathbf{P}_T] = \partial_{\mathbf{X}}[\mathbf{U}] \cdot \mathbf{P}_T + \mathbf{U} \cdot \partial_{\mathbf{X}}[\mathbf{P}_T] = 0 + \mathbf{U} \cdot \partial_{\mathbf{X}}[\mathbf{P}_T] = d/d\tau[\mathbf{P}_T]$
 Thus: $(d/d\tau)[\mathbf{P}_T] = (\partial/\partial \mathbf{X})[H_0]$

Relativistic Hamilton's Equations (4-Vector):
 $(d/d\tau)[\mathbf{X}] = (\partial/\partial \mathbf{P}_T)[H_0]$
 $(d/d\tau)[\mathbf{P}_T] = (\partial/\partial \mathbf{X})[H_0]$

$(d/d\tau)[\mathbf{X}] = \gamma(d/dt)[\mathbf{X}] = (\partial/\partial \mathbf{P}_T)[H_0] = (\partial/\partial \mathbf{P}_T)[(\mathbf{P}_T \cdot \mathbf{U})] = \mathbf{U}$
 $(d/d\tau)[\mathbf{P}_T] = \gamma(d/dt)[\mathbf{P}_T] = (\partial/\partial \mathbf{X})[H_0] = (\partial/\partial \mathbf{X})[(\mathbf{P}_T \cdot \mathbf{U})] = (\partial/\partial \mathbf{X})[\gamma(\mathbf{H} - \mathbf{p}_T \cdot \mathbf{u})]$

Taking just the spatial components:
 $\gamma(d/dt)[\mathbf{x}] = (-\partial/\partial \mathbf{p}_T)[H_0] = (-\partial/\partial \mathbf{p}_T)[H/\gamma]$ {hard}
 $\gamma(d/dt)[\mathbf{p}_T] = (-\partial/\partial \mathbf{x})[H_0] = (-\partial/\partial \mathbf{x})[H/\gamma]$ {easy because $(\partial/\partial \mathbf{x})[\gamma]=0$ }

$\gamma^2(d/dt)[\mathbf{p}_T] = (-\partial/\partial \mathbf{x})[H]$
 Take the Classical limit $\{\gamma \rightarrow 1\}$

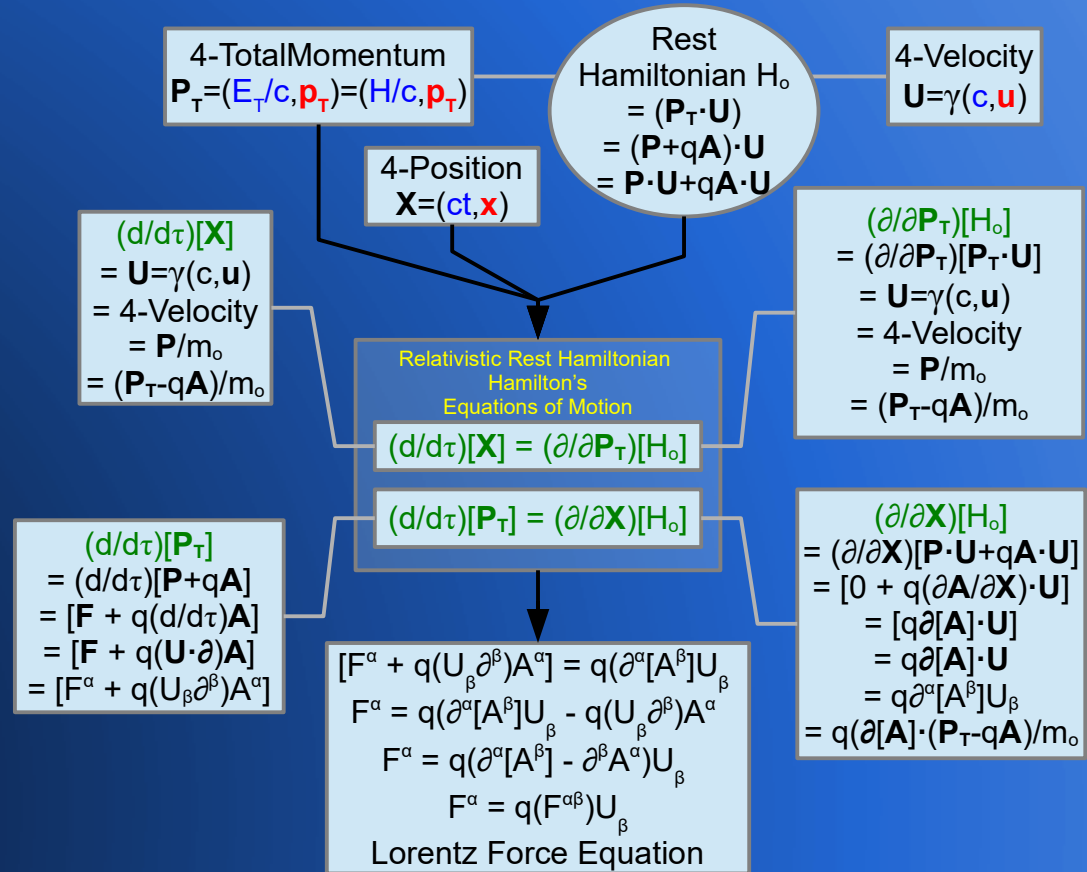
Classical Hamilton's Equations (3-vector):
 $(d/dt)[\mathbf{x}] = (+\partial/\partial \mathbf{p}_T)[H]$
 $(d/dt)[\mathbf{p}_T] = (-\partial/\partial \mathbf{x})[H]$

Sign-flip difference is interaction of $(-\partial/\partial \mathbf{p}_T)$ with $[1/\gamma]$

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SRQM Diagram: EM Lorentz Force Eqn

→ Force = - Grad[Potential]

A Tensor Study of Physical 4-Vectors

Lorentz EM Force Equation:

$$F^\alpha = q(F^{\alpha\beta})U_\beta$$

$$F^\alpha = q(\partial^\alpha A^\beta - \partial^\beta A^\alpha)U_\beta$$

Examine just the spatial components of 4-Force F :

$$F^i = q(\partial^i A^0 - \partial^0 A^i)U_0 + q(\partial^i A^j - \partial^j A^i)U_j$$

$$\mathbf{f} = q(-\nabla[\phi/c] - (\partial^t/c)\mathbf{a})(\gamma c) + q(-\nabla[\mathbf{a}\cdot\mathbf{u}] - \mathbf{u}\cdot\nabla[\mathbf{a}])\gamma$$

$$\mathbf{f} = q(-\nabla[\phi/c] - (\partial^t/c)\mathbf{a})(c) + q(\mathbf{u}\cdot\nabla[\mathbf{a}] - \nabla[\mathbf{a}\cdot\mathbf{u}])$$

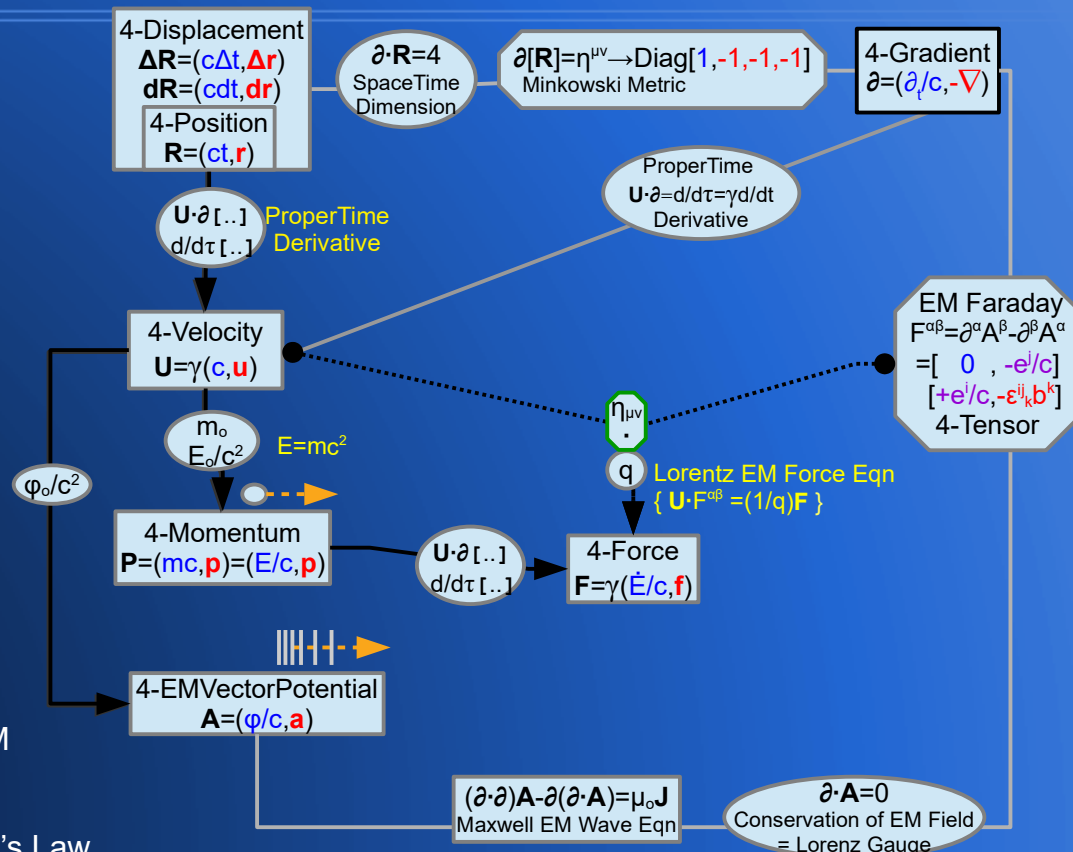
$$\mathbf{f} = q(-\nabla[\phi] - \partial^t\mathbf{a} + \mathbf{u}\cdot\nabla[\mathbf{a}] - \nabla[\mathbf{a}\cdot\mathbf{u}])$$

Take the limit of $\{ |\nabla[\phi]| \gg |\partial^t\mathbf{a}| + |\mathbf{u} \times \mathbf{b}| \}$
 $\mathbf{f} \sim q(-\nabla[\phi]) = -\nabla[q\phi] = -\nabla[U]$

The Classical Force = -Grad[Potential]
 when $\{ |\nabla[\phi]| \gg |\partial^t\mathbf{a}| + |\mathbf{u} \times \mathbf{b}| \}$ or when $\{\mathbf{a} = \mathbf{0}\}$

The majority of non-gravity, non-nuclear potentials dealt with in CM are those mediated by the EM potential.

ex. Spring Potential $\{ U = kx^2/2 \}$, then $\{ \mathbf{f} = -\nabla[kx^2/2] = -kx \}$ Hooke's Law



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(0,0)-Tensor S
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SRQM: The Speed-of-Light (c)

c² Invariant Relations (part 1)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The Speed-of-Light (c) is THE connection between Time and Space: $dR = (cdt, dr)$

This physical constant appears in several seemingly unrelated places. You don't notice these cool relations when you set $c \rightarrow 1$. Also notice that the set of all these relations definitely rules out a variable speed-of-light. (c) is an Invariant Lorentz Scalar constant.

$\mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2$ Speed of all things into the Future

$(E_0/m_0) = (\gamma E_0/\gamma m_0) = (E/m) = c^2$ Mass is concentrated Energy, $E = mc^2$

$|u * v_{phase}| = |v_{group} * v_{phase}| = c^2$ Particle-Wave "Duality" Correlation

$\lambda^2(\omega^2 - \omega_0^2) = \lambda^2(f^2 - f_0^2) = c^2$ Wavelength-Frequency Relation: $\lambda f = c$ for photons

$(1/\epsilon_0 \mu_0) = c^2$ Electric (ϵ_0) and Magnetic (μ_0) EM Field Constants

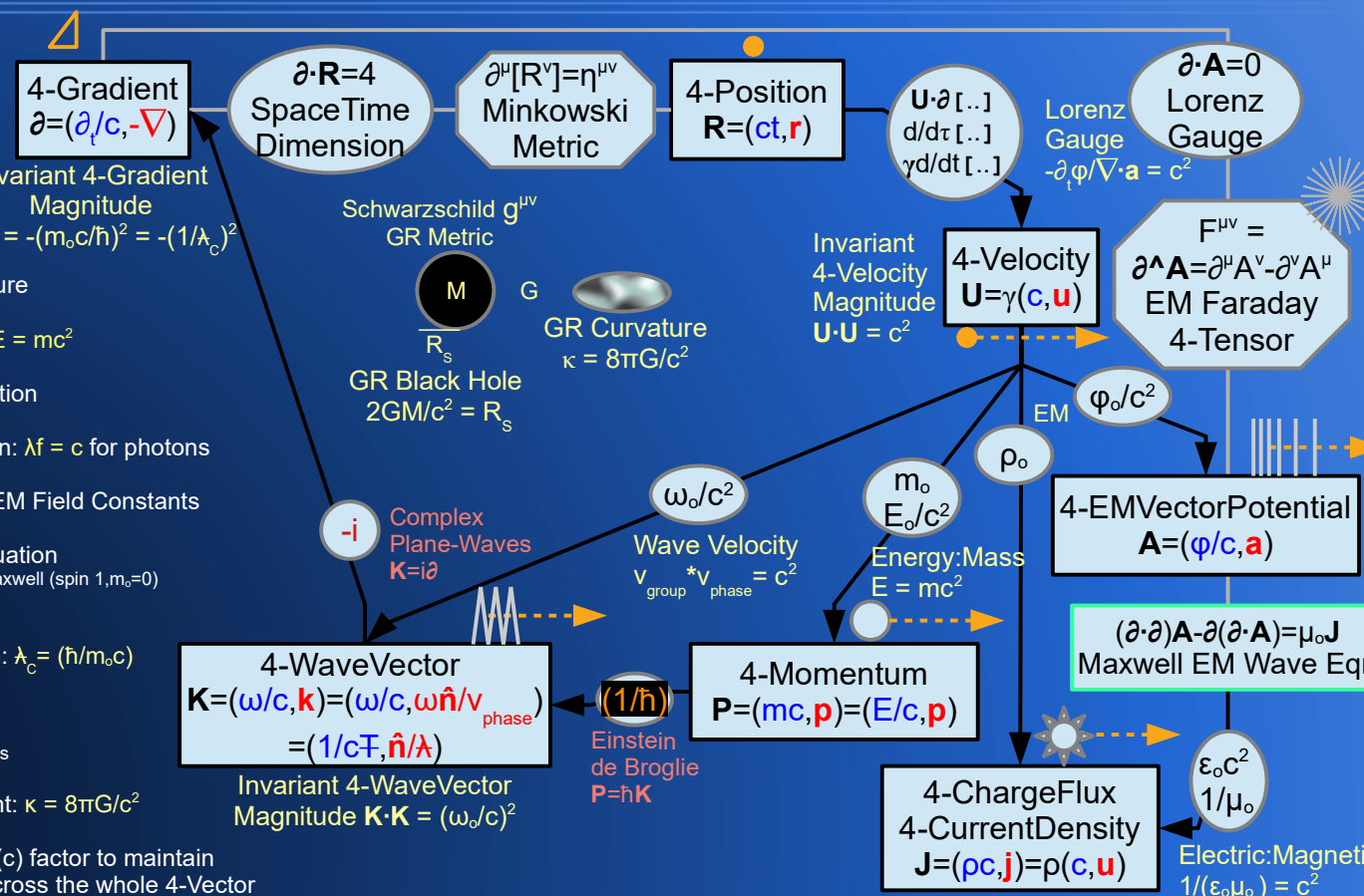
Relativistic Quantum Wave Equation
Klein-Gordon (spin 0), Proca (spin 1), Maxwell (spin 1, $m_0=0$)
Factors to Dirac (spin 1/2)
Classical-limit ($|v| \ll c$) to Schrödinger

$(\hbar/\lambda_c m_0)^2 = c^2$ Reduced Compton Wavelength: $\lambda_c = (\hbar/m_0 c)$

GR Black Hole Equation
 R_s = Schwarzschild Radius
G = GR Gravitational Const, M = BH Mass

$8\pi G/\kappa = c^2$ GR Einstein Curvature Constant: $\kappa = 8\pi G/c^2$

$(c^{\pm 1} * \text{scalar}, \mathbf{3}\text{-vector}) = \mathbf{4}\text{-Vector}$ Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector



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SRQM: The Speed-of-Light (c)

c² Invariant Relations (part 2)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

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$2GM/R_s = c^2$

$8\pi G/\kappa = c^2$ GR Einstein Curvature Constant (mass density form): $\kappa = 8\pi G/c^2$

$(c^{\pm 1} * \text{scalar, 3-vector}) = \text{4-Vector}$
Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector

$\partial^\mu [R^\nu] = \eta^{\mu\nu}$
Minkowski Metric

$\eta_{\mu\nu}$
4-Vector Scalar Product

EM
 $u_{\text{photon}}^2 = u_{\text{EMwave}}^2$

Electric:Magnetic
 $1/(\epsilon_o \mu_o) = c^2$

Energy:Mass
 $E = mc^2$
 $E_o/m_o = \hbar\omega_o/m_o = (\hbar/\lambda_c m_o)^2$

Invariant 4-Velocity Magnitude
 $U \cdot U = c^2$

$(\mathbf{e} \cdot \mathbf{b})^2 / \text{Det}[F^{\mu\nu}]$

$-\partial_t \phi / \nabla \cdot \mathbf{a}$
in Lorenz Gauge

$|u * v_{\text{phase}}| = |v_{\text{group}} * v_{\text{phase}}|$

Waves
 $\lambda^2(\omega^2 - \omega_o^2) = \lambda_c^2 \omega_o^2 = \lambda^2 \omega^2$ (for photon)

ProperTime Differential
 $R \cdot R / \tau^2 = dR \cdot dR / d\tau^2$

$-S_{\text{action, free}} / (m_o [d\tau])$

GR
 $8\pi G/\kappa$

$2GM/R_s$

$U \cdot U$

$P \cdot P / m_o^2$

$E_o^2 / P \cdot P$

$\omega_o^2 / K \cdot K$

$(\hbar/m_o)^2 K \cdot K$

$-(\hbar/m_o)^2 \partial \cdot \partial$

SRQM

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SRQM 4-Vector Study: 4-ThermalVector Relativistic Thermodynamics

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

The 4-ThermalVector is used in Relativistic Thermodynamics.

My prime motivation for the form of this 4-Vector is that the probability distributions calculated by statistical mechanics ought to be covariant functions since they are based on counting arguments.

$F(\text{state}) \sim e^{-E/k_B T} = e^{-\beta E}$, with this $\beta = 1/k_B T$, (not v/c)

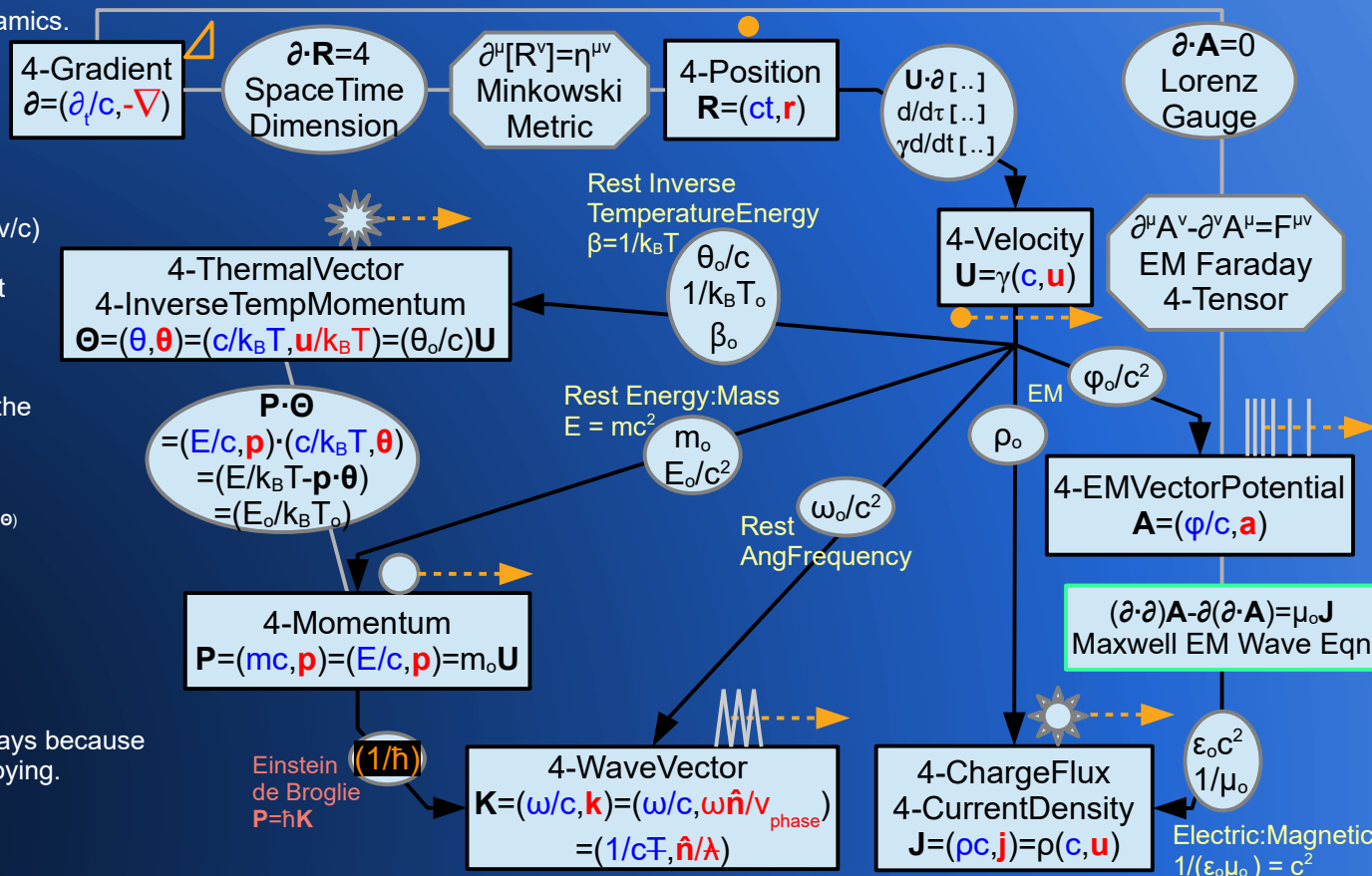
A covariant way to get this is the Lorentz Scalar Product of the 4-Momentum \mathbf{P} with the 4-ThermalVector Θ .
 $F(\text{state}) \sim e^{-\mathbf{P} \cdot \Theta} = e^{-(E_0/k_B T_0)}$

This also gets Boltzmann's constant (k_B) out there with the other Lorentz Scalars like (c) and (\hbar)

see (Relativistic) Maxwell-Jüttner distribution
 $f[\mathbf{P}] = N_0 / (2c(m_0 c)^d K_{[(d+1)/2]}[m_0 c \Theta_0])^* (m_0 c \Theta_0 / 2\pi)^{(d-1)/2} * e^{-(\mathbf{P} \cdot \Theta)}$

$f[\mathbf{P}] = N_0 / (2c(m_0 c)^3 K_{[2]}[m_0 c \Theta_0])^* (m_0 c \Theta_0 / 2\pi) * e^{-(\mathbf{P} \cdot \Theta)}$
 $f[\mathbf{P}] = (\Theta_0) N_0 / (4\pi c(m_0 c)^2 K_{[2]}[m_0 c \Theta_0]) * e^{-(\mathbf{P} \cdot \Theta)}$
 $f[\mathbf{P}] = c N_0 / (4\pi k_B T_0 (m_0 c)^2 K_{[2]}[m_0 c \Theta_0]) * e^{-(\mathbf{P} \cdot \Theta)}$
 $f[\mathbf{P}] = N_0 / (4\pi k_B T_0 m_0^2 c K_{[2]}[m_0 c^2 / k_B T_0]) * e^{-(\mathbf{P} \cdot \Theta)}$

It is possible to find this distribution written in multiple ways because many authors don't show constants, which is quite annoying. Show the damn constants people!
(k_B), (c), (\hbar) deserve at least that much respect.



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SR 4-Scalar
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Lorentz Scalar

Be careful not to confuse (unfortunate symbol clash):
Thermal $\beta = 1/k_B T$
Relativistic $\beta = v/c$
These are totally separate uses of (β)

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
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SRQM 4-Vector Study: 4-ThermalVector Unruh-Hawking Radiation

A Tensor Study
of Physical 4-Vectors

The 4-ThermalVector is used in Relativistic Thermodynamics.
It can be used in a partial derivation of Unruh-Hawking Radiation (up to a numerical constant).

Let a "Unruh-DeWitt thermal detector" be in the Momentarily-Comoving-Rest-Frame (MCRF) of a constant spatial acceleration (\mathbf{a}), in which $|\mathbf{u}| \rightarrow 0$, $\gamma \rightarrow 1$, $\gamma' \rightarrow 0$.

4-Acceleration_{MCRF} = $\mathbf{A}_{MCRF} = A_{MCRF}^\mu = (0, \mathbf{a})_{MCRF}$

Take the Lorentz Scalar Product with the 4-ThermalVector
 $\mathbf{A}_{MCRF} \cdot \Theta = (0, \mathbf{a})_{MCRF} \cdot (c/k_B T, \mathbf{u}/k_B T) = (-\mathbf{a} \cdot \mathbf{u}/k_B T) = \text{Lorentz Scalar Invariant}$

The (\mathbf{u}) here is part of the 4-ThermalVector: the 3-velocity of the thermal radiation. (not from \mathbf{A}_{MCRF})
Let the thermal radiation be photonic:EM in nature, so $|\mathbf{u}| = c$, and in a direction opposing the acceleration of the "thermal detector", which removes the minus sign.

$(ac/k_B T) = \text{Invariant}$

Use Dimensional Analysis to find appropriate Lorentz Scalar Invariant with same units:
[Invariant Units] = $[m/s^2] \cdot [m/s] / [kg \cdot m^2/s^2] = [1/kg \cdot s] \sim c^2/\hbar$

$(ac/k_B T) = \text{Invariant} \sim c^2/\hbar$

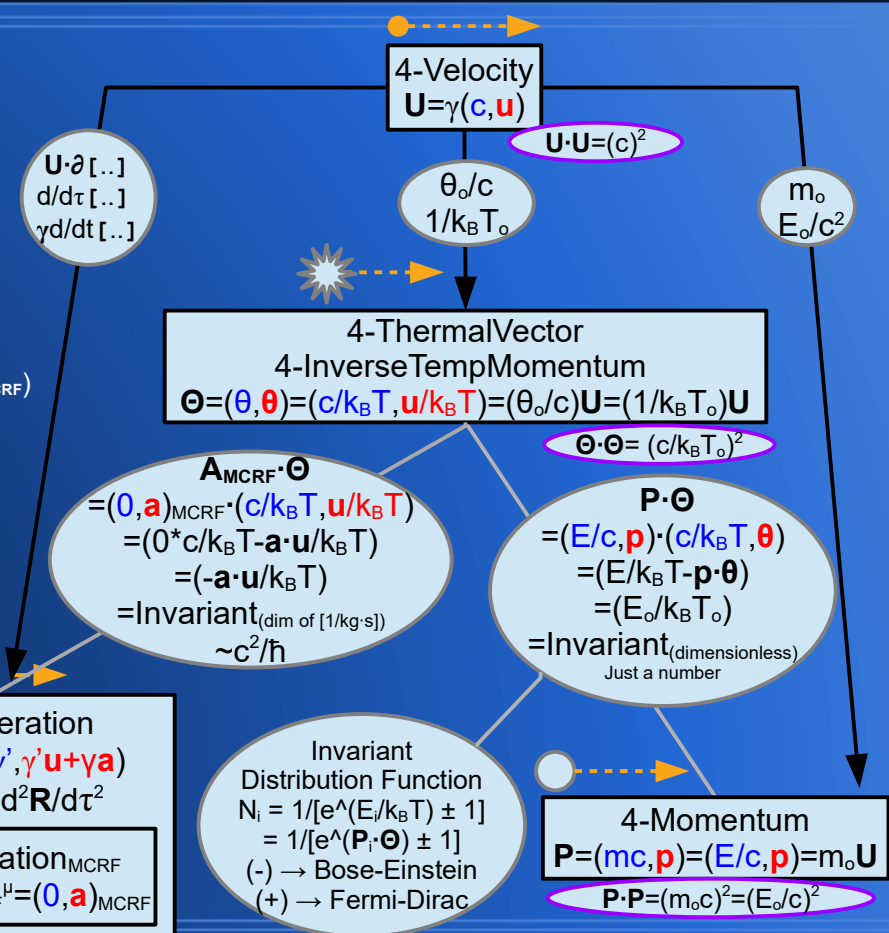
Temperature $T \sim \hbar a/k_B c$, {from EM radiation, only from the dir. of acceleration}

Further methods give the constant of proportionality $(1/2\pi)$:

$T_{Unruh} = \hbar a/2\pi k_B c$ {due to constant Minkowski-hyperbolic acceleration}

$T_{Hawking} = \hbar g/2\pi k_B c$ {due to gravitational acceleration $a=g$ }

$T_{SR} = -\hbar(\mathbf{a} \cdot \mathbf{u})/2\pi k_B c^2$ {correct version from 4-Vector derivation $\mathbf{A}_{MCRF} \cdot \Theta = 2\pi c^2/\hbar$ }



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Lorentz Scalar

Note that the temperature here is relativistically direction-specific, unlike in the classical use of temperature.

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SRQM 4-Vector Study:

4-EntropyFlux

Relativistic Thermodynamics

A Tensor Study of Physical 4-Vectors

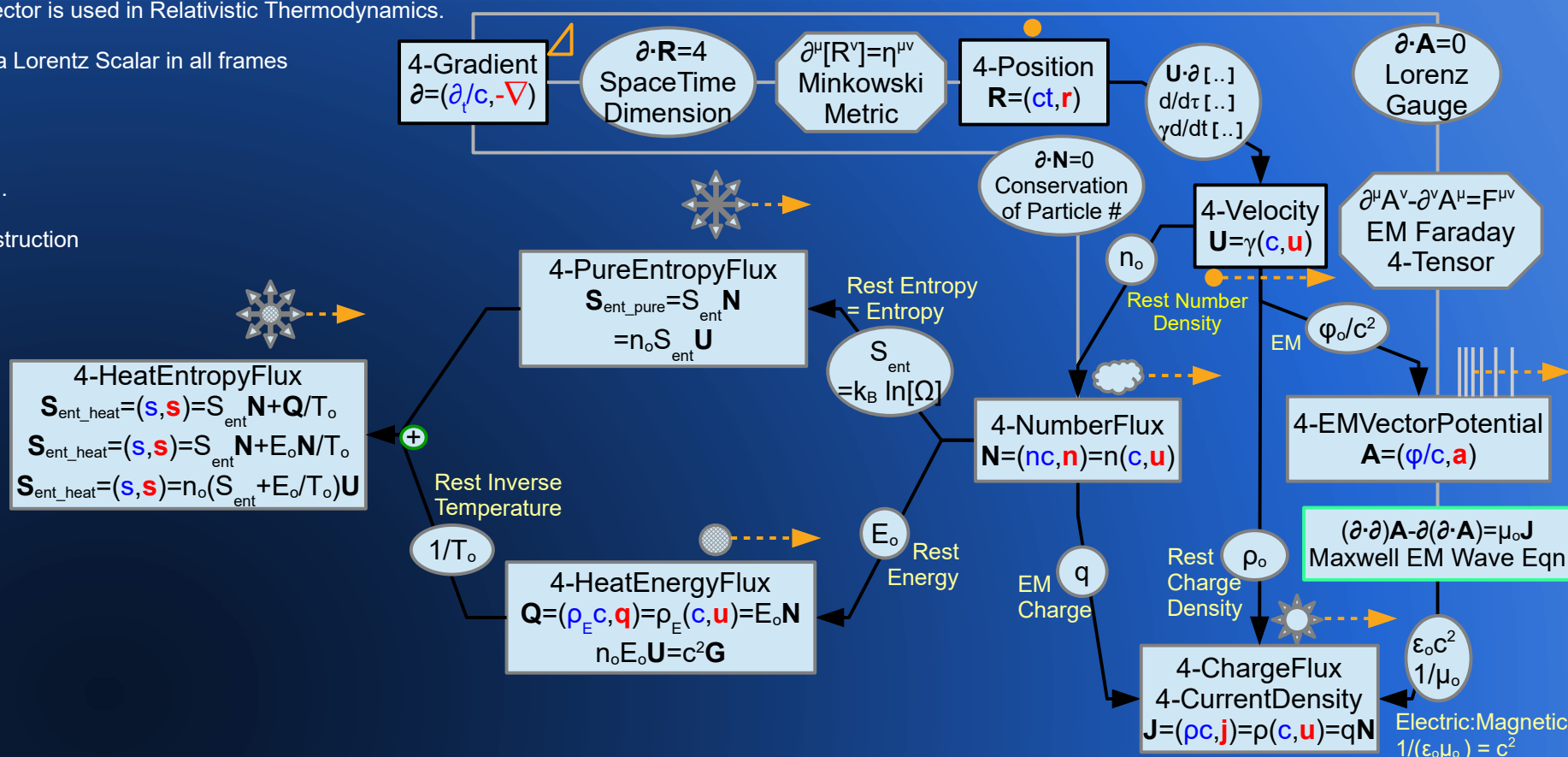
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The 4-EntropyVector is used in Relativistic Thermodynamics.

Pure Entropy is a Lorentz Scalar in all frames

not finished yet...

Page under construction



SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor $T^{\mu\nu}$ or $T_{\mu\nu}$ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$	SR 4-Scalar (0,0)-Tensor S Lorentz Scalar
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SRQM Interpretation:

**** Transition to QM ****

Up to this point, we have basically been exploring the SR aspects of 4-Vectors.

It is now time to show how RQM and QM fit into the works...

This is SRQM, [SR → QM]

RQM & QM are derivable from SR

SRQM: A treatise by John B. Wilson (SciRealm@aol.com)

SRQM Basic Idea (part 1)

SR → Relativistic Wave Eqn

The basic idea is to show that Special Relativity plus a few empirical facts lead to Relativistic Wave Equations, and thus RQM, without using any assumptions or axioms from Quantum Mechanics.

Start only with the concepts of SR, no concepts from QM

(1) SR provides the ideas of Invariant Intervals and (c) as a Physical Constant, as well as: Poincaré Invariance, Minkowski 4D SpaceTime, ProperTime, and Physical SR 4-Vectors

Note empirical facts which can relate the SR 4-Vectors from the following:

(2a) Elementary matter particles each have RestMass, (m_0), which can be measured by experiment: eg. collision, cyclotrons, Compton Scattering, etc.

(2b) There is a constant, (\hbar), which can be measured by classical experiment – eg. the Photoelectric Effect, the inverse Photoelectric Effect, LED's=Injection Electroluminescence, Duane-Hunt Law in Bremsstrahlung, the Watt/Kibble-Balance, etc. All known particles obey this constant.

(2c) The use of complex numbers (i) and differential operators { ∂_t and $\nabla = (\partial_x, \partial_y, \partial_z)$ } in wave-type equations comes from pure mathematics: not necessary to assume any QM Axioms

These few things are enough to derive the RQM Klein-Gordon equation, the most basic of the relativistic wave equations. Taking the low-velocity limit { $|\mathbf{v}| \ll c$ } (a standard SR technique) leads to the Schrödinger Equation.

SRQM Basic Idea (part 2)

Klein-Gordon RWE implies QM

If one has a Relativistic Wave Equation, such as the Klein-Gordon equation, then one has RQM, and thence QM via the low-velocity limit $\{ |\mathbf{v}| \ll c \}$.

The physical and mathematical properties of QM, usually regarded as axiomatic, are inherent in the Klein-Gordon RWE itself.

QM Principles emerge not from $\{ \text{QM Axioms} + \text{SR} \rightarrow \text{RQM} \}$,
but from $\{ \text{SR} + \text{Empirical Facts} \rightarrow \text{RQM} \}$.

The result is a paradigm shift from the idea of $\{ \text{SR and QM as separate theories} \}$
to $\{ \text{QM derived from SR} \}$ – leading to a new interpretation of QM:
The SRQM or [SR→QM] Interpretation.

GR → (low-mass limit = $\{ \text{curvature} \sim 0 \}$ limit) → SR
 SR → (+ a few empirical facts) → RQM
 RQM → (low-velocity limit $\{ |\mathbf{v}| \ll c \}$) → QM

The results of this analysis will be facilitated by the use of SR 4-Vectors

SRQM 4-Vector Path to QM

SR 4-Vector	Definition Component Notation	Unites
4-Position	$\mathbf{R} = R^\mu = (ct, \mathbf{r})$	Time, Space <i>-when & where</i>
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c, \mathbf{u})$	Lorentz Gamma * (c, Velocity) <i>-nothing faster than c</i>
4-Momentum	$\mathbf{P} = P^\mu = (E/c, \mathbf{p}) = (mc, \mathbf{p})$	Mass:Energy, Momentum <i>-used in 4-Momenta Conservation</i> $\Sigma \mathbf{P}_{\text{final}} = \Sigma \mathbf{P}_{\text{initial}}$
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c, \mathbf{k}) = (\omega/c, \omega \hat{\mathbf{n}}/v_{\text{phase}})$	Ang. Frequency, WaveNumber <i>-used in Relativistic Doppler Shift</i> $\omega_{\text{obs}} = \omega_{\text{emit}} / [\gamma(1 - \beta \cos[\theta])]$, $k = \omega/c$ for photons
4-Gradient	$\partial = \partial^\mu = (\partial_t/c, -\nabla)$ $= (\partial_t/c, -\partial_x, -\partial_y, -\partial_z)$ $= (\partial/\partial ct, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$	Temporal Partial, Spatial Partial <i>-used in SR Continuity Eqns., ProperTime</i> <i>-eg. $\partial \cdot \mathbf{A} = 0$ means \mathbf{A} is conserved</i>

All of these are standard SR 4-Vectors, which can be found and used in a totally relativistic context, with no mention or need of QM.

I want to emphasize that these objects are ALL relativistic in origin.

SRQM 4-Vector Invariants

SR 4-Vector	Lorentz Invariant	What it means in SR...
4-Position	$\mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct_0)^2 = (c\tau)^2$	SR Invariant Interval
4-Velocity	$\mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2$	Events move into future at magnitude c
4-Momentum	$\mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2$	Einstein Mass:Energy Relation
4-WaveVector	$\mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_0/c)^2$	Dispersion Invariance Relation
4-Gradient	$\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = (\partial_t/c)^2$	The d'Alembert Operator

All 4-Vectors have invariant magnitudes, found by taking the scalar product of the 4-Vector with itself. Quite often a simple expression can be found by examining the case when the spatial part is zero. This is usually found when the 3-velocity is zero. The temporal part is then specified by its “rest” value.

For example: $\mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2 = (m_0 c)^2$

$E = \text{Sqrt}[(E_0)^2 + \mathbf{p} \cdot \mathbf{p} c^2]$, from above relation

$E = \gamma E_0$, using $\{\gamma = 1/\text{Sqrt}[1-\beta^2] = \text{Sqrt}[1+\gamma^2\beta^2]\}$ and $\{\beta=v/c\}$

meaning the relativistic energy E is equal to the relative gamma factor γ * the rest energy E_0 .

SR + A few empirical facts: SRQM Overview

SR 4-Vector	Empirical Fact	SI Dimensional Units
4-Position $\mathbf{R} = (ct, \mathbf{r})$; alt. $\mathbf{X} = (ct, \mathbf{x})$	$\mathbf{R} = \langle \text{Event} \rangle$; alt. \mathbf{X}	[m]
4-Velocity $\mathbf{U} = \gamma(c, \mathbf{u})$	$\mathbf{U} = d\mathbf{R}/d\tau$	[m/s]
4-Momentum $\mathbf{P} = (E/c, \mathbf{p}) = (mc, \mathbf{p})$	$\mathbf{P} = m_0\mathbf{U}$	[kg·m/s]
4-WaveVector $\mathbf{K} = (\omega/c, \mathbf{k})$	$\mathbf{K} = \mathbf{P}/\hbar$	[{rad}/m]
4-Gradient $\partial = (\partial_t/c, -\nabla)$	$\partial = -i\mathbf{K}$	[1/m]

The Axioms of SR, which are actually GR limiting-cases, lead us to the use of Minkowski Space and Physical 4-Vectors, which are elements of Minkowski Space (4D SpaceTime).

Empirical Observation leads us to the transformation relations between the components of these SR 4-Vectors, and to the chain of relations between the 4-Vectors themselves

These relations all turn out to be Lorentz Invariant Constants, whose values are measured empirically.

The combination of these SR objects and their relations is enough to derive RQM.

SRQM:

SR→QM Interpretation Simplified

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson
SciRealm@aol.com
<http://scirealm.org/SRQM.pdf>

SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR,
although technically SR is itself the low-curvature limiting-case of GR

{c,τ,m₀,ħ,i}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors:

Related by these SR Lorentz Invariants

4-Position	$\mathbf{R} = (ct, \mathbf{r})$	= <Event>	$(\mathbf{R} \cdot \mathbf{R}) = (c\tau)^2$
4-Velocity	$\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$	= $(\mathbf{U} \cdot \partial)\mathbf{R} = (d/d\tau)\mathbf{R} = d\mathbf{R}/d\tau$	$(\mathbf{U} \cdot \mathbf{U}) = (c)^2$
4-Momentum	$\mathbf{P} = (E/c, \mathbf{p})$	= $m_0\mathbf{U}$	$(\mathbf{P} \cdot \mathbf{P}) = (m_0c)^2$
4-WaveVector	$\mathbf{K} = (\omega/c, \mathbf{k})$	= \mathbf{P}/\hbar	$(\mathbf{K} \cdot \mathbf{K}) = (m_0c/\hbar)^2$
4-Gradient	$\partial = (\partial_t/c, -\nabla)$	= $-i\mathbf{K}$	$(\partial \cdot \partial) = -(m_0c/\hbar)^2 = KG \text{ Eqn} \rightarrow RQM \rightarrow QM$

$|v| \ll c$

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn,
and thence to QM via the low-velocity limit { $|v| \ll c$ }, giving the Schrödinger Eqn.

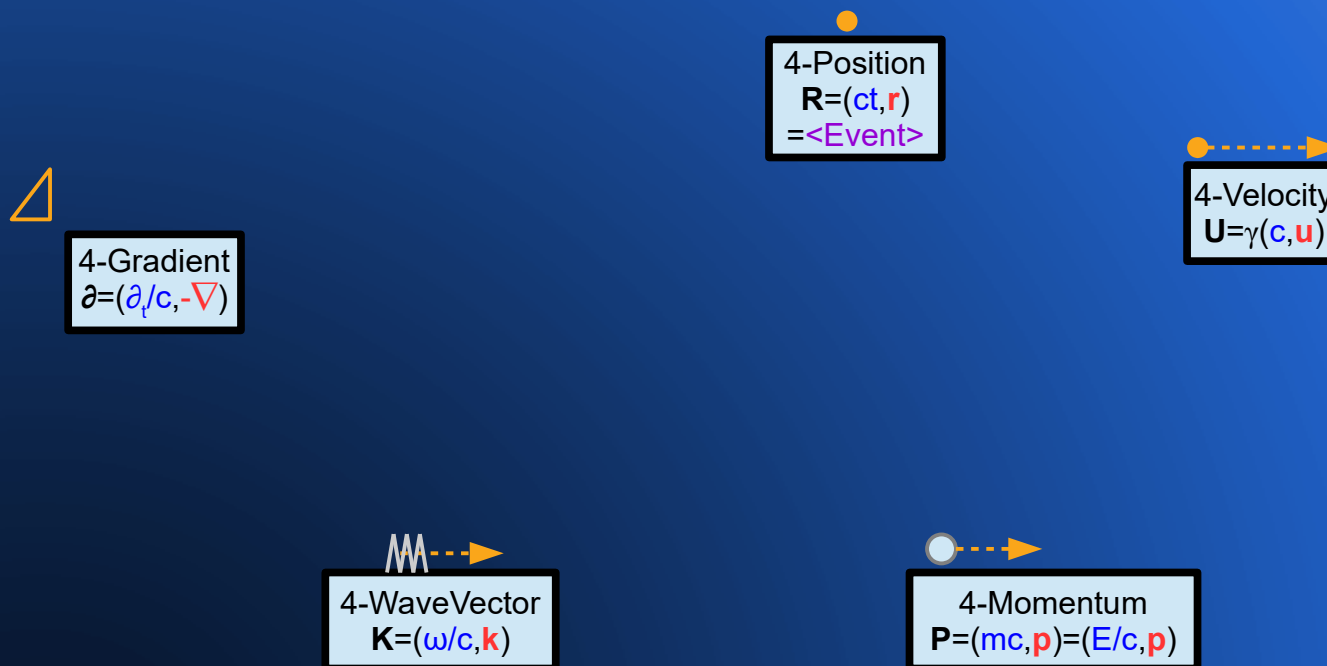
The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

SRQM Diagram: RoadMap of SR (4-Vectors)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or T_μ^ν
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

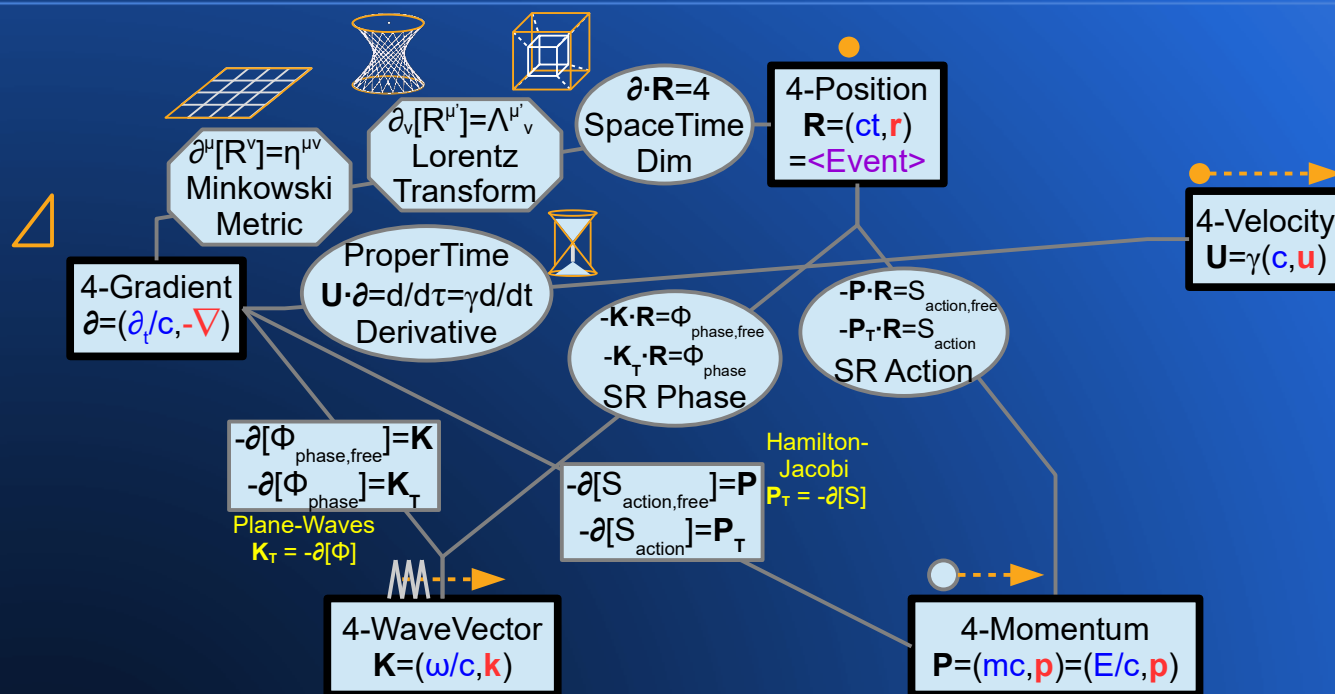
$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$$

SRQM Diagram: RoadMap of SR (Connections)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
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(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$$

SRQM Diagram: RoadMap of SR (Free Particle)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

4-Gradient=**Alteration** of SR <Events>

SR SpaceTime Dimension=4

SR SpaceTime Metric

SR Lorentz Transforms

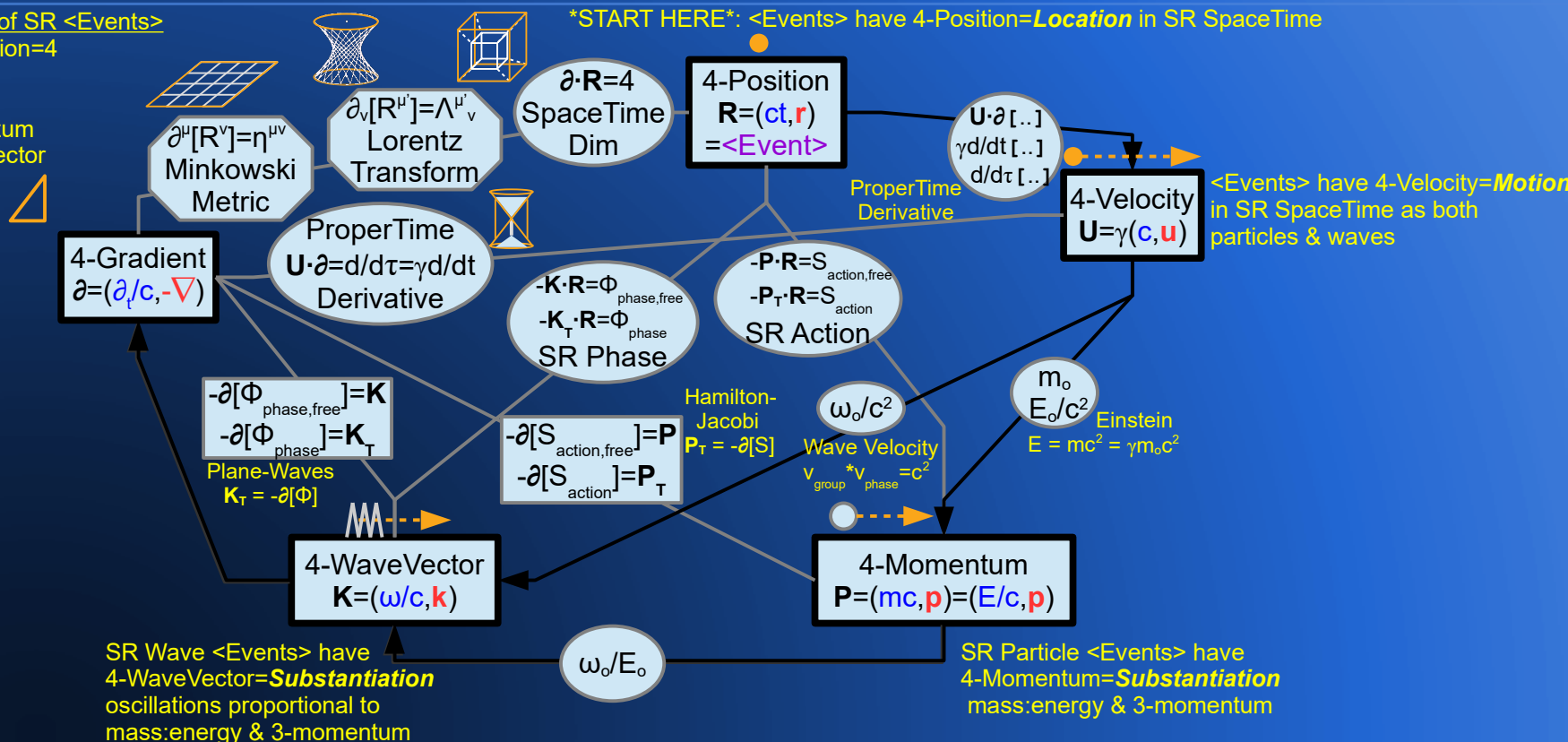
SR Action → 4-Momentum

SR Phase → 4-WaveVector

SR Proper Time

SR & QM Waves

START HERE: <Events> have 4-Position=**Location** in SR SpaceTime



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

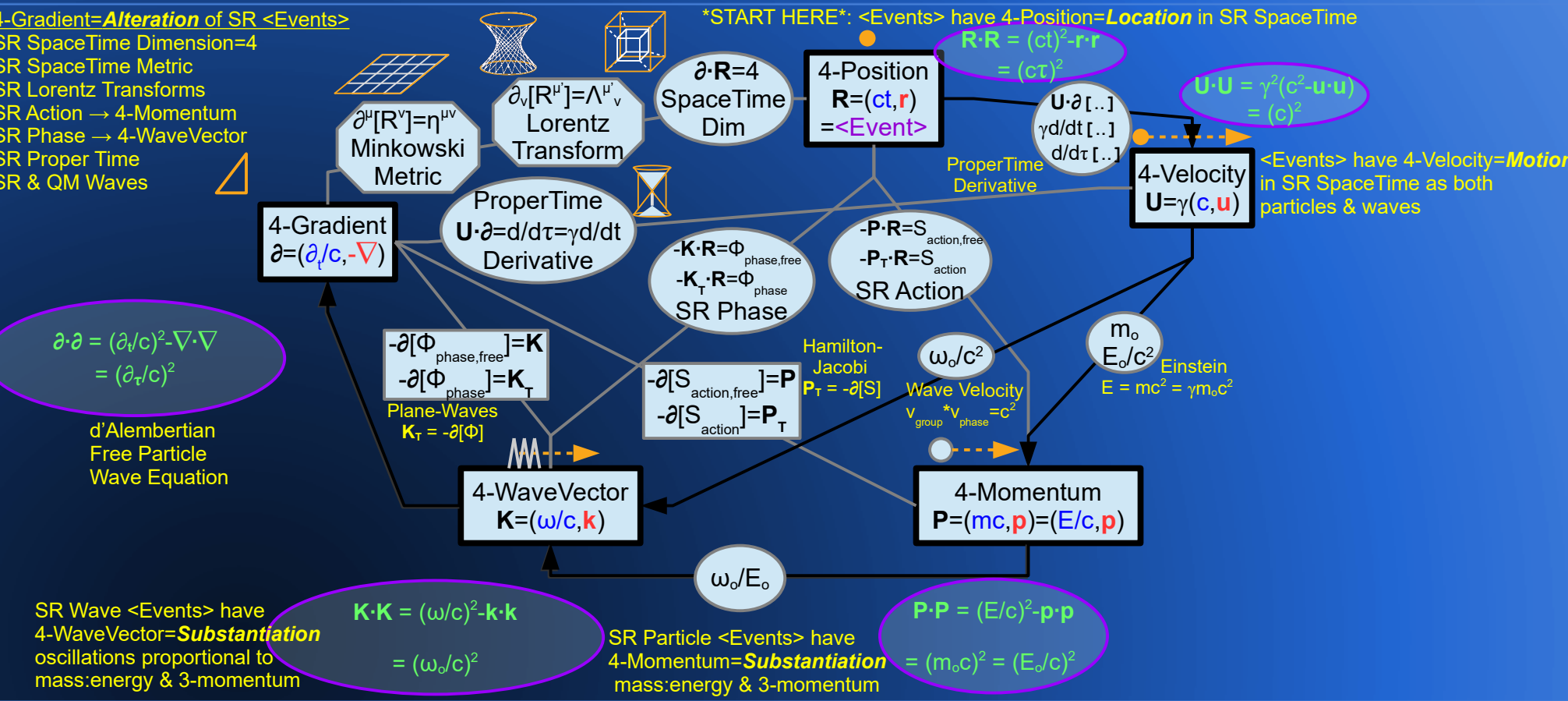
$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

SRQM Diagram: RoadMap of SR (Free Particle) with Magnitudes

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John B. Wilson

A Tensor Study of Physical 4-Vectors

- 4-Gradient = **Alteration** of SR <Events>
- SR SpaceTime Dimension = 4
- SR SpaceTime Metric
- SR Lorentz Transforms
- SR Action → 4-Momentum
- SR Phase → 4-WaveVector
- SR Proper Time
- SR & QM Waves



SR Wave <Events> have 4-WaveVector = **Substantiation** oscillations proportional to mass:energy & 3-momentum

SR Particle <Events> have 4-Momentum = **Substantiation** mass:energy & 3-momentum

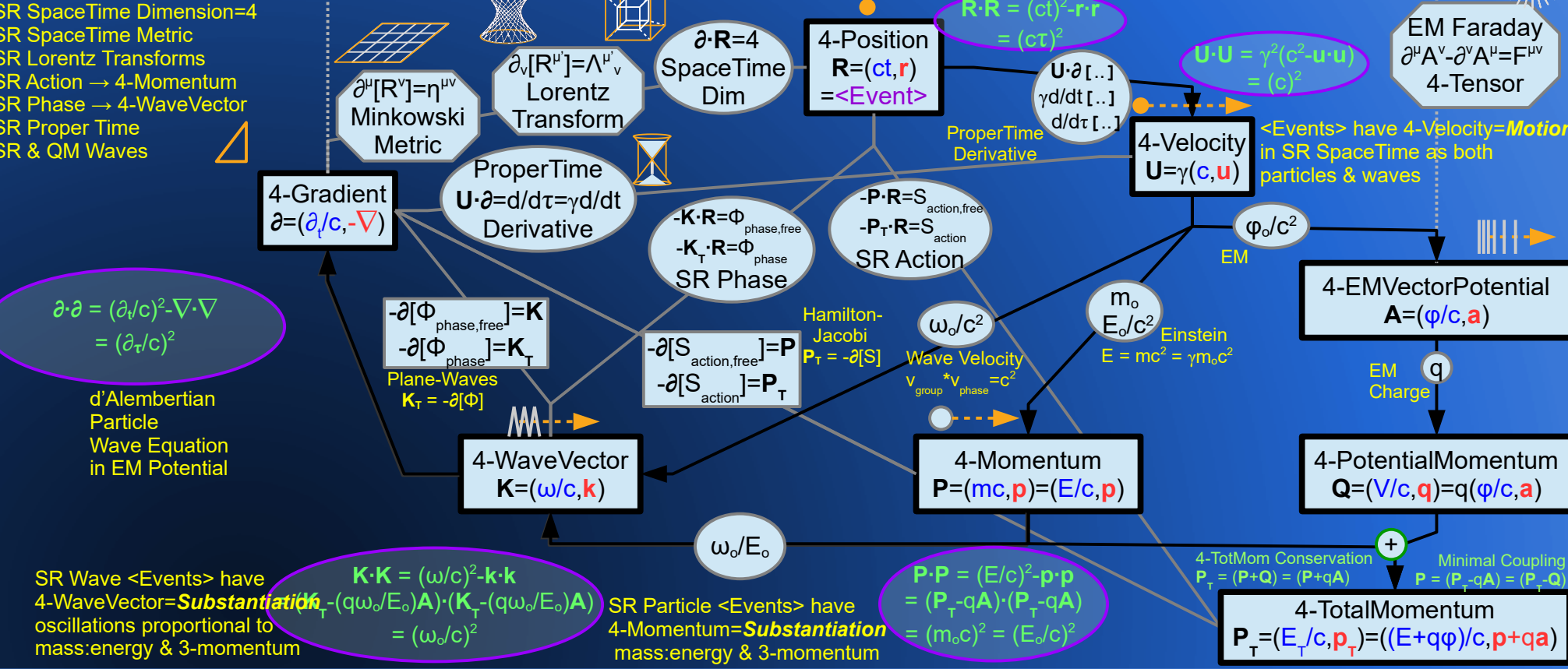
SRQM Diagram: RoadMap of SR (EM Potential)

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

4-Gradient=**Alteration** of SR <Events>
 SR SpaceTime Dimension=4
 SR SpaceTime Metric
 SR Lorentz Transforms
 SR Action → 4-Momentum
 SR Phase → 4-WaveVector
 SR Proper Time
 SR & QM Waves

START HERE: <Events> have 4-Position=**Location** in SR SpaceTime



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
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 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

$Trace[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$
 $V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0/c)^2 = \text{Lorentz Scalar}$

SRQM Diagram:

Special Relativity → Quantum Mechanics RoadMap of SR→QM (EM Potential)

A Tensor Study of Physical 4-Vectors

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John B. Wilson

4-Gradient=**Alteration** of SR <Events>

- SR SpaceTime Dimension=4
- SR SpaceTime Metric
- SR Lorentz Transforms
- SR Action → 4-Momentum
- SR Phase → 4-WaveVector
- SR Proper Time
- SR & QM Waves

SR → RQM Klein-Gordon
Relativistic Quantum
Particle in EM Potential
d'Alembertian Wave Equation

$$\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = (\partial_t + (iq/\hbar)\mathbf{A}) \cdot (\partial_t + (iq/\hbar)\mathbf{A}) = -(\omega_0/c)^2 = -(m_0c/\hbar)^2 = (\partial_t/c)^2$$

Limit: $\{ |v| \ll c \}$
 $(i\hbar\partial_{tT}) \sim [q\phi + (m_0c^2) + (i\hbar\nabla_T + q\mathbf{a})^2/(2m_0)]$
 $(i\hbar\partial_{tT}) \sim [V + (i\hbar\nabla_T + q\mathbf{a})^2/(2m_0)]$
 with potential $V = q\phi + (m_0c^2)$
 =Schrödinger QM Equation (EM potential)
****[SR → QM]****

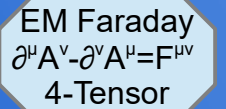
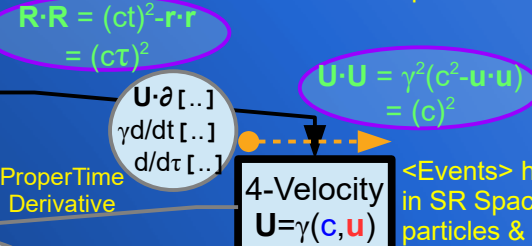
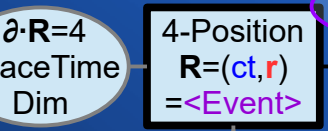
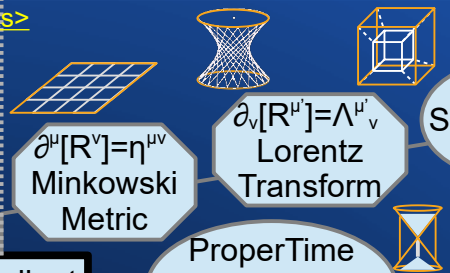
SR Wave <Events> have 4-WaveVector=**Substantiation** oscillations proportional to mass:energy & 3-momentum

$$\mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\mathbf{K}_T - (q/\hbar)\mathbf{A}) \cdot (\mathbf{K}_T - (q/\hbar)\mathbf{A}) = (m_0c/\hbar)^2 = (\omega_0/c)^2$$

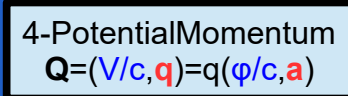
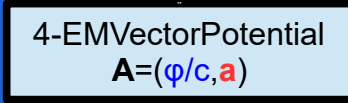
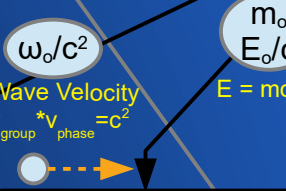
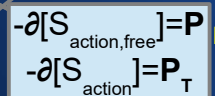
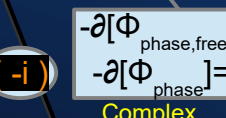
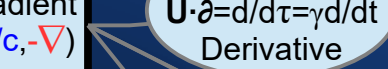
SR Particle <Events> have 4-Momentum=**Substantiation** mass:energy & 3-momentum

$$\mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (\mathbf{P}_T - q\mathbf{A}) \cdot (\mathbf{P}_T - q\mathbf{A}) = (m_0c)^2 = (E_0/c)^2$$

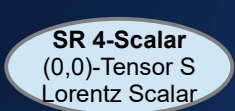
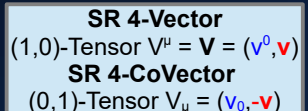
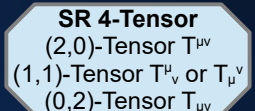
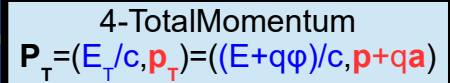
START HERE: <Events> have 4-Position=**Location** in SR SpaceTime



<Events> have 4-Velocity=**Motion** in SR SpaceTime as both particles & waves



4-TotMom Conservation $\mathbf{P}_T = (\mathbf{P} + \mathbf{Q}) = (\mathbf{P} + q\mathbf{A})$
Minimal Coupling $\mathbf{P} = (\mathbf{P}_T - q\mathbf{A}) = (\mathbf{P}_T - \mathbf{Q})$



Existing SR Rules
Quantum Principles

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM: The Empirical 4-Vector Facts

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

SR 4-Vector	Empirical Fact	Discoverer	Physics
4-Position	$\mathbf{R} = \langle \text{Event} \rangle$	Newton+ Einstein	[t & \mathbf{r}] Time & Space Dimensions [$\mathbf{R}=(c\mathbf{t},\mathbf{r})$] SpaceTime
4-Velocity	$\mathbf{U} = d\mathbf{R}/d\tau$	Newton Einstein	[$\mathbf{v}=d\mathbf{r}/dt$] Calculus of motion [$\mathbf{U}=\gamma(\mathbf{c},\mathbf{u})=d\mathbf{R}/d\tau$] Gamma & Proper Time
4-Momentum	$\mathbf{P} = m_0\mathbf{U}$	Newton Einstein	[$\mathbf{p}=m\mathbf{v}$] Classical Mechanics [$\mathbf{P}=(E/c,\mathbf{p})=m_0\mathbf{U}$] SR Mechanics
4-WaveVector	$\mathbf{K} = \mathbf{P}/\hbar$	Planck Einstein de Broglie	[h] Thermal Distribution [$E=h\nu=\hbar\omega$] Photoelectric Effect ($\hbar=h/2\pi$) [$\mathbf{p}=\hbar\mathbf{k}$] Matter Waves
4-Gradient	$\partial = -i\mathbf{K}$	Schrödinger	[$\omega=i\partial_t, \mathbf{k}=-i\nabla$] (SR) Wave Mechanics

- (1) The SR 4-Vectors and their components are related to each other via constants
- (2) We have not taken any 4-vector relation as axiomatic, the constants come from experiment.
- (3) c, τ, m_0, \hbar come from physical experiments, $(-i)$ comes from the general mathematics of waves

The SRQM 4-Vector Relations Explained

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

SR 4-Vector	Empirical Fact	What it means in SRQM...	Lorentz Invariant
4-Position $\mathbf{R} = (ct, \mathbf{r})$	$\mathbf{R} = \langle \text{Event} \rangle$	SpaceTime as Unified Concept	$c = \text{LightSpeed}$
4-Velocity $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$	$\mathbf{U} = d\mathbf{R}/d\tau$	Velocity is ProperTime Derivative	$\tau = t_0 = \text{ProperTime}$
4-Momentum $\mathbf{P} = (E/c, \mathbf{p})$	$\mathbf{P} = m_0\mathbf{U}$	Mass:Energy-Momentum Equivalence	$m_0 = \text{RestMass}$
4-WaveVector $\mathbf{K} = (\omega/c, \mathbf{k})$	$\mathbf{K} = \mathbf{P}/\hbar$	Wave-Particle Duality	$\hbar = \text{UniversalAction}$
4-Gradient $\partial = (\partial_t/c, -\nabla)$	$\partial = -i\mathbf{K}$	Unitary Evolution, Operator Formalism	$i = \text{ComplexSpace}$

Three old-paradigm QM Axioms:

Particle-Wave Duality $[(\mathbf{P})=\hbar(\mathbf{K})]$, Unitary Evolution $[\partial=(-i)\mathbf{K}]$, Operator Formalism $[(\partial)=-i\mathbf{K}]$ are actually just empirically-found constant relations between known SR 4-Vectors.

Note that these constants are in fact all Lorentz Scalar Invariants.

Minkowski Space and 4-Vectors also lead to idea of Lorentz Invariance. A Lorentz Invariant is a quantity that always has the same value, independent of the motion of inertial observers.

Lorentz Invariants can typically be derived using the scalar product relation.

$\mathbf{U} \cdot \mathbf{U} = c^2$, $\mathbf{U} \cdot \partial = d/d\tau$, $\mathbf{P} \cdot \mathbf{U} = m_0c^2$, etc.

A very important Lorentz invariant is the Proper Time τ , which is defined as the time displacement between two points on a worldline that is at rest wrt. an observer. It is used in the relations between 4-Position \mathbf{R} , 4-Velocity $\mathbf{U} = d\mathbf{R}/d\tau$, and 4-Acceleration $\mathbf{A} = d\mathbf{U}/d\tau$.

SRQM: The SR Path to RQM

Follow the Invariants...

SR 4-Vector	Lorentz Invariant	What it means in SRQM...
4-Position	$\mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (c\tau)^2$	SR Invariant Interval
4-Velocity	$\mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2$	Events move into future at magnitude c
4-Momentum	$\mathbf{P} \cdot \mathbf{P} = (m_0c)^2$	Einstein Mass:Energy Relation
4-WaveVector	$\mathbf{K} \cdot \mathbf{K} = (m_0c/\hbar)^2 = (\omega_0/c)^2$	Matter-Wave Dispersion Relation
4-Gradient	$\partial \cdot \partial = (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2$	The Klein-Gordon Equation → RQM!

$$\mathbf{U} = d\mathbf{R}/d\tau$$

Remember, everything after 4-Velocity was just a constant times the last 4-vector, and the Invariant Magnitude of the 4-Velocity is itself a constant

$$\mathbf{P} = m_0\mathbf{U}, \mathbf{K} = \mathbf{P}/\hbar, \partial = -i\mathbf{K}, \text{ so e.g. } \mathbf{P} \cdot \mathbf{P} = m_0\mathbf{U} \cdot m_0\mathbf{U} = m_0^2\mathbf{U} \cdot \mathbf{U} = (m_0c)^2$$

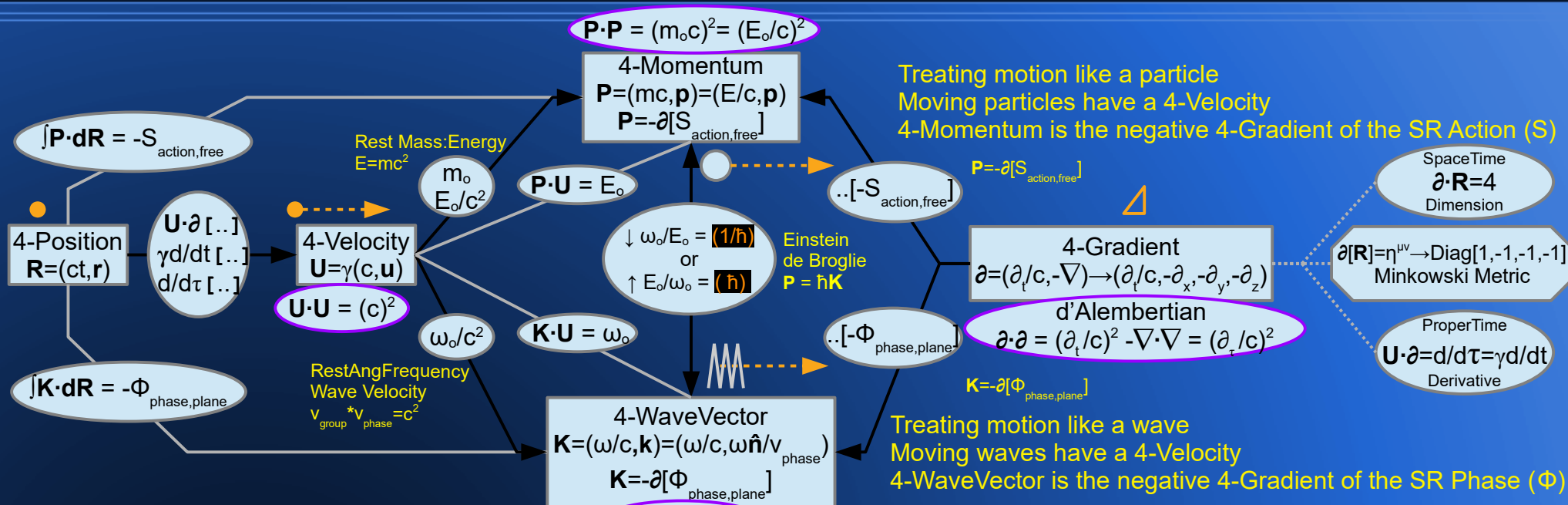
The last equation is the Klein-Gordon RQM Equation, which we have just derived without invoking any QM axioms, only SR plus a few empirical facts

SRQM: Some Basic 4-Vectors

4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, **Wave-Particle**

A Tensor Study of Physical 4-Vectors

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John B. Wilson



See **Hamilton-Jacobi Formulation of Mechanics** for info on the Lorentz Scalar Invariant SR Action.
 $\{ \mathbf{P} = (E/c, \mathbf{p}) = -\partial[\mathbf{S}] = (-\partial/c \partial t[\mathbf{S}], \nabla[\mathbf{S}]) \}$
 {temporal component} $E = -\partial/\partial t[\mathbf{S}] = -\partial_t[\mathbf{S}]$
 {spatial component} $\mathbf{p} = \nabla[\mathbf{S}]$
 Note This is the Action (\mathbf{S}_{action}) for a free particle.
 Generally Action is for the 4-TotalMomentum \mathbf{P}_T of a system.

See **SR Wave Definition** for info on the Lorentz Scalar Invariant SR WavePhase.
 $\{ \mathbf{K} = (\omega/c, \mathbf{k}) = -\partial[\Phi] = (-\partial/c \partial t[\Phi], \nabla[\Phi]) \}$
 {temporal component} $\omega = -\partial/\partial t[\Phi] = -\partial_t[\Phi]$
 {spatial component} $\mathbf{k} = \nabla[\Phi]$
 Note This is the Phase (Φ) for a single plane-wave.
 Generally WavePhase is for the 4-TotalWaveVector \mathbf{K}_T of a system.

SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or $T_{\mu\nu}$
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

Existing SR Rules
Quantum Principles

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM: Wave-Particle Diffraction/Interference Types

A Tensor Study of Physical 4-Vectors

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John B. Wilson

The 4-Vector Wave-Particle relation is inherent in all particle types: **Einstein-de Broglie $\mathbf{P} = (E/c, \mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c, \mathbf{k})$.**

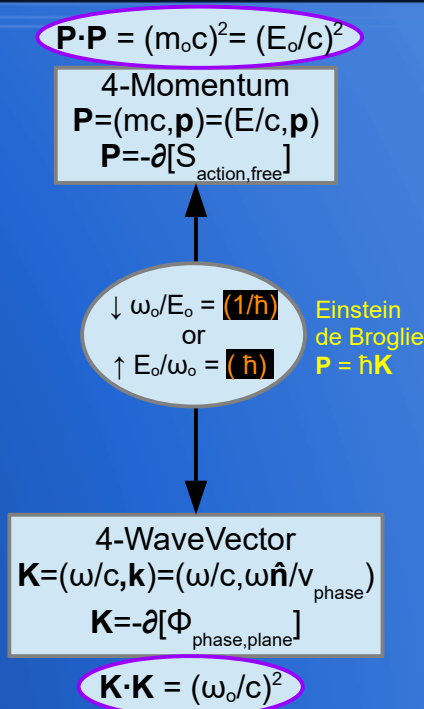
All waves can diffract: Water waves, gravitational waves, photonic waves of all frequencies, etc.
In all cases: experiments using single particles build the diffraction/interference pattern over the course many iterations.

Photon/light Diffraction: Photonic particles diffracted by matter particles.
Photons of any frequency encounter a “solid” object or grating.
Most often encountered are diffraction gratings and the famous double-slit experiment

Matter Diffraction: Matter particles diffracted by matter particles.
Electrons, neutrons, atoms, small molecules, buckyballs (fullerenes), macromolecules, etc. have been shown to diffract through crystals.
Crystals may be solid single pieces or in powder form.

Kapitsa-Dirac Diffraction: Matter particles diffracted by photonic standing waves.
Electrons, atoms, super-sonic atom beams have been diffracted from resonant standing waves of light.

Photonic-Photonic Diffraction?: Delbruck scattering
Light-by-light scattering/two-photon physics/gamma-gamma physics.
Normally, photons do not interact, but at high enough relative energy, virtual particles can form which allow interaction.



SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^μ_ν or T_μ^ν
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V}\cdot\mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

Hold on, aren't you getting the “ \hbar ” from a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-WaveVector	$\mathbf{K} = (\omega/c, \mathbf{k}) = (\omega/c, \omega\hbar/v_{\text{phase}}) = (\omega_0/c^2)\mathbf{U}$	Wave-Particle Duality

\hbar is actually an empirically measurable quantity, just like e or c . It can be measured classically from the photoelectric effect, the inverse photoelectric effect, from LED's (injection electroluminescence), from the Duane-Hunt Law in Bremsstrahlung, Electron Diffraction in crystals, the Watt/Kibble-Balance, etc.

For the LED experiment, one uses several different LED's, each with its own characteristic wavelength.

One then makes a chart of wavelength (λ) vs threshold voltage (V) needed to make each individual LED emit.

*One finds that: $\{\lambda = h*c/(eV)\}$, where e =ElectronCharge and c =LightSpeed. h is found by measuring the slope.*

Consider this as a blackbox where no assumption about QM is made. However, we know the SR relations $\{E = eV\}$, and $\{\lambda f = c\}$.

The data force one to conclude that $\{E = hf = \hbar\omega\}$.

Applying our 4-Vector knowledge, we recognize this as the temporal components of a 4-Vector relation. $(E/c, \dots) = \hbar(\omega/c, \dots)$

*Due to manifest tensor invariance, this means that 4-Momentum $\mathbf{P} = (E/c, \mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c, \mathbf{k}) = \hbar$ *4-WaveVector \mathbf{K} .*

The spatial component (due to De Broglie) follows naturally from the temporal component (due to Einstein) via to the nature of 4-Vector mathematics.

This is also derivable from pure SR 4-Vector (Tensor) arguments: $\mathbf{P} = m_0\mathbf{U} = (E_0/c^2)\mathbf{U}$ and $\mathbf{K} = (\omega_0/c^2)\mathbf{U}$

Since \mathbf{P} and \mathbf{K} are both Lorentz Scalar proportional to \mathbf{U} , then by the rules of tensor mathematics, \mathbf{P} must also be Lorentz Scalar proportional to \mathbf{K} .

i.e. Tensors obey certain mathematical structures:

Transitivity{if $a \sim b$ and $b \sim c$, then $a \sim c$ } & Euclideaness: {if $a \sim c$ and $b \sim c$, then $a \sim b$ } **Not to be confused with the Euclidean Metric**

This invariant proportional constant is empirically measured to be (\hbar) for each known particle type, massive ($m_0 > 0$) or massless ($m_0 = 0$):

$$\mathbf{P} = m_0\mathbf{U} = (E_0/c^2)\mathbf{U} = (E_0/c^2)/(\omega_0/c^2)\mathbf{K} = (E_0/\omega_0)\mathbf{K} = (\gamma E_0/\gamma\omega_0)\mathbf{K} = (E/\omega)\mathbf{K} = (\hbar)\mathbf{K}$$

Hold on, aren't you getting the “K” from a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-WaveVector	$\mathbf{K} = (\omega/c, \mathbf{k}) = (\omega/c, \omega \hat{\mathbf{n}}/v_{\text{phase}}) = (\omega_0/c^2)\mathbf{U}$	Wave-Particle Duality

\mathbf{K} is a standard SR 4-Vector, used in generating the SR formulae:

Relativistic Doppler Effect:

$$\omega_{\text{obs}} = \omega_{\text{emit}} / [\gamma(1 - \beta \cos[\theta])], \quad k = \omega/c \text{ for photons}$$

Relativistic Aberration Effect:

$$\cos[\theta_{\text{obs}}] = (\cos[\theta_{\text{emit}}] + |\beta|) / (1 + |\beta|\cos[\theta_{\text{emit}}])$$

The 4-WaveVector \mathbf{K} can be derived in terms of periodic motion, where families of surfaces move through space as time increases, or alternately, as families of hypersurfaces in SpaceTime, formed by all events passed by the wave surface. The 4-WaveVector is everywhere in the direction of propagation of the wave surfaces.

$$\mathbf{K} = -\partial[\Phi_{\text{phase}}]$$

From this structure, one obtains relativistic/wave optics without ever mentioning QM.

Hold on, aren't you getting the “-i” from a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-Gradient	$\partial = (\partial_t/c, -\nabla) = -i\mathbf{K}$	Unitary Evolution of States Operator Formalism

$[\partial = -i\mathbf{K}]$ gives the sub-equations $[\partial_t = -i\omega]$ and $[\nabla = i\mathbf{k}]$, and is certainly the main equation that relates QM and SR by allowing Operator Formalism. But, this is a basic equation regarding the general mathematics of plane-waves; not just quantum-waves, but anything that can be mathematically described by plane-waves and superpositions of plane-waves...

This includes purely SR waves, an example of which would be EM plane-waves (i.e. photons)...

$\psi(t, \mathbf{r}) = ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$: Standard mathematical plane-wave equation

$$\partial_t[\psi(t, \mathbf{r})] = \partial_t[ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}] = (-i\omega)[ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}] = (-i\omega)\psi(t, \mathbf{r}), \text{ or } [\partial_t = -i\omega]$$

$$\nabla[\psi(t, \mathbf{r})] = \nabla[ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}] = (i\mathbf{k})[ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}] = (i\mathbf{k})\psi(t, \mathbf{r}), \text{ or } [\nabla = i\mathbf{k}]$$

In the more economical SR notation:

$$\partial[\psi(\mathbf{R})] = \partial[ae^{i(-\mathbf{K}\cdot\mathbf{R})}] = (-i\mathbf{K})[ae^{i(-\mathbf{K}\cdot\mathbf{R})}] = (-i\mathbf{K})\psi(\mathbf{R}), \text{ or } [\partial = -i\mathbf{K}]$$

This one is more of a mathematical empirical fact, but regardless, it is not axiomatic. It can describe purely SR waves, again without any mention of QM.

Hold on, aren't you getting the “ ∂ ” from a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-Gradient	$\partial = (\partial_t/c, -\nabla) = -i\mathbf{K}$	4D Gradient Operator

$[\partial = (\partial_t/c, -\nabla)]$ is the SR 4-Vector Gradient Operator. It occurs in a purely relativistic context without ever mentioning QM.

$$\partial \cdot \mathbf{X} = (\partial_t/c, -\nabla) \cdot (ct, \mathbf{x}) = (\partial_t/c[ct] - (-\nabla \cdot \mathbf{x})) = (\partial_t[t] + \nabla \cdot \mathbf{x}) (1)+(3) = 4$$

The 4-Divergence of the 4-Position ($\partial \cdot \mathbf{X} = \partial^\mu \eta_{\mu\nu} X^\nu$) gives the dimensionality of SpaceTime.

$$\partial[\mathbf{X}] = (\partial_t/c, -\nabla)(ct, \mathbf{x}) = (\partial_t/c[ct], -\nabla[\mathbf{x}]) = \text{Diag}[1, -1] = \eta^{\mu\nu}$$

The 4-Gradient acting on the 4-Position ($\partial[\mathbf{X}] = \partial^\mu[X^\nu]$) gives the Minkowski Metric Tensor

$$\partial \cdot \mathbf{J} = (\partial_t/c, -\nabla) \cdot (\rho c, \mathbf{j}) = (\partial_t/c[\rho c] - (-\nabla \cdot \mathbf{j})) = (\partial_t[\rho] + \nabla \cdot \mathbf{j}) = 0$$

The 4-Divergence of the 4-CurrentDensity is equal to 0 for a conserved current. It can be rewritten as $(\partial_t[\rho] = -\nabla \cdot \mathbf{j})$, which means that the time change of ChargeDensity is balanced by the space change or divergence of CurrentDensity. It is a Continuity Equation, giving local conservation of ChargeDensity. It is related to Noether's Theorem.

Hold on, doesn't using “ ∂ ” in an Equation of Motion presume a QM Axiom?

SR 4-Vector	SR Empirical Fact	What it means...
4-(Position)Gradient	$\partial_R = \partial = (\partial_t/c, -\nabla) = -i\mathbf{K}$	4D Gradient Operator

Klein-Gordon Relativistic Quantum Wave Equation

$$\partial \cdot \partial[\Psi] = -(m_0 c/\hbar)^2[\Psi] = -(\omega_0/c)^2[\Psi]$$

Relativistic Euler-Lagrange Equations

$$\partial_R[L] = (d/d\tau)\partial_U[L]: \{\text{particle format}\}$$

$$\partial_{[\Phi]}[\mathcal{L}] = (\partial_R) \partial_{[\partial_R(\Phi)]}[\mathcal{L}]: \{\text{density format}\}$$

$[\partial = (\partial_t/c, -\nabla)]$ is the SR 4-Vector (Position)Gradient Operator.

It occurs in a purely relativistic context without ever mentioning QM.

There is a long history of using the gradient operator on classical physics functions, in this case the Lagrangian. And, in fact, it is another area where the same mathematics is used in both classical and quantum contexts.

SRQM Diagram: RoadMap of SR→QM

QM Schrödinger Relation

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The QM Schrödinger Relation
 $\mathbf{P} = i\hbar\partial$

This is derived from the combination of:

The Einstein-de Broglie Relation
 $\mathbf{P} = \hbar\mathbf{K}$

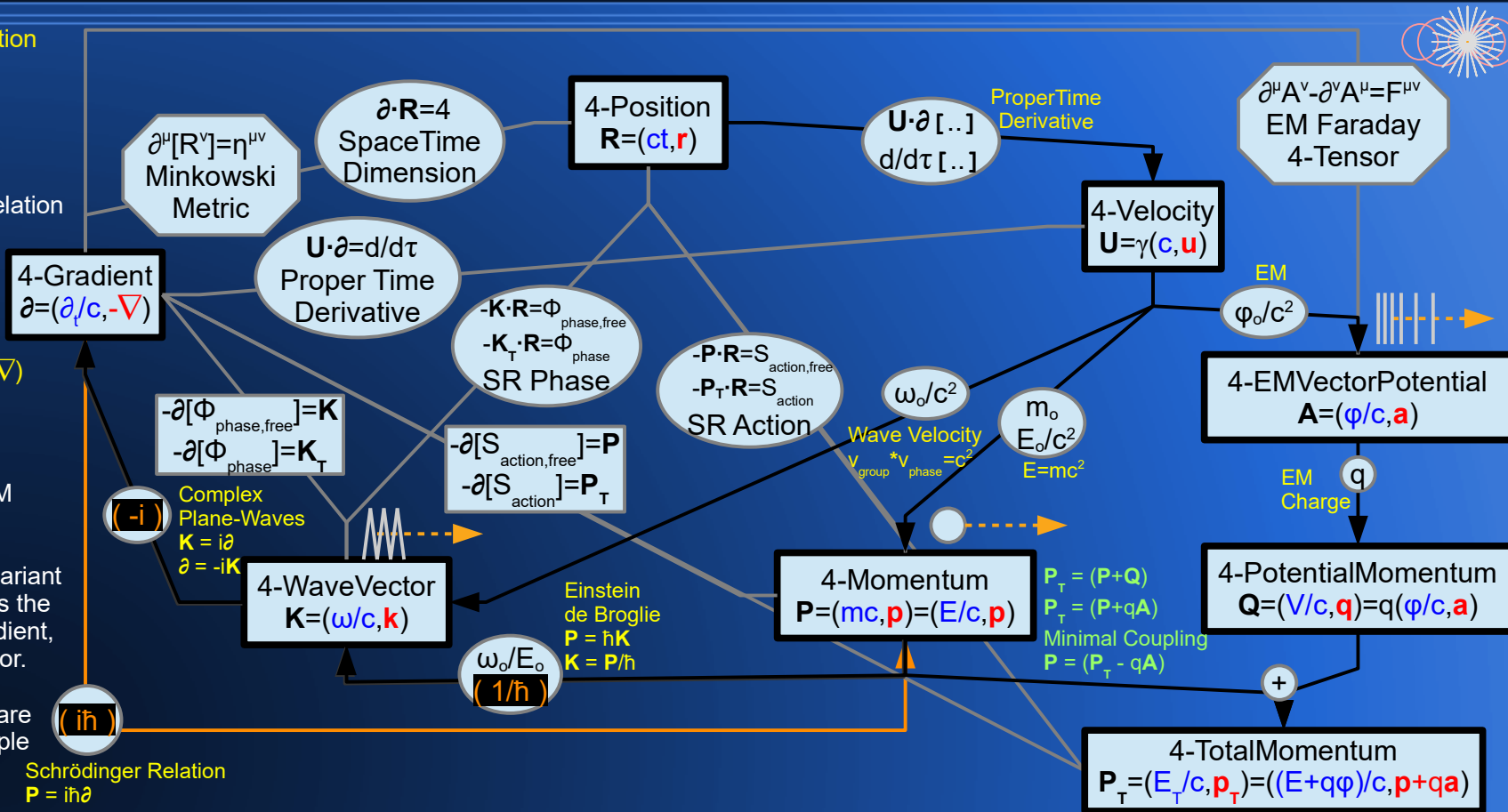
Complex Plane-Waves
 $\mathbf{K} = i\partial$

$\mathbf{P} = (E/c, \mathbf{p}) = i\hbar\partial = i\hbar(\partial/c, -\nabla)$
{temporal} $E = i\hbar\partial_t$
{spatial} $\mathbf{p} = -i\hbar\nabla$

These are the standard QM Schrödinger Relations.

It is this Lorentz Scalar Invariant relation ($i\hbar$) which connects the 4-Momentum to the 4-Gradient, making it into a QM operator.

Note that these 4-Vectors are already connected in multiple ways in standard SR.



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(0,2)-Tensor $T_{\mu\nu}$

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(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
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(0,0)-Tensor S
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$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$
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Review of SR 4-Vector Mathematics

A Tensor Study
of Physical 4-Vectors

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$$4\text{-Gradient } \partial = (\partial_t/c, -\nabla)$$

$$4\text{-Position } \mathbf{X} = (ct, \mathbf{x})$$

$$4\text{-Velocity } \mathbf{U} = \gamma(c, \mathbf{u})$$

$$4\text{-Momentum } \mathbf{P} = (E/c, \mathbf{p}) = (E_o/c^2)\mathbf{U}$$

$$4\text{-WaveVector } \mathbf{K} = (\omega/c, \mathbf{k}) = (\omega_o/c^2)\mathbf{U}$$

$$\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(\omega_o/c)^2$$

$$\mathbf{X} \cdot \mathbf{X} = ((ct)^2 - \mathbf{x} \cdot \mathbf{x}) = (ct_o)^2 = (c\tau)^2: \text{Invariant Interval Measure}$$

$$\mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = (c)^2$$

$$\mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_o/c)^2$$

$$\mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_o/c)^2$$

$$\partial \cdot \mathbf{X} = (\partial_t/c, -\nabla) \cdot (ct, \mathbf{x}) = (\partial_t/c[ct] - (-\nabla \cdot \mathbf{x})) = 1 - (-3) = 4:$$

$$\mathbf{U} \cdot \partial = \gamma(c, \mathbf{u}) \cdot (\partial_t/c, -\nabla) = \gamma(\partial_t + \mathbf{u} \cdot \nabla) = \gamma(d/dt) = d/d\tau:$$

$$\partial[\mathbf{X}] = (\partial_t/c, -\nabla)(ct, \mathbf{x}) = (\partial_t/c[ct], -\nabla[\mathbf{x}]) = \text{Diag}[1, -\mathbf{1}] = \eta^{\mu\nu}:$$

$$\partial[\mathbf{K}] = (\partial_t/c, -\nabla)(\omega/c, \mathbf{k}) = (\partial_t/c[\omega/c], -\nabla[\mathbf{k}]) = [[\mathbf{0}]]$$

$$\mathbf{K} \cdot \mathbf{X} = (\omega/c, \mathbf{k}) \cdot (ct, \mathbf{x}) = (\omega t - \mathbf{k} \cdot \mathbf{x}) = \phi:$$

$$\partial[\mathbf{K} \cdot \mathbf{X}] = \partial[\mathbf{K} \cdot \mathbf{X} + \mathbf{K} \cdot \partial[\mathbf{X}]] = \mathbf{K} = -\partial[\phi]:$$

Dimensionality of SpaceTime

Derivative wrt. ProperTime is Lorentz Scalar

The Minkowski Metric

Phase of SR Wave

Neg 4-Gradient of Phase gives 4-WaveVector

$$(\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}] = ((\partial_t/c)^2 - \nabla \cdot \nabla)(\omega t - \mathbf{k} \cdot \mathbf{x}) = 0$$

$$(\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}]) = \partial \cdot \mathbf{K} = 0:$$

Wave Continuity Equation, No sources or sinks

$$\text{let } f = ae^{i\mathbf{b}(\mathbf{K} \cdot \mathbf{X})}:$$

$$\text{then } \partial[f] = (-i\mathbf{K})ae^{i\mathbf{b}(\mathbf{K} \cdot \mathbf{X})} = (-i\mathbf{K})f: \quad (\partial = -i\mathbf{K}):$$

$$\text{and } \partial \cdot \partial[f] = (-i)^2(\mathbf{K} \cdot \mathbf{K})f = -(\omega_o/c)^2 f:$$

$$(\partial \cdot \partial) = (\partial_t/c)^2 - \nabla \cdot \nabla = -(\omega_o/c)^2 :$$

Standard mathematical plane-waves if { b = -i }

Unitary Evolution, Operator Formalism

The Klein-Gordon Equation → RQM

Note that no QM Axioms are assumed: This is all just pure SR 4-vector (tensor) manipulation

Review of SR 4-Vector Mathematics

$$\text{Klein-Gordon Equation: } \partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_0 c/\hbar)^2 = -(\omega_0/c)^2 = -(1/\lambda_C)^2$$

$$\text{Let } \mathbf{X}_T = (ct+c\Delta t, \mathbf{x}), \text{ then } \partial[\mathbf{X}_T] = (\partial_t/c, -\nabla)(ct+c\Delta t, \mathbf{x}) = \text{Diag}[1, -\mathbf{I}_{(3)}] = \partial[\mathbf{X}] = \eta^{\mu\nu}$$

$$\text{so } \partial[\mathbf{X}_T] = \partial[\mathbf{X}] \text{ and } \partial[\mathbf{K}] = [[\mathbf{0}]]$$

let $f = ae^{-i(\mathbf{K} \cdot \mathbf{X}_T)}$, the time translated version

$$(\partial \cdot \partial)[f]$$

$$\partial \cdot (\partial[f])$$

$$\partial \cdot (\partial[e^{-i(\mathbf{K} \cdot \mathbf{X}_T)}])$$

$$\partial \cdot (e^{-i(\mathbf{K} \cdot \mathbf{X}_T)} \partial[-i(\mathbf{K} \cdot \mathbf{X}_T)])$$

$$-i\partial \cdot (f\partial[\mathbf{K} \cdot \mathbf{X}_T])$$

$$-i\partial[f]\partial[\mathbf{K} \cdot \mathbf{X}_T] + \Psi(\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_T]$$

$$(-i)^2 f (\partial[\mathbf{K} \cdot \mathbf{X}_T])^2 + 0$$

$$(-i)^2 f (\partial[\mathbf{K}] \cdot \mathbf{X}_T + \mathbf{K} \cdot \partial[\mathbf{X}_T])^2$$

$$(-i)^2 f (0 + \mathbf{K} \cdot \partial[\mathbf{X}])^2$$

$$(-i)^2 f (\mathbf{K})^2$$

$$-(\mathbf{K} \cdot \mathbf{K})f$$

$$-(\omega_0/c)^2 f$$

What does the Klein-Gordon Equation give us?... **A lot of RQM!**

Relativistic Quantum Wave Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_0 c/\hbar)^2 = (im_0 c/\hbar)^2 = -(\omega_0/c)^2$

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles (Scalars)

Factoring the KG Eqn leads to the RQM Dirac Equation for spin=1/2 particles (Spinors)

Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Taking the low-velocity-limit of the KG leads to the standard QM non-relativistic Schrödinger Eqn, for spin=0

Taking the low-velocity-limit of the Dirac leads to the standard QM non-relativistic Pauli Eqn, for spin=1/2

Setting RestMass $\{m_0 \rightarrow 0\}$ leads to the RQM Free Wave, Weyl, and Free Maxwell Eqns

In all of these cases, the equations can be modified to work with various potentials by using more

SR 4-Vectors, and more empirically found relations between them, e.g. the Minimal Coupling Relations:

4-TotalMomentum $\mathbf{P}_{\text{tot}} = \mathbf{P} + q\mathbf{A}$, where \mathbf{P} is the particle 4-Momentum, (q) is a charge, and \mathbf{A} is a 4-VectorPotential, typically the 4-EMVectorPotential.

Also note that generating QM from RQM (via a low-energy limit) is much more natural than attempting to “relativize or generalize” a given NRQM equation. Facts assumed from a non-relativistic equation may or may not be applicable to a relativistic one, whereas the relativistic facts are still true in the low-velocity limiting-cases. This leads to the idea that QM is an approximation only of a more general RQM, just as SR is an approximation only of GR.

Relativistic Quantum Wave Eqns.

A Tensor Study of Physical 4-Vectors

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Spin-(Statistics) Bose-Einstein=n Fermi-Dirac=n/2	Relativistic Light-like Mass = 0	Relativistic Matter-like Mass > 0	Non-Relativistic Limit (v ≪c) Mass > 0	Field Representation
0-(Boson)	<p>Free Wave N-G Bosons</p> $(\partial \cdot \partial)\Psi = 0$	<p>Klein-Gordon Higgs Bosons, maybe Axions</p> $(\partial \cdot \partial + (m_0 c/\hbar)^2)\Psi = [\partial_\mu + im_0 c/\hbar][\partial^\mu - im_0 c/\hbar]\Psi = 0$ <p>with minimal coupling $((i\hbar\partial_t - q\phi)^2 - (m_0 c)^2 - c^2(-i\hbar\nabla - q\mathbf{a})^2)\Psi = 0$</p> <p>?Axions? are KG with EM invariant src term $(\partial \cdot \partial + (m_{a0})^2)\Psi = -\mathbf{k} \cdot \mathbf{b} = -kc\text{Sqrt}[\text{Det}[F^{\mu\nu}]]$</p> $L = (-\hbar^2/m_0)\partial^\mu\Psi^*\partial_\nu\Psi - m_0 c^2\Psi^*\Psi$	<p>Schrödinger Common NRQM Systems</p> $(i\hbar\partial_t + [\hbar^2\nabla^2/2m_0 - V])\Psi = 0$ <p>with minimal coupling $(i\hbar\partial_t - q\phi - [(\mathbf{p} - q\mathbf{a})^2/2m_0])\Psi = 0$</p>	<p>Scalar (0-Tensor) $\Psi = \Psi[K_\mu X^\mu]$ $= \Psi[\Phi]$</p>
1/2-(Fermion)	<p>Weyl Idealized Matter Neutinos</p> $(\boldsymbol{\sigma} \cdot \partial)\Psi = 0$ <p>factored to Right & Left Spinors $(\boldsymbol{\sigma} \cdot \partial)\Psi_R = 0, (\bar{\boldsymbol{\sigma}} \cdot \partial)\Psi_L = 0$</p> $L = i\Psi_R^\dagger \boldsymbol{\sigma}^\mu \partial_\mu \Psi_R, L = i\Psi_L^\dagger \bar{\boldsymbol{\sigma}}^\mu \partial_\mu \Psi_L$	<p>Dirac Matter Leptons/Quarks</p> $(i\boldsymbol{\gamma} \cdot \partial - m_0 c/\hbar)\Psi = 0$ $(\boldsymbol{\gamma} \cdot \partial + im_0 c/\hbar)\Psi = 0$ <p>with minimal coupling $(i\boldsymbol{\gamma} \cdot (\partial + iq\mathbf{A}) - m_0 c/\hbar)\Psi = 0$</p> $L = i\hbar c \bar{\Psi} \boldsymbol{\gamma}^\mu \partial_\mu \Psi - m_0 c^2 \bar{\Psi} \Psi$	<p>Pauli Common NRQM Systems w Spin</p> $(i\hbar\partial_t - [(\boldsymbol{\sigma} \cdot \mathbf{p})^2/2m_0])\Psi = 0$ <p>with minimal coupling $(i\hbar\partial_t - q\phi - [(\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{a}))^2/2m_0])\Psi = 0$</p>	<p>Spinor $\Psi = \Psi[K_\mu X^\mu]$ $= \Psi[\Phi]$</p>
1-(Boson)	<p>Maxwell Photons/Gluons</p> $(\partial \cdot \partial)\mathbf{A} = 0 \text{ free}$ $(\partial \cdot \partial)\mathbf{A} = \mu_0 \mathbf{J} \text{ w current src}$ <p>where $\partial \cdot \mathbf{A} = 0$</p> $(\partial \cdot \partial)\mathbf{A} = \mu_0 e \bar{\Psi} \boldsymbol{\gamma} \Psi \text{ QED}$	<p>Proca Force Bosons</p> $(\partial \cdot \partial + (m_0 c/\hbar)^2)\mathbf{A} = 0$ <p>where $\partial \cdot \mathbf{A} = 0$</p> $\partial^\mu(\partial^\nu A^\nu - \partial^\nu A^\mu) + (m_0 c/\hbar)^2 A^\nu = 0$		<p>4-Vector (1-Tensor) $\mathbf{A} = A^\nu = A^\nu[K_\mu X^\mu]$ $= A^\nu[\Phi]$</p>

Factoring the KG Equation → Dirac Eqn

A Tensor Study
of Physical 4-Vectors

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Klein-Gordon Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_0c/\hbar)^2$

Since the 4-vectors are related by constants, we can go back to the 4-Momentum description:

$$\begin{aligned} (\partial_t/c)^2 - \nabla \cdot \nabla &= -(m_0c/\hbar)^2 \\ (E/c)^2 - \mathbf{p} \cdot \mathbf{p} &= (m_0c)^2 \\ E^2 - c^2 \mathbf{p} \cdot \mathbf{p} - (m_0c^2)^2 &= 0 \end{aligned}$$

Factoring: $[E - c \boldsymbol{\alpha} \cdot \mathbf{p} - \beta(m_0c^2)] [E + c \boldsymbol{\alpha} \cdot \mathbf{p} + \beta(m_0c^2)] = 0$

E & \mathbf{p} are quantum operators,
 $\boldsymbol{\alpha}$ & β are matrices which must obey $\boldsymbol{\alpha}_i \beta = -\beta \boldsymbol{\alpha}_i$, $\boldsymbol{\alpha}_i \boldsymbol{\alpha}_j = -\boldsymbol{\alpha}_j \boldsymbol{\alpha}_i$, $\boldsymbol{\alpha}_i^2 = \beta^2 = \mathbf{I}$

The left hand term can be set to 0 by itself, giving...
 $[E - c \boldsymbol{\alpha} \cdot \mathbf{p} - \beta(m_0c^2)] = 0$, which is one form of the Dirac equation

Remember: $P^\mu = (\mathbf{p}^0, \mathbf{p}) = (E/c, \mathbf{p})$ and $\alpha^\mu = (\alpha^0, \boldsymbol{\alpha})$ where $\alpha^0 = I_{(2)}$

$$\begin{aligned} [E - c \boldsymbol{\alpha} \cdot \mathbf{p} - \beta(m_0c^2)] &= [c\alpha^0\mathbf{p}^0 - c \boldsymbol{\alpha} \cdot \mathbf{p} - \beta(m_0c^2)] = [c\alpha^\mu P_\mu - \beta(m_0c^2)] = 0 \\ [\alpha^\mu P_\mu - \beta(m_0c)] &= [i\hbar \alpha^\mu \partial_\mu - \beta(m_0c)] = 0 \\ \alpha^\mu \partial_\mu &= -\beta(im_0c/\hbar) \end{aligned}$$

Transforming from Pauli Spinor (2 component) to Dirac Spinor (4 component) form:
Dirac Equation: $(\gamma^\mu \partial_\mu)[\psi] = -(im_0c/\hbar)\psi$

Thus, the Dirac Eqn is guaranteed by construction to be one solution of the KG Eqn

The KG Equation is at the heart of all the various relativistic wave equations, which differ based on mass and spin values, but all of them respect $E^2 - c^2 \mathbf{p} \cdot \mathbf{p} - (m_0c^2)^2 = 0$

SRQM Study: Lots of Relativistic Quantum Wave Equations: **A lot of RQM!**

A Tensor Study
of Physical 4-Vectors

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Relativistic Quantum Wave Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_0 c/\hbar)^2 = (im_0 c/\hbar)^2 = -(\omega_0/c)^2$
 $\partial \cdot \partial = -(m_0 c/\hbar)^2$

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles {Higgs} (4-Scalars)

Factoring the KG Eqn leads to the RQM Dirac Equation for spin=1/2 particles (4-Spinors)

Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Setting RestMass $\{m_0 \rightarrow 0\}$ leads to the:

RQM Free Wave (4-Scalar massless)

RQM Weyl (4-Spinor massless)

Free Maxwell Eqns (4-Vector massless)

So, the same Relativistic Quantum Wave Equation is simply applied to different SR Tensorial Quantum Fields

See [Mathematical_formulation_of_the_Standard_Model](#) at Wikipedia:

4-Scalar (massive)	Higgs Field ϕ	$[\partial \cdot \partial = -(m_0 c/\hbar)^2]\phi$	Free Field Eqn → Klein-Gordon Eqn	$\partial \cdot \partial[\phi] = -(m_0 c/\hbar)^2 \phi$
4-Vector (massive)	Weak Field $Z^\mu, W^{\pm\mu}$	$[\partial \cdot \partial = -(m_0 c/\hbar)^2]Z^\mu$	Free Field Eqn → Proca Eqn	$\partial \cdot \partial[Z^\mu] = -(m_0 c/\hbar)^2 Z^\mu$
4-Vector (massless $m_0=0$)	Photon Field A^μ	$[\partial \cdot \partial = 0]A^\mu$	Free Field Eqn → EM Wave Eqn	$\partial \cdot \partial[A^\mu] = 0^\mu$
4-Spinor (massive)	Fermion Field ψ	$[\gamma \cdot \partial = -im_0 c/\hbar]\Psi$	Free Field Eqn → Dirac Eqn	$\gamma \cdot \partial[\Psi] = -(im_0 c/\hbar)\Psi$

*The Fermion field is a special case, the Dirac Gamma Matrices γ^μ and 4-Spinor field Ψ work together to preserve Lorentz Invariance.

SRQM Study: Lots of Relativistic Quantum Wave Equations: **A lot of RQM!**

A Tensor Study
of Physical 4-Vectors

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In relativistic quantum mechanics and quantum field theory, the Bargmann–Wigner equations describe free particles of arbitrary spin j , an integer for bosons ($j = 1, 2, 3 \dots$) or half-integer for fermions ($j = 1/2, 3/2, 5/2 \dots$). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields.

Bargmann–Wigner equations: $(-\gamma^\mu P_\mu + mc)_{\alpha_1 \dots \alpha_j} \Psi_{\alpha_1 \dots \alpha_j} = 0$

In relativistic quantum mechanics and quantum field theory, the Joos–Weinberg equation is a relativistic wave equations applicable to free particles of arbitrary spin j , an integer for bosons ($j = 1, 2, 3 \dots$) or half-integer for fermions ($j = 1/2, 3/2, 5/2 \dots$). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields. The spin quantum number is usually denoted by s in quantum mechanics, however in this context j is more typical in the literature.

Joos–Weinberg equation: $[\gamma^{\mu_1 \mu_2 \dots \mu_j} P_{\mu_1} P_{\mu_2} \dots P_{\mu_j} + (mc)^{2j}] \Psi = 0$

The primary difference appears to be the expansion in either the wavefunctions for (BW) or the Dirac Gamma's for (JW)

For both of these: A state or quantum field in such a representation would satisfy no field equation except the Klein-Gordon equation.

Yet another form is the Duffin-Kemmer-Petiau Equation vs Dirac Equation

DKP Eqn {spin 0 or 1}: $(i\hbar\beta^\alpha\partial_\alpha - m_0c)\Psi = 0$, with β^α as the DKP matrices

Dirac Eqn (spin $1/2$): $(i\hbar\gamma^\alpha\partial_\alpha - m_0c)\Psi = 0$, with γ^α as the Dirac Gamma matrices

A few more SR 4-Vectors

SR 4-Vector	Definition	Unites
4-Position	$\mathbf{R} = (ct, \mathbf{r}); \text{ alt. } \mathbf{X} = (ct, \mathbf{x})$	Time, Space
4-Velocity	$\mathbf{U} = \gamma(c, \mathbf{u})$	Gamma, Velocity
4-Momentum	$\mathbf{P} = (E/c, \mathbf{p}) = (mc, \mathbf{p})$	Energy:Mass, Momentum
4-WaveVector	$\mathbf{K} = (\omega/c, \mathbf{k}) = (\omega/c, \omega \hat{\mathbf{n}}/v_{\text{phase}})$	Frequency, WaveNumber
4-Gradient	$\partial = (\partial_t/c, -\nabla)$	Temporal Partial, Space Partial
4-VectorPotential	$\mathbf{A} = (\phi/c, \mathbf{a})$	Scalar Potential, Vector Potential
4-TotalMomentum	$\mathbf{P}_{\text{tot}} = (E/c + q\phi/c, \mathbf{p} + q\mathbf{a})$	Energy-Momentum inc. EM fields
4-TotalWaveVector	$\mathbf{K}_{\text{tot}} = (\omega/c + (q/\hbar)\phi/c, \mathbf{k} + (q/\hbar)\mathbf{a})$	Freq-WaveNum inc. EM fields
4-CurrentDensity	$\mathbf{J} = (c\rho, \mathbf{j}) = q\mathbf{J}_{\text{prob}}$	Charge Density, Current Density
4-ProbabiltyCurrentDensity <small>can have complex values</small>	$\mathbf{J}_{\text{prob}} = (c\rho_{\text{prob}}, \mathbf{j}_{\text{prob}})$	QM Probability (Density, Current Density)

More SR 4-Vectors Explained

SR 4-Vector	Empirical Fact	What it means...
4-Position	$\mathbf{R} = (ct, \mathbf{r})$	SpaceTime as Single United Concept
4-Velocity	$\mathbf{U} = d\mathbf{R}/d\tau$	Velocity is Proper Time Derivative
4-Momentum	$\mathbf{P} = m_0\mathbf{U} = (E_0/c^2)\mathbf{U}$	Mass-Energy-Momentum Equivalence
4-WaveVector	$\mathbf{K} = \mathbf{P}/\hbar = (\omega_0/c^2)\mathbf{U}$	Wave-Particle Duality
4-Gradient	$\partial = -i\mathbf{K}$	Unitary Evolution of States Operator Formalism, Complex Waves
4-VectorPotential	$\mathbf{A} = (\varphi/c, \mathbf{a}) = (\varphi_0/c^2)\mathbf{U}$	Potential Fields...
4-TotalMomentum	$\mathbf{P}_{\text{tot}} = \mathbf{P} + q\mathbf{A}$	Energy-Momentum inc. Potential Fields
4-TotalWaveVector	$\mathbf{K}_{\text{tot}} = \mathbf{K} + (q/\hbar)\mathbf{A}$	Freq-WaveNum inc. Potential Fields
4-CurrentDensity	$\mathbf{J} = \rho_0\mathbf{U} = q\mathbf{J}_{\text{prob}}$ $\partial \cdot \mathbf{J} = 0$	ChargeDensity-CurrentDensity Equivalence CurrentDensity is conserved
4-Probability CurrentDensity	$\mathbf{J}_{\text{prob}} = (c\rho_{\text{prob}}, \mathbf{j}_{\text{prob}})$ $\partial \cdot \mathbf{J}_{\text{prob}} = 0$	QM Probability from SR Probability Worldlines are conserved

Minimal Coupling = Potential Interaction

Klein-Gordon Eqn → Schrödinger Eqn

$$\mathbf{P}_T = \mathbf{P} + \mathbf{Q} = \mathbf{P} + q\mathbf{A}$$

$$\mathbf{K} = i\partial$$

$$\mathbf{P} = \hbar\mathbf{K}$$

$$\mathbf{P} = i\hbar\partial$$

Minimal Coupling: Total = Dynamic + Charge_Coupled to 4-(EM)VectorPotential

Complex Plane-Waves

Einstein-de Broglie QM Relations

Schrödinger Relations

$$\mathbf{P} = (E/c, \mathbf{p}) = \mathbf{P}_T - q\mathbf{A} = (E_T/c - q\phi/c, \mathbf{p}_T - q\mathbf{a}) = \hbar\mathbf{K} = i\hbar\partial$$

$$\partial = (\partial/c, -\nabla) = \partial_T + (iq/\hbar)\mathbf{A} = (\partial_{TT}/c + (iq/\hbar)\phi/c, -\nabla_T + (iq/\hbar)\mathbf{a}) = -i\mathbf{K} = (-i/\hbar)\mathbf{P}$$

$$\partial \cdot \partial = (\partial/c)^2 - \nabla^2 = -(m_0c/\hbar)^2 :$$

$$\mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p}^2 = (m_0c)^2 :$$

The Klein-Gordon RQM Wave Equation (relativistic QM)

Einstein Mass:Energy:Momentum Equivalence

$$E^2 = (m_0c^2)^2 + c^2\mathbf{p}^2 :$$

$$E \sim [(m_0c^2)^2 + \mathbf{p}^2/2m_0] :$$

Relativistic

Low velocity limit { $|\mathbf{v}| \ll c$ } from $(1+x)^n \sim [1 + nx + O(x^2)]$ for $|x| \ll 1$

$$(E_T - q\phi)^2 = (m_0c^2)^2 + c^2(\mathbf{p}_T - q\mathbf{a})^2 :$$

$$(E_T - q\phi) \sim [(m_0c^2)^2 + (\mathbf{p}_T - q\mathbf{a})^2/2m_0] :$$

Relativistic with Minimal Coupling

Low velocity with Minimal Coupling

$$(i\hbar\partial_{TT} - q\phi)^2 = (m_0c^2)^2 + c^2(-i\hbar\nabla_T - q\mathbf{a})^2 :$$

$$(i\hbar\partial_{TT} - q\phi) \sim [(m_0c^2)^2 + (-i\hbar\nabla_T - q\mathbf{a})^2/2m_0] :$$

Relativistic with Minimal Coupling

Low velocity with Minimal Coupling

$$(i\hbar\partial_{TT}) \sim [q\phi + (m_0c^2)^2 + (i\hbar\nabla_T + q\mathbf{a})^2/2m_0] :$$

$$(i\hbar\partial_{TT}) \sim [V + (i\hbar\nabla_T + q\mathbf{a})^2/2m_0] :$$

$$(i\hbar\partial_{TT}) \sim [V - (\hbar\nabla_T)^2/2m_0] :$$

Low velocity with Minimal Coupling

$$V = q\phi + (m_0c^2)$$

Typically the 3-vector_potential $\mathbf{a} \sim 0$ in many situations

The better statement is that the Schrödinger Eqn is the limiting low-velocity case of the more general KG Eqn, not that the KG Eqn is the relativistic generalization of the Schrödinger Eqn

$$(i\hbar\partial_{TT})|\Psi\rangle \sim [V - (\hbar\nabla_T)^2/2m_0]|\Psi\rangle :$$

The Schrödinger NRQM Wave Equation (non-relativistic QM)

Once one has a **Relativistic Wave Eqn...**

A Tensor Study
of Physical 4-Vectors

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Klein-Gordon Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2$

Once we have derived a RWE, what does it imply?

The KG Eqn. was derived from the physics of SR plus a few empirical facts. It is a 2nd order, linear, wave PDE that pertains to physical objects of reality from SR.

Just being a linear wave PDE implies all the mathematical techniques that have been discovered to solve such equations generally: Hilbert Space, Superpositions, $\langle \text{Bra} |, | \text{Ket} \rangle$ notation, wavevectors, wavefunctions, etc. These things are from mathematics in general, not only and specifically from an Axiom of QM.

Therefore, if one has a physical RWE, it implies the mathematics of waves, the formalism of the mathematics, and thus the mathematical Principles and Formalism of QM. Again, QM Axioms are not required – they emerge from the physics and math...

Once one has a Relativistic Wave Eqn...

Examine **Photon Polarization**

From the Wikipedia page on [Photon Polarization]

Photon polarization is the quantum mechanical description of the classical polarized sinusoidal plane electromagnetic wave. An individual photon can be described as having right or left circular polarization, or a superposition of the two. Equivalently, a photon can be described as having horizontal or vertical linear polarization, or a superposition of the two.

The description of photon polarization contains many of the physical concepts and much of the mathematical machinery of more involved quantum descriptions and forms a fundamental basis for an understanding of more complicated quantum phenomena. Much of the mathematical machinery of quantum mechanics, such as state vectors, probability amplitudes, unitary operators, and Hermitian operators, emerge naturally from the classical Maxwell's equations in the description. The quantum polarization state vector for the photon, for instance, is identical with the Jones vector, usually used to describe the polarization of a classical wave. Unitary operators emerge from the classical requirement of the conservation of energy of a classical wave propagating through lossless media that alter the polarization state of the wave. Hermitian operators then follow for infinitesimal transformations of a classical polarization state.

Many of the implications of the mathematical machinery are easily verified experimentally. In fact, many of the experiments can be performed with two pairs (or one broken pair) of polaroid sunglasses.

The connection with quantum mechanics is made through the identification of a minimum packet size, called a photon, for energy in the electromagnetic field. The identification is based on the theories of Planck and the interpretation of those theories by Einstein. The correspondence principle then allows the identification of momentum and angular momentum (called spin), as well as energy, with the photon.

Principle of Superposition:

From the mathematics of waves

Klein-Gordon Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_0 c/\hbar)^2 = -(\omega_0/c)^2$

The Extended Superposition Principle for Linear Equations

=====

Suppose that the non-homogeneous equation, where L is linear, is solved by some particular u_p

Suppose that the associated homogeneous problem is solved by a sequence of u_i .

$L(u_p) = C$; $L(u_0) = 0$, $L(u_1) = 0$, $L(u_2) = 0$...

Then u_p plus any linear combination of the u_n satisfies the original non-homogeneous equation:

$L(u_p + \sum a_n u_n) = C$,

where a_n is a sequence of (possibly complex) constants and the sum is arbitrary.

=====

Note that there is no mention of partial differentiation. Indeed, it's true for any linear equation, algebraic or integro-partial differential-whatever.

QM superposition is not axiomatic, it emerges from the mathematics of the Linear PDE

Klein-Gordon obeys Principle of Superposition

Klein-Gordon Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_0 c/\hbar)^2 = -(\omega_0/c)^2$

$\mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_0/c)^2$: The particular solution (w rest mass)

$\mathbf{K}_n \cdot \mathbf{K}_n = (\omega_n/c)^2 - \mathbf{k}_n \cdot \mathbf{k}_n = 0$: The homogenous solution for a (virtual photon?) microstate n

Note that $\mathbf{K}_n \cdot \mathbf{K}_n = 0$ is a null 4-vector (photonic)

Let $\Psi_p = A e^{-i(\mathbf{K} \cdot \mathbf{X})}$, then $\partial \cdot \partial[\Psi_p] = (-i)^2(\mathbf{K} \cdot \mathbf{K})\Psi_p = -(\omega_0/c)^2\Psi_p$
which is the Klein-Gordon Equation, the particular solution...

Let $\Psi_n = A_n e^{-i(\mathbf{K}_n \cdot \mathbf{X})}$, then $\partial \cdot \partial[\Psi_n] = (-i)^2(\mathbf{K}_n \cdot \mathbf{K}_n)\Psi_n = (0)\Psi_n$
which is the Klein-Gordon Equation homogeneous solution for a microstate n

We may take $\Psi = \Psi_p + \sum_n \Psi_n$

Hence, the Principle of Superposition is not required as an QM Axiom, it follows from SR and our empirical facts which lead to the Klein-Gordon Equation. The Klein-Gordon equation is a linear wave PDE, which has overall solutions which can be the complex linear sums of individual solutions – i.e. it obeys the Principle of Superposition.

This is not an axiom – it is a general mathematical property of linear PDE's.

This property continues over as well to the limiting case $\{ |\mathbf{v}| \ll c \}$ of the Schrödinger Equation.

QM Hilbert Space:

From the mathematics of waves

Klein-Gordon Equation: $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_0c/\hbar)^2$

Hilbert Space (HS) representation:

if $|\Psi\rangle \in \text{HS}$, then $c|\Psi\rangle \in \text{HS}$, where c is complex number

if $|\Psi_1\rangle$ and $|\Psi_2\rangle \in \text{HS}$, then $|\Psi_1\rangle + |\Psi_2\rangle \in \text{HS}$

if $|\Psi\rangle = c_1|\Psi_1\rangle + c_2|\Psi_2\rangle$, then $\langle\Phi|\Psi\rangle = c_1\langle\Phi|\Psi_1\rangle + c_2\langle\Phi|\Psi_2\rangle$ and $\langle\Psi| = c_1^*\langle\Psi_1| + c_2^*\langle\Psi_2|$

$\langle\Phi|\Psi\rangle = \langle\Psi|\Phi\rangle$

$\langle\Psi|\Psi\rangle \geq 0$

if $\langle\Psi|\Psi\rangle = 0$, then $|\Psi\rangle = \mathbf{0}$

etc.

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. They are indispensable tools in the theories of partial differential equations, Fourier analysis, signal processing, heat transfer, ergodic theory, and Quantum Mechanics.

The QM Hilbert Space emerges from the fact that the KG Equation is a linear wave PDE – Hilbert spaces as solutions to PDE's are a purely mathematical phenomenon – no QM Axiom is required.

Likewise, this introduces the $\langle\text{bra}|, |\text{ket}\rangle$ notation, wavevectors, wavefunctions, etc.

Note:

One can use Hilbert Space descriptions of Classical Mechanics using the Koopman-von Neumann formulation. One can not use Hilbert Space descriptions of Quantum Mechanics by using the Phase Space formulation of QM.

Canonical Commutation Relation: Viewed from standard QM

Standard QM Canonical Commutation Relation: $[\mathbf{x}^j, \mathbf{p}^k] = i\hbar\delta^{jk}$

The Standard QM Canonical Commutation Relation is simply an axiom in standard QM. It is just given, with no explanation. You just had to accept it.

I always found that unsatisfactory.

There are at least 4 parts to it:

Where does the commutation ($[,]$) come from?

Where does the imaginary constant (i) come from?

Where does the Planck constant (\hbar) come from?

Where does the Kronecker Delta (δ^{jk}) come from?

See the next page for SR enlightenment...

SRQM Diagram: Canonical QM Commutation Relation Derived from SR

A Tensor Study of Physical 4-Vectors

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Let (f) be an arbitrary SR function
 $\mathbf{X}[f] = \mathbf{X}f$, $\partial[f] = \partial[f]$
 \mathbf{X} , function or not, has no effect on (f)
 $\partial = \partial[\]$ is definitely an SR function/operator

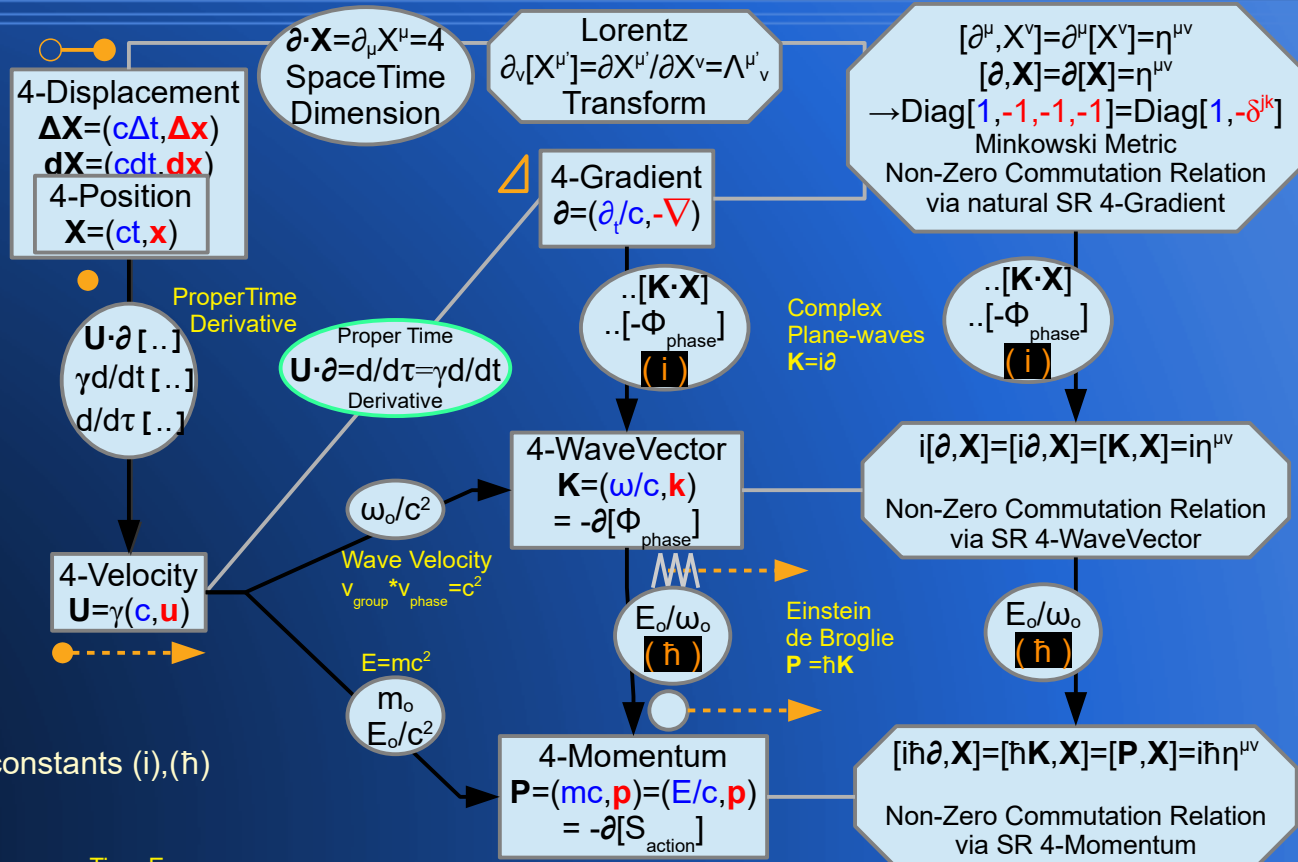
$\mathbf{X}[\partial[f]] = \mathbf{X}\partial[f]$
 $\partial[\mathbf{X}f] = \partial[\mathbf{X}]f + \mathbf{X}\partial[f]$
 $\partial[\mathbf{X}f] - \mathbf{X}\partial[f] = \partial[\mathbf{X}]f$
 $\partial[\mathbf{X}[f]] - \mathbf{X}[\partial[f]] = \partial[\mathbf{X}]f$

Recognize this as a commutation relation
 $[\partial, \mathbf{X}]f = \partial[\mathbf{X}]f$

$[\partial, \mathbf{X}] = \partial[\mathbf{X}]$
 $= \partial^\mu[X^\nu]$
 $= (\partial/c, -\nabla)(ct, \mathbf{x})$
 $= (\partial/c, -\partial_x, -\partial_y, -\partial_z)(ct, x, y, z)$
 $= \text{Diag}\{1, -1, -1, -1\} = \text{Diag}[1, -\delta^{jk}]$
 $= \eta^{\mu\nu} = \text{Minkowski Metric}$

$[\partial^\mu, X^\nu] = \eta^{\mu\nu}$ Tensor form: true for all observers
 $[P^\mu, X^\nu] = i\hbar\eta^{\mu\nu}$ Independently true from empirical constants (i), (ħ)
 $[p^k, x^j] = -i\hbar\delta^{kj}$ $[p^0, x^0] = [E/c, ct] = [E, t] = i\hbar$

$[x^j, p^k] = i\hbar\delta^{jk}$ Position:Momentum QM Commutation Relation
 $[t, E] = -i\hbar$ Time:Energy QM Commutation Relation



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^μ_ν or T_μ^ν
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

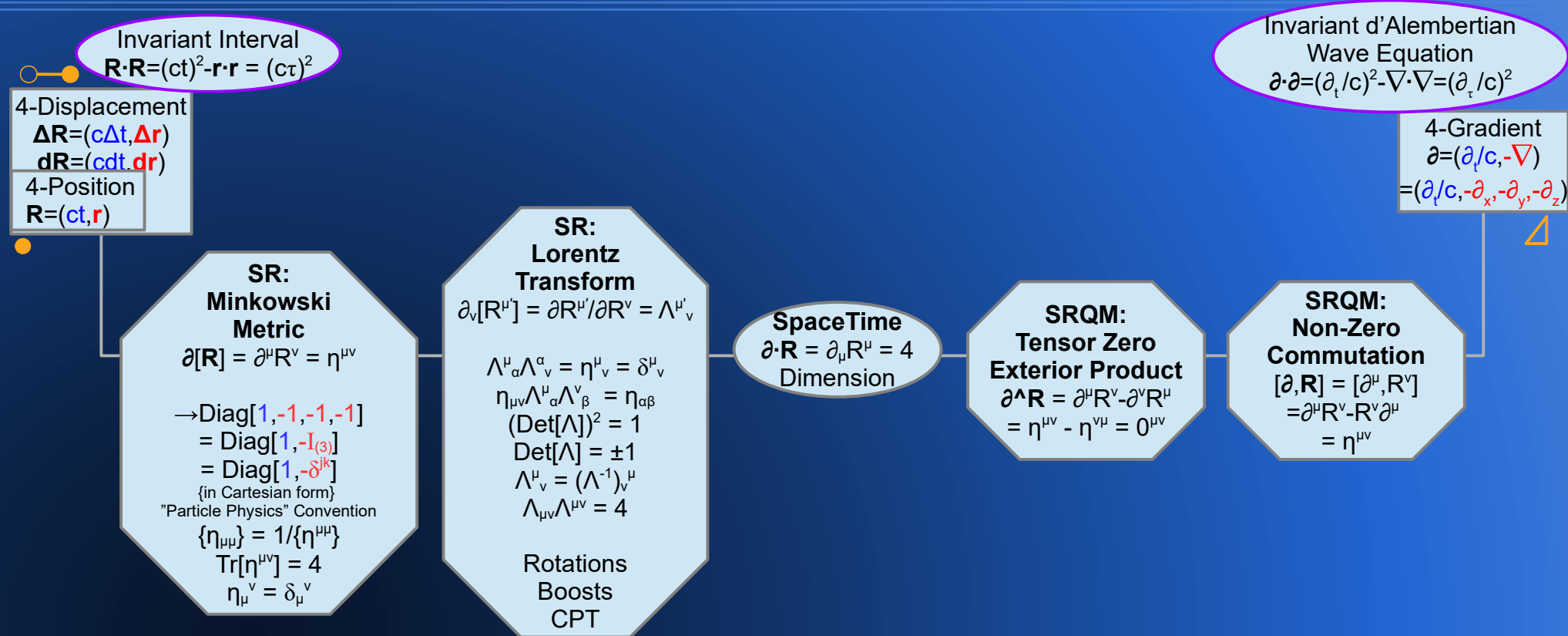
Existing SR Rules
Quantum Principles

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V}\cdot\mathbf{V} = V^\mu\eta_{\mu\nu}V^\nu = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM Study: 4-Position and 4-Gradient

A Tensor Study of Physical 4-Vectors

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SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\mu{}_\nu$ or $T_\mu{}^\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2$
= Lorentz Scalar

Heisenberg Uncertainty Principle: Viewed from SRQM

A Tensor Study
of Physical 4-Vectors

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Heisenberg Uncertainty $\{ \sigma_A^2 \sigma_B^2 \} \geq (1/2) | \langle [A, B] \rangle |$
arises from the non-commuting nature of certain operators.

The commutator is $[A, B] = AB - BA$, where A & B are functional “measurement” operators.
The Operator Formalism arose naturally from our SR → QM path: $[\partial = -i\mathbf{K}]$.

The Generalized Uncertainty Relation: $\sigma_f^2 \sigma_g^2 = (\Delta F) * (\Delta G) \geq (1/2) | \langle i[F, G] \rangle |$

The uncertainty relation is a very general mathematical property, which applies to both
classical or quantum systems. From Wikipedia: Photon Polarization: "This is a purely
mathematical result. No reference to a physical quantity or principle is required."

The Cauchy–Schwarz inequality asserts that (for all vectors f and g of an inner product
space, with either real or complex numbers):
 $\sigma_f^2 \sigma_g^2 = \langle f | f \rangle \cdot \langle g | g \rangle \geq | \langle f | g \rangle |^2$

But first, let's back up a bit; Using standard complex number math, we have:

$$z = a + ib$$

$$z^* = a - ib$$

$$\text{Re}(z) = a = (z + z^*) / (2)$$

$$\text{Im}(z) = b = (z - z^*) / (2i)$$

$$z^* z = |z|^2 = a^2 + b^2 = [\text{Re}(z)]^2 + [\text{Im}(z)]^2 = [(z + z^*) / (2)]^2 + [(z - z^*) / (2i)]^2$$

or

$$|z|^2 = [(z + z^*) / (2)]^2 + [(z - z^*) / (2i)]^2$$

Now, generically, based on the rules of a complex inner product space we can arbitrarily
assign:

$$z = \langle f | g \rangle, z^* = \langle g | f \rangle$$

Which allows us to write:

$$| \langle f | g \rangle |^2 = [\langle f | g \rangle + \langle g | f \rangle] / (2)]^2 + [\langle f | g \rangle - \langle g | f \rangle] / (2i)]^2$$

We can also note that:
 $| f \rangle = F | \Psi \rangle$ and $| g \rangle = G | \Psi \rangle$

Thus,

$$| \langle f | g \rangle |^2 = [[\langle \Psi | F^* G | \Psi \rangle + \langle \Psi | G^* F | \Psi \rangle] / (2)]^2 + [[\langle \Psi | F^* G | \Psi \rangle - \langle \Psi | G^* F | \Psi \rangle] / (2i)]^2$$

For Hermetian Operators...

$$F^* = +F, G^* = +G$$

For Anti-Hermetian (Skew-Hermetian) Operators...

$$F^* = -F, G^* = -G$$

Assuming that F and G are either both Hermetian, or both anti-Hermetian...

$$| \langle f | g \rangle |^2 = [[\langle \Psi | (\pm) F G | \Psi \rangle + \langle \Psi | (\pm) G F | \Psi \rangle] / (2)]^2 + [[\langle \Psi | (\pm) F G | \Psi \rangle - \langle \Psi | (\pm) G F | \Psi \rangle] / (2i)]^2$$

$$| \langle f | g \rangle |^2 = [(\pm) \langle \Psi | F G | \Psi \rangle + \langle \Psi | G F | \Psi \rangle] / (2)]^2 + [(\pm) \langle \Psi | F G | \Psi \rangle - \langle \Psi | G F | \Psi \rangle] / (2i)]^2$$

We can write this in commutator and anti-commutator notation...

$$| \langle f | g \rangle |^2 = [(\pm) \langle \Psi | \{F, G\} | \Psi \rangle] / (2)]^2 + [(\pm) \langle \Psi | [F, G] | \Psi \rangle] / (2i)]^2$$

Due to the squares, the (±)'s go away, and we can also multiply the commutator by an (i^2)

$$| \langle f | g \rangle |^2 = [[\langle \Psi | \{F, G\} | \Psi \rangle] / 2]^2 + [[\langle \Psi | i[F, G] | \Psi \rangle] / 2]^2$$

$$| \langle f | g \rangle |^2 = [[\langle \Psi | \{F, G\} \rangle] / 2]^2 + [[\langle \Psi | i[F, G] \rangle] / 2]^2$$

The Cauchy–Schwarz inequality again...

$$\sigma_f^2 \sigma_g^2 = \langle f | f \rangle \cdot \langle g | g \rangle \geq | \langle f | g \rangle |^2 = [[\langle \Psi | \{F, G\} \rangle] / 2]^2 + [[\langle \Psi | i[F, G] \rangle] / 2]^2$$

Taking the root:

$$\sigma_f^2 \sigma_g^2 \geq (1/2) | \langle i[F, G] \rangle |$$

Which is what we had for the generalized Uncertainty Relation.

Note This is not a QM axiom - This is just pure math. At this stage we already see the
hints of commutation and anti-commutation.

It is true generally, whether applying to a physical or purely mathematical situation.

Heisenberg Uncertainty Principle: Simultaneous vs Sequential

Heisenberg Uncertainty $\{ \sigma_A^2 \sigma_B^2 \geq (1/2) | \langle [A, B] \rangle | \}$ arises from the non-commuting nature of certain operators.

$$[\partial^\mu, X^\nu] = \partial[X^\nu] = \eta^{\mu\nu} = \text{Minkowski Metric}$$

$$[P^\mu, X^\nu] = [i\hbar\partial^\mu, X^\nu] = i\hbar[\partial^\mu, X^\nu] = i\hbar\eta^{\mu\nu}$$

Consider the following:

Operator A acts on System $|\Psi\rangle$ at SR Event A: $A|\Psi\rangle \rightarrow |\Psi'\rangle$

Operator B acts on System $|\Psi'\rangle$ at SR Event B: $B|\Psi'\rangle \rightarrow |\Psi''\rangle$

or $BA|\Psi\rangle = B|\Psi'\rangle = |\Psi''\rangle$

If measurement Events A & B are space-like separated, then there are observers who can see {A before B, A simultaneous with B, A after B}, which of course does not match the quantum description of how Operators act on Kets

If Events A & B are time-like separated, then all observers will always see A before B. This does match how the operators act on Kets, and also matches how $|\Psi\rangle$ would be evolving along its worldline, starting out as $|\Psi\rangle$, getting hit with operator A at Event A to become $|\Psi'\rangle$, then getting hit with operator B at Event B to become $|\Psi''\rangle$.

The Uncertainty Relation here does NOT refer to simultaneous (space-like separated) measurements, it refers to sequential (time-like separated) measurements. This removes the need for ideas about the particles not having simultaneous properties. There are simply no “simultaneous measurements” of non-zero commuting properties on an individual system, a single worldline – they are sequential, and the first measurement places the system in such a state that the outcome of the second measurement will be altered wrt. if the order of the operations had been reversed.

Pauli Exclusion Principle: Requires SR for the detailed explanation

A Tensor Study
of Physical 4-Vectors

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The Pauli Exclusion Principle is a result of the empirical fact that nature uses identical particles, and this combined with the Spin-Statistics theorem from SR, leads to an exclusion principle for fermions (anti-symmetric, Fermi-Dirac statistics) and an aggregation principle for bosons (symmetric, Bose-Einstein statistics). The Spin-Statistics Theorem is related as well to the CPT Theorem.

For large numbers and/or mixed states these both tend to the Maxwell-Boltzmann statistics. In the $\{kT \gg (\epsilon_i - \mu)\}$ limit, Bose-Einstein reduces to Rayleigh-Jeans. The commutation relations here are based on space-like separation particle exchanges, unlike the time-like separation for measurement operator exchanges in the Uncertainty Principle.

Spin	Particle Type	Quantum Statistics	Classical $\{kT \gg (\epsilon_i - \mu)\}$
spin:(0,1,...,N)	Indistinguishable, Commutation relation ($ab = ba$)	Bose-Einstein: $n_i = g_i / [e^{(\epsilon_i - \mu)/kT} - 1]$ aggregation principle	Rayleigh-Jeans: from $e^x \sim (1 + x + \dots)$ $n_i = g_i / [(\epsilon_i - \mu)/kT]$
		↓ Limit as $e^{(\epsilon_i - \mu)/kT} \gg 1$ ↓	
Multi-particle Mixed	Distinguishable, or high temp, or low density	Maxwell-Boltzmann: $n_i = g_i / [e^{(\epsilon_i - \mu)/kT} + 0]$	Maxwell-Boltzmann: $n_i = g_i / [e^{(\epsilon_i - \mu)/kT}]$
		↑ Limit as $e^{(\epsilon_i - \mu)/kT} \gg 1$ ↑	
spin:(1/2,3/2,...,N/2)	Indistinguishable, Anti-commutation relation ($ab = -ba$)	Fermi-Dirac: $n_i = g_i / [e^{(\epsilon_i - \mu)/kT} + 1]$ exclusion principle	

4-Vectors & Minkowski Space Review

Complex 4-Vectors

Complex 4-vectors are simply 4-Vectors where the components may be complex-valued

$$\mathbf{A} = A^\mu = (a^0, \mathbf{a}) = (a^0, a^1, a^2, a^3) \rightarrow (a^t, a^x, a^y, a^z)$$

$$\mathbf{B} = B^\mu = (b^0, \mathbf{b}) = (b^0, b^1, b^2, b^3) \rightarrow (b^t, b^x, b^y, b^z)$$

Examples of 4-Vectors with complex components are the 4-Polarization and the 4-ProbabilityCurrentDensity

Minkowski Metric $g^{\mu\nu} \rightarrow \eta^{\mu\nu} = \eta_{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -\mathbf{I}_{(3)}]$,
which is the {curvature~0 limit = low-mass limit} of the GR metric $g^{\mu\nu}$.

Applying the Metric to raise or lower an index also applies a complex-conjugation *

Scalar Product = Lorentz Invariant → Same value for all inertial observers

$$\mathbf{A} \cdot \mathbf{B} = \eta_{\mu\nu} A^\mu B^\nu = A_\nu^* B^\nu = A^\mu B_\mu^* = (a^{0*} b^0 - \mathbf{a}^* \cdot \mathbf{b}) \text{ using the Einstein summation convention}$$

This reverts to the usual rules for real components

However, it does imply that $\mathbf{A} \cdot \mathbf{B} = \overline{\mathbf{B} \cdot \mathbf{A}}$

SRQM: CPT Theorem

Phase Connection, Lorentz Invariance

A Tensor Study of Physical 4-Vectors

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The Phase is a Lorentz Scalar Invariant – all observers must agree on its value.
 $K \cdot X = (\omega/c, \mathbf{k}) \cdot (ct, \mathbf{x}) = (\omega t - \mathbf{k} \cdot \mathbf{x}) = -\Phi$: Phase of SR Wave

We take the point of view of an observer operating on a particle at 4-Position X , which has an initial 4-WaveVector K . The 4-Position X of the particle, the operation's event, will not change: we are applying the various operations only to the particle's 4-Momentum K .

Note that for matter particles $K = (\omega/c)\mathbf{T}$, where \mathbf{T} is the Unit-Temporal 4-Vector $\mathbf{T} = \gamma(1, \boldsymbol{\beta})$, which defines the particle's worldline at each point. The gamma factor (γ) will be unaffected in the following operations, since it uses the square of $\boldsymbol{\beta}$: $\gamma = 1/\sqrt{1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}}$. For photonic particles, $K = (\omega/c)\mathbf{N}$, where \mathbf{N} is the "Unit"-Null 4-Vector $\mathbf{N} = (1, \mathbf{n})$ and \mathbf{n} is a unit-spatial 3-vector. All operations listed below work similarly on the Null 4-Vector.

Do a Time Reversal Operation: T
 The particle's temporal direction is reversed & complex-conjugated:
 $\mathbf{T}_T = -\mathbf{T}^* = \gamma(-1, \boldsymbol{\beta})^*$

Do a Parity Operation (Space Reflection): P
 Only the spatial directions are reversed:
 $\mathbf{T}_P = \gamma(1, -\boldsymbol{\beta})$

Do a Charge Conjugation Operation: C
 Charge Conjugation actually changes all internal quantum #: charge, lepton #, etc.
 Feynman showed this is the equivalent of a world-line reversal & complex-conjugation:
 $\mathbf{T}_C = \gamma(-1, -\boldsymbol{\beta})^*$

Pairwise combinations:
 $\mathbf{T}_{TP} = \mathbf{T}_{PT} = \mathbf{T}_C = \gamma(-1, -\boldsymbol{\beta})^*$
 $\mathbf{T}_{TC} = \mathbf{T}_{CT} = \mathbf{T}_P = \gamma(1, -\boldsymbol{\beta})$
 $\mathbf{T}_{PC} = \mathbf{T}_{CP} = \mathbf{T}_T = \gamma(-1, \boldsymbol{\beta})^*$, a CP event is mathematically the same as a T event
 $\mathbf{T}_{CPT} = \mathbf{T} = \gamma(1, \boldsymbol{\beta})$ $\mathbf{T}_{CC} = \mathbf{T} = \gamma(1, \boldsymbol{\beta})$ $\mathbf{T}_{PP} = \mathbf{T} = \gamma(1, \boldsymbol{\beta})$ $\mathbf{T}_{TT} = \mathbf{T} = \gamma(1, \boldsymbol{\beta})$

It is only the combination of all three ops: {C,P,T}, or pairs of singles: {CC}, {PP}, {TT} that leave the Unit-Temporal 4-Vector, and thus the Phase, Invariant.

Matter-like
 $\mathbf{T} = \gamma(1, \boldsymbol{\beta})$
 $\mathbf{T} \cdot \mathbf{T} = \gamma(1, \boldsymbol{\beta})^* \cdot \gamma(1, \boldsymbol{\beta}) = \gamma^2(1^2 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}) = 1$: It's a temporal 4-vector

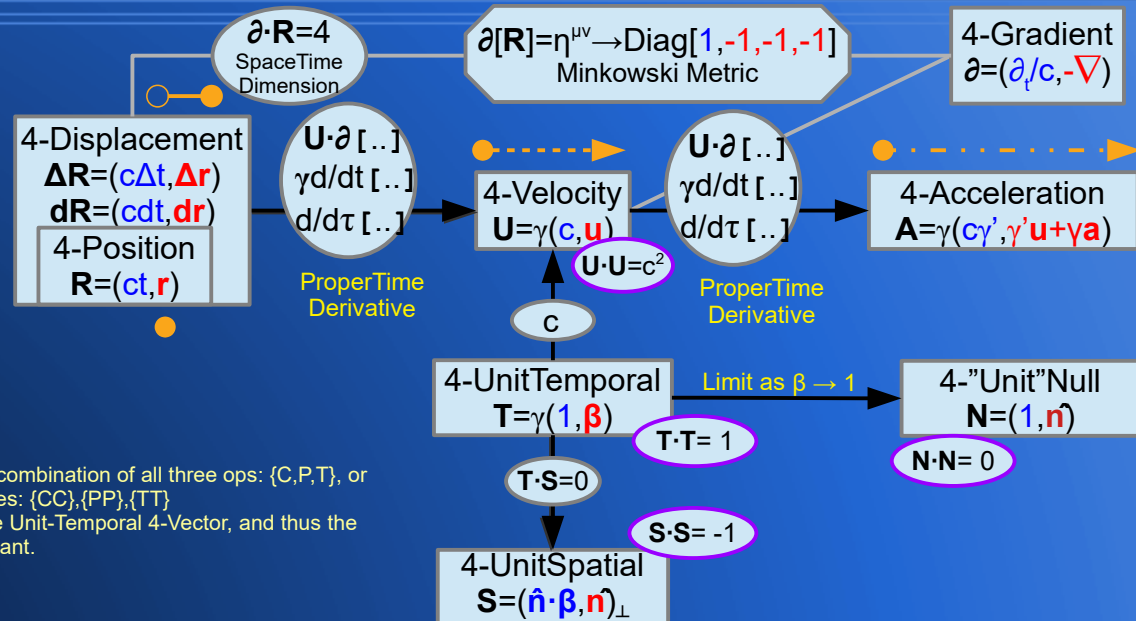
$\mathbf{T}_C \cdot \mathbf{T}_C = \gamma(-1, -\boldsymbol{\beta}) \cdot \gamma(-1, -\boldsymbol{\beta})^* = \gamma^2((-1)^2 - (-\boldsymbol{\beta}) \cdot (-\boldsymbol{\beta})) = \gamma^2(1^2 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}) = 1$
 $\mathbf{T}_P \cdot \mathbf{T}_P = \gamma(1, -\boldsymbol{\beta}) \cdot \gamma(1, -\boldsymbol{\beta})^* = \gamma^2(1^2 - (-\boldsymbol{\beta}) \cdot (-\boldsymbol{\beta})) = \gamma^2(1^2 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}) = 1$
 $\mathbf{T}_T \cdot \mathbf{T}_T = \gamma(-1, \boldsymbol{\beta}) \cdot \gamma(-1, \boldsymbol{\beta})^* = \gamma^2((-1)^2 - (\boldsymbol{\beta}) \cdot (\boldsymbol{\beta})) = \gamma^2(1^2 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}) = 1$
 They all remain temporal 4-vectors

$\mathbf{T}_{CPT} = \mathbf{T} = \gamma(1, \boldsymbol{\beta})$
 $\mathbf{T}_{CPT} \cdot \mathbf{T}_{CPT} = \mathbf{T} \cdot \mathbf{T} = 1$

Light-like/Photonic
 $\mathbf{N} = (1, \mathbf{n})$
 $\mathbf{N} \cdot \mathbf{N} = (1, \mathbf{n})^* \cdot (1, \mathbf{n}) = (1^2 - \mathbf{n} \cdot \mathbf{n}) = (1-1) = 0$: It's a null 4-vector

$\mathbf{N}_C \cdot \mathbf{N}_C = (-1, -\mathbf{n}) \cdot (-1, -\mathbf{n})^* = ((-1)^2 - (-\mathbf{n}) \cdot (-\mathbf{n})) = (1^2 - \mathbf{n} \cdot \mathbf{n}) = (1-1) = 0$
 $\mathbf{N}_P \cdot \mathbf{N}_P = (1, -\mathbf{n}) \cdot (1, -\mathbf{n})^* = (1^2 - (-\mathbf{n}) \cdot (-\mathbf{n})) = (1^2 - \mathbf{n} \cdot \mathbf{n}) = (1-1) = 0$
 $\mathbf{N}_T \cdot \mathbf{N}_T = (-1, \mathbf{n}) \cdot (-1, \mathbf{n})^* = ((-1)^2 - (\mathbf{n}) \cdot (\mathbf{n})) = (1^2 - \mathbf{n} \cdot \mathbf{n}) = (1-1) = 0$
 They all remain null 4-vectors

$\mathbf{N}_{CPT} = \mathbf{N} = (1, \mathbf{n})$
 $\mathbf{N}_{CPT} \cdot \mathbf{N}_{CPT} = \mathbf{N} \cdot \mathbf{N} = 0$



SR 4-Tensor
 (2,0)-Tensor $T^{\mu\nu}$
 (1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
 (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
 (1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
 (0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

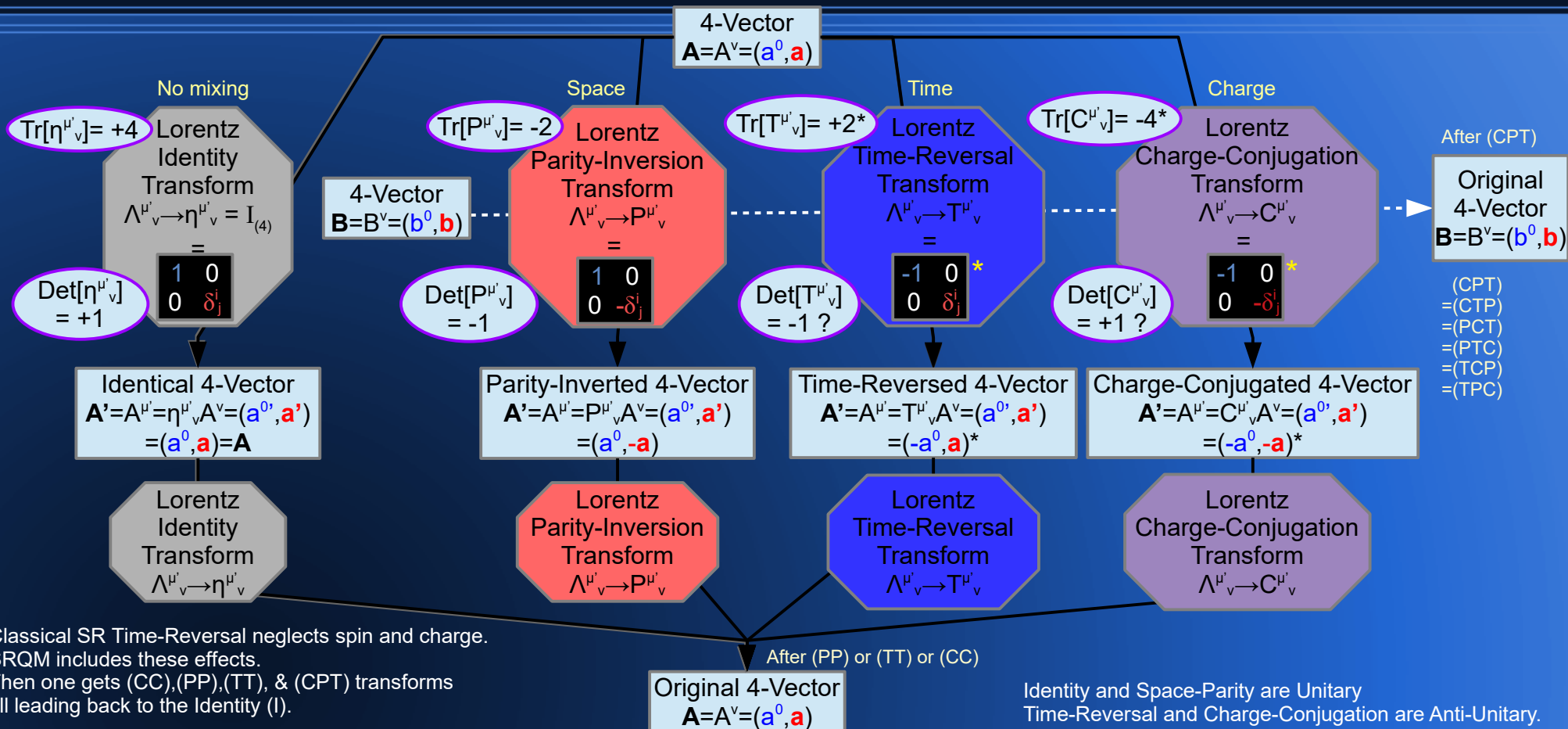
SR 4-Scalar
 (0,0)-Tensor S
 Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V} \cdot \mathbf{V} = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM: CPT Theorem (Charge) vs (Parity) vs (Time)

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SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$
SR 4-CoVector
(0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}_{\mu} = T$
 $\mathbf{V}\cdot\mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v}\cdot\mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM Transforms: Venn Diagram

Poincaré = Lorentz + Translations

(10)

(6)

(4)

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Transformations

(# of independent parameters = # continuous symmetries = # Lie Dimensions)

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation
Lie group of all affine isometries of SR:Minkowski TimeSpace (preserve quadratic form $\eta_{\mu\nu}$)
General Linear, Affine Transform $X^\mu = \Lambda^\mu_\nu X^\nu + \Delta X^\mu$ with $\text{Det}[\Lambda^\mu_\nu] = \pm 1$
(6+4=10)

Lorentz Transform Λ^μ_ν

(3+3=6) 4-Tensor {mixed type-(1,1)}

Discrete

Time-reversal
 $\Lambda^\mu_\nu \rightarrow T^\mu_\nu$
(0)
 $t \rightarrow -t^*$
time parity
anti-unitary

Spatial Flip Combs
 $\Lambda^\mu_\nu \rightarrow F^\mu_\nu$
(0)
 $\{x|y|z\} \rightarrow -\{x|y|z\}$
unitary

Parity-Inversion
 $\Lambda^\mu_\nu \rightarrow P^\mu_\nu$
(0)
 $\mathbf{r} \rightarrow -\mathbf{r}$
space parity
unitary

Charge-Conjugation
 $\Lambda^\mu_\nu \rightarrow C^\mu_\nu$
(0)
 $\mathbf{R} \rightarrow -\mathbf{R}^*, q \rightarrow -q$
charge parity
anti-unitary

Identity $I_{(4)}$
 $\Lambda^\mu_\nu \rightarrow \eta^\mu_\nu = \delta^\mu_\nu$
(0)
no mixing
unitary

CPT Symmetry
{Charge}
{Parity}
{Time}

Continuous

Rotation
 $\Lambda^\mu_\nu \rightarrow R^\mu_\nu$
(3)
 $x:y | x:z | y:z$

Boost
 $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$
(3)
 $t:x | t:y | t:z$

Isotropy
{same all directions}

Translation Transform ΔX^μ

(1+3=4) 4-Vector

Discrete

4-Zero
 $\Delta X^\mu \rightarrow (0,0)$
(0)
no motion

Continuous

Temporal
 $\Delta X^\mu \rightarrow (c\Delta t, \mathbf{0})$
(1)
 Δt

Spatial
 $\Delta X^\mu \rightarrow (0, \Delta \mathbf{x})$
(3)
 $\Delta x | \Delta y | \Delta z$

Homogeneity
{same all points}

	M^{01}	M^{02}	M^{03}		P^0
M^{10}		M^{12}	M^{13}		P^1
M^{20}	M^{21}		M^{23}		P^2
M^{30}	M^{31}	M^{32}			P^3

4-AngularMomentum $M^{\mu\nu} = X^\mu \wedge P^\nu = X^\mu P^\nu - X^\nu P^\mu$
= Generator of Lorentz Transformations (6)
= { $\Lambda^\mu_\nu \rightarrow R^\mu_\nu$ Rotations (3) + $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$ Boosts (3) }

4-LinearMomentum P^μ
= Generator of Translation Transformations (4)
= { $\Delta X^\mu \rightarrow (c\Delta t, \mathbf{0})$ Time (1) + $\Delta X^\mu \rightarrow (0, \Delta \mathbf{x})$ Space (3) }

$\text{Det}[\Lambda^\mu_\nu] = +1$ for Proper Lorentz Transforms
 $\text{Det}[\Lambda^\mu_\nu] = -1$ for Improper Lorentz Transforms

Lorentz Matrices can be generated by a matrix M with $\text{Tr}[M]=0$ which gives:
{ $\Lambda = e^\wedge M = e^\wedge (+\theta \cdot \mathbf{J} - \zeta \cdot \mathbf{K})$ }
{ $\Lambda^T = (e^\wedge M)^T = e^\wedge M^T$ }
{ $\Lambda^{-1} = (e^\wedge M)^{-1} = e^\wedge -M$ }



$M = +\theta \cdot \mathbf{J} - \zeta \cdot \mathbf{K}$
 $B[\zeta] = e^\wedge(-\zeta \cdot \mathbf{K})$
 $R[\theta] = e^\wedge(+\theta \cdot \mathbf{J})$
 $\Lambda = e^\wedge M = e^\wedge (+\theta \cdot \mathbf{J} - \zeta \cdot \mathbf{K})$

SR:Lorentz Transform

$\partial_\nu[R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu$
 $\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu$
 $\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$
Det $[\Lambda^\mu_\nu] = \pm 1$ **$\Lambda_{\mu\nu} \Lambda^{\mu\nu} = 4$**

Rotations $J_i = -\epsilon_{imn} M^{mn} / 2$, Boosts $K_i = M_{i0}$

[$(\mathbf{R} \rightarrow -\mathbf{R}^*)$] or [$(t \rightarrow -t^*)$ & $(\mathbf{r} \rightarrow -\mathbf{r})$] imply $q \rightarrow -q$
Feynman-Stueckelberg Interpretation
Amusingly, Inhomogeneous Lorentz adds homogeneity.

Hermitian Generators

Noether's Theorem - Continuity

The Hermitian Generators that lead to translations and rotations via unitary operators in QM...

These all ultimately come from the Poincaré Invariance → Lorentz Invariance that is at the heart of SR and Minkowski Space.

Infinitesimal Unitary Transformation

$$\hat{U}_\varepsilon(\hat{\mathbf{G}}) = \mathbf{I} + i\varepsilon\hat{\mathbf{G}}$$

Finite Unitary Transformation

$$\hat{U}_\alpha(\hat{\mathbf{G}}) = e^{i\alpha\hat{\mathbf{G}}}$$

let $\hat{\mathbf{G}} = \mathbf{P}/\hbar = \mathbf{K}$

let $\alpha = \Delta\mathbf{x}$

$$\hat{U}_{\Delta\mathbf{x}}(\mathbf{P}/\hbar)\Psi(\mathbf{X}) = e^{i\Delta\mathbf{x}\cdot\mathbf{P}/\hbar}\Psi(\mathbf{X}) = e^{i\Delta\mathbf{x}\cdot\hbar\mathbf{K}}\Psi(\mathbf{X}) = \Psi(\mathbf{X} - \Delta\mathbf{x})$$

$$\text{Time component: } \hat{U}_{\Delta ct}(\mathbf{P}/\hbar)\Psi(ct) = e^{i\Delta ct E/\hbar}\Psi(ct) = e^{i\Delta ct \partial_t}\Psi(ct) = \Psi(ct - c\Delta t) = c\Psi(t - \Delta t)$$

$$\text{Space component: } \hat{U}_{\Delta\mathbf{x}}(\mathbf{p}/\hbar)\Psi(\mathbf{x}) = e^{i\Delta\mathbf{x}\cdot\mathbf{p}/\hbar}\Psi(\mathbf{x}) = e^{i\Delta\mathbf{x}\cdot\nabla}\Psi(\mathbf{x}) = \Psi(\mathbf{x} + \Delta\mathbf{x})$$

By Noether's Theorem, this leads to $\partial\cdot\mathbf{K} = 0$

We had already calculated

$$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}] = ((\partial_t/c)^2 - \nabla\cdot\nabla)(\omega t - \mathbf{k}\cdot\mathbf{x}) = 0$$

$$(\partial\cdot\partial)[\mathbf{K}\cdot\mathbf{X}] = \partial\cdot(\partial[\mathbf{K}\cdot\mathbf{X}]) = \partial\cdot\mathbf{K} = 0$$

Poincaré Invariance also gives the Casimir invariants of mass and spin, and ultimately leads to the spin-statistics theorem of RQM.

QM Correspondence Principle: Analogous to the GR and SR limits

Basically, the old school QM Correspondence Principle says that QM should give the same results as classical physics in the realm of large quantum systems, i.e. where macroscopic behavior overwhelms quantum effects. Perhaps a better way to state it is when the change of system by a single quantum has a negligible effect on the overall state.

There is a way to derive this limit, by using Hamilton-Jacobi Theory:

$(i\hbar\partial_t)\Psi \sim [V - (\hbar\nabla_T)^2/2m_0]\Psi$: The Schrödinger NRQM Equation for a point particle (non-relativistic QM)

Examine solutions of form $\Psi = \Psi_0 e^{i\Phi} = \Psi_0 e^{iS/\hbar}$, where S is the QM Action
 $\partial_t[\Psi] = (i/\hbar)\Psi\partial_t[S]$ and $\partial_x[\Psi] = (i/\hbar)\Psi\partial_x[S]$ and $\nabla^2[\Psi] = (i/\hbar)\Psi\nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2$

$$(i\hbar)(i/\hbar)\Psi\partial_t[S] = V\Psi - (\hbar^2/2m_0)((i/\hbar)\Psi\nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2)$$

$$(i)(i)\Psi\partial_t[S] = V\Psi - ((i\hbar/2m_0)\Psi\nabla^2[S] - (\Psi/2m_0)(\nabla[S])^2)$$

$$\partial_t[S] = -V + (i\hbar/2m_0)\nabla^2[S] - (1/2m_0)(\nabla[S])^2$$

$$\partial_t[S] + [V + (1/2m_0)(\nabla[S])^2] = (i\hbar/2m_0)\nabla^2[S] : \text{Quantum Single Particle Hamilton-Jacobi}$$

$$\partial_t[S] + [V + (1/2m_0)(\nabla[S])^2] = 0 : \text{Classical Single Particle Hamilton-Jacobi}$$

Thus, the classical limiting case is:

$$\nabla^2[\Phi] \ll (\nabla[\Phi])^2$$

$$\hbar\nabla^2[S] \ll (\nabla[S])^2$$

$$\hbar\nabla \cdot \mathbf{p} \ll (\mathbf{p} \cdot \mathbf{p})$$

$$(\rho\lambda)\nabla \cdot \mathbf{p} \ll (\mathbf{p} \cdot \mathbf{p})$$

QM Correspondence Principle: Analogous to the GR and SR limits

$\partial_t[S] + [V + (1/2m_0)(\nabla[S])^2] = (i\hbar/2m_0)\nabla^2[S]$: Quantum Single Particle Hamilton-Jacobi

$\partial_t[S] + [V + (1/2m_0)(\nabla[S])^2] = 0$: Classical Single Particle Hamilton-Jacobi

Thus, the quantum → classical limiting-case is: {all equivalent representations}

$$\begin{array}{ll} \hbar \nabla^2[S_{\text{action}}] \ll (\nabla[S_{\text{action}}])^2 & \nabla^2[\Phi_{\text{phase}}] \ll (\nabla[\Phi_{\text{phase}}])^2 \\ \hbar \nabla \cdot \nabla[S_{\text{action}}] \ll (\nabla[S_{\text{action}}])^2 & \nabla \cdot \nabla[\Phi_{\text{phase}}] \ll (\nabla[\Phi_{\text{phase}}])^2 \\ \hbar \nabla \cdot \mathbf{p} \ll (\mathbf{p} \cdot \mathbf{p}) & \nabla \cdot \mathbf{k} \ll (\mathbf{k} \cdot \mathbf{k}) \\ (\rho\lambda) \nabla \cdot \mathbf{p} \ll (\mathbf{p} \cdot \mathbf{p}) & \end{array}$$

with

$$\mathbf{P} = (E/c, \mathbf{p}) = -\partial[S_{\text{action}}] = -(\partial_t/c, -\nabla)[S_{\text{action}}] = (-\partial_t/c, \nabla)[S_{\text{action}}]$$

$$\mathbf{K} = (\omega/c, \mathbf{k}) = -\partial[\Phi_{\text{phase}}] = -(\partial_t/c, -\nabla)[\Phi_{\text{phase}}] = (-\partial_t/c, \nabla)[\Phi_{\text{phase}}]$$

It is analogous to GR → SR in limit of low curvature (low mass), or SR → CM in limit of low velocity { $|v| \ll c$ }.
It still applies, but is now understood as the same type of limiting-case as these others.

Note The commonly seen form of ($c \rightarrow \infty, \hbar \rightarrow 0$) as limits are incorrect!

c and \hbar are universal constants – they never change.

If $c \rightarrow \infty$, then photons (light-waves) would have infinite energy { $E = pc$ }. This is not true classically.

If $\hbar \rightarrow 0$, then photons (light-waves) would have zero energy { $E = \hbar\omega$ }. This is not true classically.

Always better to write the SR Classical limit as { $|v| \ll c$ }, the QM Classical limit as { $\nabla^2[\Phi_{\text{phase}}] \ll (\nabla[\Phi_{\text{phase}}])^2$ }

Again, it is more natural to find a limiting-case of a more general system than to try to unite two separate theories which may or may not ultimately be compatible. From logic, there is always the possibility to have a paradox result from combination of arbitrary axioms, whereas deductions from a single true axiom will always give true results.

This page needs some
work. Source was from
Goldstein

SRQM: 4-Vector Quantum Probability Conservation of Probability Density

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson

Conservation of Probability : Probability Current : Charge Current
Consider the following purely mathematical argument
(based on Green's Vector Identity):

$\partial \cdot (f \partial[g] - \partial[f] g) = f \partial \cdot \partial[g] - \partial \cdot \partial[f] g$
with (f) and (g) as SR Lorentz Scalar functions

Proof:

$$\begin{aligned} & \partial \cdot (f \partial[g] - \partial[f] g) \\ &= \partial \cdot (f \partial[g]) - \partial \cdot (\partial[f] g) \\ &= (f \partial \cdot \partial[g] + \partial[f] \cdot \partial[g]) - (\partial[f] \cdot \partial[g] + \partial \cdot \partial[f] g) \\ &= f \partial \cdot \partial[g] - \partial \cdot \partial[f] g \end{aligned}$$

We can also multiply this by a Lorentz Invariant Scalar Constant s
 $s (f \partial \cdot \partial[g] - \partial \cdot \partial[f] g) = s \partial \cdot (f \partial[g] - \partial[f] g) = \partial \cdot s (f \partial[g] - \partial[f] g)$

Ok, so we have the math that we need...

Now, on to the physics... Start with the Klein-Gordon Eqn.

$$\begin{aligned} \partial \cdot \partial &= (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2 \\ \partial \cdot \partial + (m_0c/\hbar)^2 &= 0 \end{aligned}$$

Let it act on SR Lorentz Invariant function g

$$\partial \cdot \partial[g] + (m_0c/\hbar)^2[g] = 0 [g]$$

Then pre-multiply by f

$$\begin{aligned} [f] \partial \cdot \partial[g] + [f] (m_0c/\hbar)^2[g] &= [f] 0 [g] \\ [f] \partial \cdot \partial[g] + (m_0c/\hbar)^2[f]g &= 0 \end{aligned}$$

Now, subtract the two equations

$$\begin{aligned} \{ [f] \partial \cdot \partial[g] + (m_0c/\hbar)^2[f]g = 0 \} - \{ \partial \cdot \partial[f]g + (m_0c/\hbar)^2[f]g = 0 \} \\ [f] \partial \cdot \partial[g] + (m_0c/\hbar)^2[f]g - \partial \cdot \partial[f]g - (m_0c/\hbar)^2[f]g = 0 \\ [f] \partial \cdot \partial[g] - \partial \cdot \partial[f]g = 0 \end{aligned}$$

And as we noted from the mathematical Green's Vector identity at the start...

$$[f] \partial \cdot \partial[g] - \partial \cdot \partial[f]g = \partial \cdot (f \partial[g] - \partial[f] g) = 0$$

Therefore,

$$\begin{aligned} s \partial \cdot (f \partial[g] - \partial[f] g) &= 0 \\ \partial \cdot s (f \partial[g] - \partial[f] g) &= 0 \end{aligned}$$

Thus, there is a conserved current 4-Vector, $\mathbf{J}_{\text{prob}} = s (f \partial[g] - \partial[f] g)$, for which $\partial \cdot \mathbf{J}_{\text{prob}} = 0$, and which also solves the Klein-Gordon equation.

Let's choose as before ($\partial = -i\mathbf{K}$) with a plane wave function $f = ae^{-i(\mathbf{K} \cdot \mathbf{X})} = \psi$, and choose $g = f^* = ae^{i(\mathbf{K} \cdot \mathbf{X})} = \psi^*$ as its complex conjugate.

At this point, I am going to choose $s = (i\hbar/2m_0)$, which is Lorentz Scalar Invariant, in order to make the probability have dimensionless units and be normalized to unity in the rest case.

4-Vector Quantum Probability

4-ProbabilityFlux, Klein-Gordon RQM Eqn

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson

4-ProbabilityCurrentDensity, a.k.a. 4-ProbabilityFlux

$$\mathbf{J}_{\text{prob}} = (c\rho_{\text{prob}}, \mathbf{j}_{\text{prob}}) = (i\hbar/2m_0)(\psi^* \partial[\psi] - \partial[\psi^*] \psi) = (\rho_{\text{prob}}, \mathbf{u}) = (\rho_{\text{prob}}, \gamma(\mathbf{c}, \mathbf{u})) = (\gamma\rho_{\text{prob}}, \mathbf{c}, \mathbf{u}) = (\rho_{\text{prob}}, \mathbf{c}, \mathbf{u})$$

with 4-Divergence of Probability $\{ \partial \cdot \mathbf{J}_{\text{prob}} = 0 \}$ by construction via Green's Vector Identity and the Klein-Gordon RQM Eqn.

The reason for $s = (i\hbar/2m_0)$ becomes more clear by examining our diagram:

Start at the 4-Gradient and follow the arrows toward the 4-ProbabilityFlux

You immediately see where the $(i\hbar/m_0)$ factor comes from.

The $\rho_{\text{prob},0}$ is then a function of the ψ 's divided by 2.

$$\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla$$

d'Alembertian
 $\partial \cdot \partial = -(m_0 c/\hbar)^2$
Klein-Gordon

4-Gradient
 $\partial = (\partial_t/c, -\nabla)$

Complex Plane-waves
 $\mathbf{K} = i\partial$

4-WaveVector
 $\mathbf{K} = (\omega/c, \mathbf{k})$
 $\mathbf{K} \cdot \mathbf{K} = (\omega/c)^2$
 $\mathbf{K} \cdot \mathbf{K} = (m_0 c/\hbar)^2$

Einstein de Broglie
 $\mathbf{P} = \hbar \mathbf{K}$

4-Momentum
 $\mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p})$
 $\mathbf{P} \cdot \mathbf{P} = (m_0 c)^2 = (E_0/c)^2$

4-Velocity
 $\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$
 $\mathbf{U} \cdot \mathbf{U} = c^2$

Rest Number Density
 n_0
 $\rho_{\text{prob},0} = \chi^* \psi$
 $= |\langle \chi | \psi \rangle|^2$
Born Probability Rule

4-NumberFlux
 $\mathbf{N} = (nc, \mathbf{n}) = n(\mathbf{c}, \mathbf{u})$
4-ProbCurrentDensity
4-ProbabilityFlux
 $\mathbf{J}_{\text{prob}} = (\rho_{\text{prob}}, c\mathbf{j}_{\text{prob}}) = \rho_{\text{prob}}(\mathbf{c}, \mathbf{u})$
 $= (i\hbar/2m_0)(\psi^* \partial[\psi] - \partial[\psi^*] \psi)$
Complex

4-ChargeFlux
4-CurrentDensity
 $\mathbf{J} = (pc, \mathbf{j}) = \rho(\mathbf{c}, \mathbf{u})$
 $\mathbf{J} \cdot \mathbf{J} = (\rho_0 c)^2$

$\partial \cdot \mathbf{N} = 0 : \partial \cdot \mathbf{J}_{\text{prob}} = 0$
Conservation of Particle # : Probability

$\partial \cdot \mathbf{J} = 0$
Conservation of Charge

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SR 4-Scalar
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Lorentz Scalar

Existing SR Rules
Quantum Principles

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_0)^2 = \text{Lorentz Scalar}$

Examine the temporal component, the Relativistic Probability Density
 $\rho_{\text{prob}} = (i\hbar/2m_0 c^2)(\psi^* \partial_t[\psi] - \partial_t[\psi^*] \psi)$
Assume wave solution in following general form:
 $\{ \psi = A f[\mathbf{k}] e(-i\omega t) \}$
 $\{ \psi^* = A^* f[\mathbf{k}]^* e(+i\omega t) \}$
then
 $\{ \partial_t[\psi] = (-i\omega) A f[\mathbf{k}] e(-i\omega t) = (-i\omega) \psi \}$
 $\{ \partial_t[\psi^*] = (+i\omega) A^* f[\mathbf{k}]^* e(+i\omega t) = (+i\omega) \psi^* \}$
then
 $\rho_{\text{prob}} = (i\hbar/2m_0 c^2)(\psi^* \partial_t[\psi] - \partial_t[\psi^*] \psi)$
 $\rho_{\text{prob}} = (i\hbar/2m_0 c^2)((-i\omega)\psi^* \psi - (+i\omega)\psi^* \psi)$
 $\rho_{\text{prob}} = (i\hbar/2m_0 c^2)((-2i\omega)\psi^* \psi)$
 $\rho_{\text{prob}} = (\hbar\omega/m_0 c^2)(\psi^* \psi)$
 $\rho_{\text{prob}} = (\hbar\omega_0/m_0 c^2)(\psi^* \psi)$
 $\rho_{\text{prob}} = (\gamma)(\psi^* \psi) = (\gamma)(\rho_{\text{prob},0})$
Finally, multiply by charge (q) to get standard SR EM
4-CurrentDensity = 4-ChargeFlux = $\mathbf{J} = (pc, \mathbf{j}) = q\mathbf{J}_{\text{prob}} = q(c\rho_{\text{prob}}, \mathbf{j}_{\text{prob}})$

4-Vector Quantum Probability

4-Probability Flux, Klein-Gordon RQM Eqn with Minimal Coupling

A Tensor Study of Physical 4-Vectors

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4-Probability Current Density, a.k.a. 4-Probability Flux

$$\mathbf{J}_{\text{prob}} = (c\rho_{\text{prob}}, \mathbf{j}_{\text{prob}}) = (i\hbar/2m_0)(\psi^*\partial[\psi] - \partial[\psi^*]\psi) = (\rho_{\text{prob}}, \mathbf{u}) = (\rho_{\text{prob}}, \gamma(\mathbf{c}, \mathbf{u})) = (\gamma\rho_{\text{prob}}, \gamma(\mathbf{c}, \mathbf{u})) = (\rho_{\text{prob}}, \mathbf{c}, \mathbf{u})$$

with 4-Divergence of Probability $\{\partial \cdot \mathbf{J}_{\text{prob}} = 0\}$ by construction via Green's Vector Identity and the Klein-Gordon RQM Eqn.

If we include minimal coupling:

$$\mathbf{J}_{\text{prob}} = (c\rho_{\text{prob}}, \mathbf{j}_{\text{prob}}) = (i\hbar/2m_0)(\psi^*\partial[\psi] - \partial[\psi^*]\psi) + (q/m_0)(\psi^*\psi)\mathbf{A}$$

Start at **A** on the chart

Follow past (q) factor to get to **Q** = q**A**

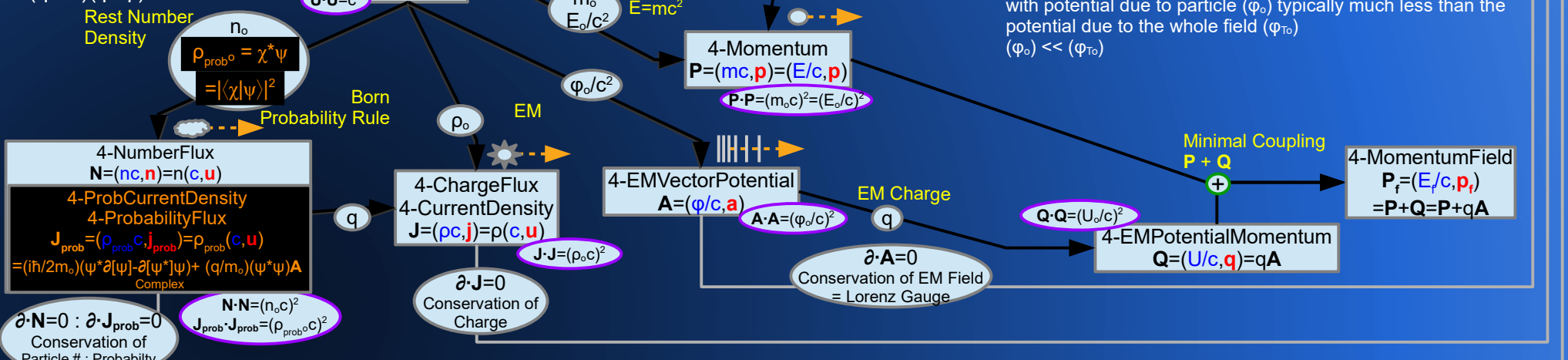
Minimal Coupling allows passage back to **P** with no factors

Follow back past (1/m₀) to get to **U**

Follow past Born Rule ($\psi^*\psi$)

Now have the additional factor:

$$+ (q/m_0)(\psi^*\psi)\mathbf{A}$$



An alternate way would be to take **A** to **U** via the direct route:
 $+ (c^2/\phi_{T_0})(\psi^*\psi)\mathbf{A}$
which would lead to a term like
 $\rho_{\text{prob}} \rightarrow (\gamma)(\psi^*\psi) + (\gamma)(\phi_0/\phi_{T_0})(\psi^*\psi) = (\gamma)[1 + \phi_0/\phi_{T_0}](\psi^*\psi)$
with potential due to particle (ϕ_0) typically much less than the potential due to the whole field (ϕ_{T_0})
($\phi_0 \ll \phi_{T_0}$)

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Existing SR Rules
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4-Vector Quantum Probability

Newtonian Limit

4-ProbabilityCurrentDensity $\mathbf{J}_{\text{prob}} = (c\rho_{\text{prob}}, \mathbf{j}_{\text{prob}}) = (i\hbar/2m_0)(\psi^*\partial[\psi]-\partial[\psi^*]\psi) + (q/m_0)(\psi^*\psi)\mathbf{A}$

Examine the temporal component:

$$\rho_{\text{prob}} = (i\hbar/2m_0c^2)(\psi^*\partial_t[\psi]-\partial_t[\psi^*]\psi) + (q/m_0)(\psi^*\psi)(\phi/c^2)$$

$$\rho_{\text{prob}} \rightarrow (\gamma)(\psi^*\psi) + (\gamma)(q\phi_0/m_0c^2)(\psi^*\psi) = (\gamma)[1 + q\phi_0/E_0](\psi^*\psi)$$

Typically, the particle EM potential energy ($q\phi_0$) is much less than the particle rest energy (E_0), else it could generate new particles. So, take ($q\phi_0 \ll E_0$), which gives the EM factor ($q\phi_0/E_0$) ~ 0

Now, taking the low-velocity limit ($\gamma \rightarrow 1$), $\rho_{\text{prob}} = \gamma[1 + \sim 0](\psi^*\psi)$, $\rho_{\text{prob}} \rightarrow (\psi^*\psi) = (\rho_{\text{prob}_0})$ for $|\mathbf{v}| \ll c$

The Standard Born Probability Interpretation, $(\psi^*\psi) = (\rho_{\text{prob}_0})$, only applies in the low-potential-energy & low-velocity limit

This is why the {non-positive-definite} probabilities and {probabilities} > 1 in the RQM Klein-Gordon equation gave physicists fits, and is the reason why one must regard the probabilities as charge conservation instead.

The original definition from SR is Continuity of Worldlines, $\partial \cdot \mathbf{J}_{\text{prob}} = 0$, for which all is good and well in the RQM version.

The definition says there are no external sources or sinks of probability = conservation of probability.

The Born idea that $(\rho_{\text{prob}_0}) \rightarrow \text{Sum}[(\psi^*\psi)] = 1$ is just the Low-Velocity QM limit.

Only the non-EM rest version $(\rho_{\text{prob}_0}) = \text{Sum}[(\psi^*\psi)] = 1$ is true.

It is not a fundamental axiom, it is an emergent property which is valid only in the NRQM limit

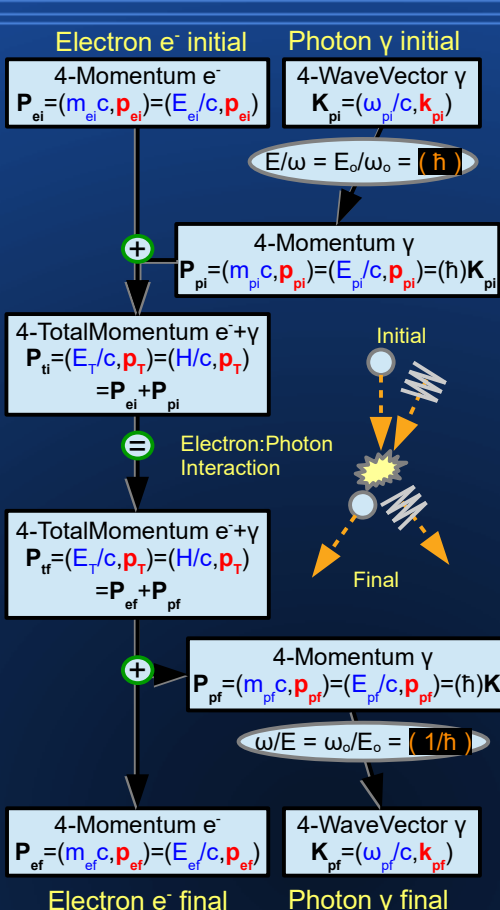
We now multiply by charge (q) to instead get a

4-"Charge"CurrentDensity $\mathbf{J} = (c\rho, \mathbf{j}) = q\mathbf{J}_{\text{prob}} = q(c\rho_{\text{prob}}, \mathbf{j}_{\text{prob}})$, which is the standard SR EM 4-CurrentDensity

SRQM 4-Vector Study: The QM Compton Effect

A Tensor Study of Physical 4-Vectors

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Compton Scattering Derivation : Compton Effect

$\mathbf{P} \cdot \mathbf{P} = (m_o c)^2$ generally → 0 for photons ($m_o=0$)

$\mathbf{P}_{phot1} \cdot \mathbf{P}_{phot2} = \hbar^2 \mathbf{K}_1 \cdot \mathbf{K}_2 = (\hbar^2 \omega_1 \omega_2 / c^2) (1 - \hat{n}_1 \cdot \hat{n}_2) = (\hbar^2 \omega_1 \omega_2 / c^2) (1 - \cos[\theta])$

$\mathbf{P}_{phot} \cdot \mathbf{P}_{mass} = \hbar \mathbf{K} \cdot \mathbf{P} = (\hbar \omega / c) (1, \hat{n}) \cdot (E/c, \mathbf{p}) = (\hbar \omega / c) (E/c - \hat{n} \cdot \mathbf{p}) = (\hbar \omega E_o / c^2) = (\hbar \omega m_o)$

$\mathbf{P}_{phot} + \mathbf{P}_{mass} = \mathbf{P}'_{phot} + \mathbf{P}'_{mass}$: 4-Momentum Conservation in Photon-Mass Interaction

$\mathbf{P}_{phot} + \mathbf{P}_{mass} - \mathbf{P}'_{phot} = \mathbf{P}'_{mass}$: rearrange

$(\mathbf{P}_{phot} + \mathbf{P}_{mass} - \mathbf{P}'_{phot})^2 = (\mathbf{P}'_{mass})^2$: square to get scalars

$(\mathbf{P}_{phot} \cdot \mathbf{P}_{phot} + 2\mathbf{P}_{phot} \cdot \mathbf{P}_{mass} - 2\mathbf{P}_{phot} \cdot \mathbf{P}'_{phot} + \mathbf{P}_{mass} \cdot \mathbf{P}_{mass} - 2\mathbf{P}_{mass} \cdot \mathbf{P}'_{mass} - 2\mathbf{P}_{mass} \cdot \mathbf{P}'_{phot} + \mathbf{P}'_{phot} \cdot \mathbf{P}'_{phot}) = (\mathbf{P}'_{mass})^2$

$(0 + 2\mathbf{P}_{phot} \cdot \mathbf{P}_{mass} - 2\mathbf{P}_{phot} \cdot \mathbf{P}'_{phot} + (m_o c)^2 - 2\mathbf{P}_{mass} \cdot \mathbf{P}'_{mass} + 0) = (m_o c)^2$

$\mathbf{P}_{phot} \cdot \mathbf{P}_{mass} - \mathbf{P}'_{phot} \cdot \mathbf{P}'_{mass} = \mathbf{P}'_{phot} \cdot \mathbf{P}'_{phot}$

$(\hbar \omega m_o) - (\hbar \omega' m_o) = (\hbar^2 \omega \omega' / c^2) (1 - \cos[\theta])$

$(\omega - \omega') / (\omega \omega') = (\hbar / m_o c^2) (1 - \cos[\theta])$

$(1/\omega' - 1/\omega) = (\hbar / m_o c^2) (1 - \cos[\theta])$

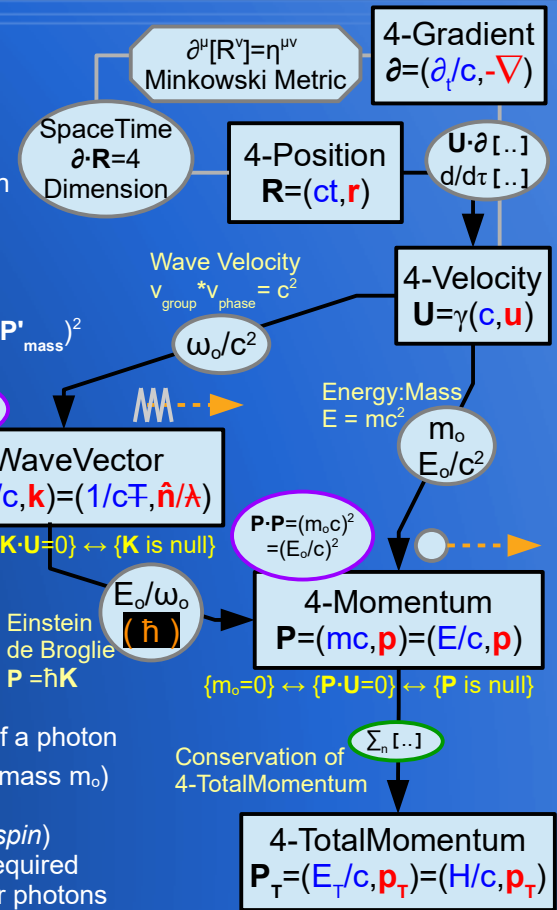
$\Delta \lambda = (\lambda' - \lambda) = (\hbar / m_o c) (1 - \cos[\theta]) = \lambda_c (1 - \cos[\theta])$

The Compton Effect: Compton Scattering

with $\lambda_c = \lambda_c / 2\pi = (\hbar / m_o c) =$ Reduced Compton Wavelength

$\lambda_c = (h / m_o c) =$ Compton Wavelength (not a rest-wavelength, but the wavelength of a photon with the energy equivalent to a massive particle of rest-mass m_o)

Calculates the wavelength shift of a photon scattering from an electron (*ignoring spin*)
Proves that light does not have a "wave-only" description, photon 4-Momentum required
 $E/\omega = \gamma E_o / \gamma \omega_o = E_o / \omega_o = \hbar$ $\mathbf{K}_{photon} = (\omega/c)(1, \mathbf{n}) = \text{null}$ $\{\omega \lambda = v \lambda = c\}$ for photons



SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or T_ν^μ (0,2)-Tensor $T_{\mu\nu}$	SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (\mathbf{v}_0, -\mathbf{v})$
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SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

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SRQM 4-Vector Study:

The QM Aharonov-Bohm Effect

QM Potential $\Delta\Phi_{pot} = -(q/\hbar) \int_{path} \mathbf{A} \cdot d\mathbf{X}$

A Tensor Study of Physical 4-Vectors

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Aharonov-Bohm Effect

The EM 4-Vector Potential gives the Aharonov-Bohm Effect.

$$\Phi_{pot} = -(q/\hbar) \mathbf{A} \cdot \mathbf{X} = -\mathbf{K}_{pot} \cdot \mathbf{X}$$

or taking the differential...

$$d\Phi_{pot} = -(q/\hbar) \mathbf{A} \cdot d\mathbf{X}$$

over a path...

$$\Delta\Phi_{pot} = \int_{path} d\Phi_{pot}$$

$$\Delta\Phi_{pot} = -(q/\hbar) \int_{path} \mathbf{A} \cdot d\mathbf{X}$$

$$\Delta\Phi_{pot} = -(q/\hbar) \int_{path} [(\varphi/c)(cdt) - \mathbf{a} \cdot d\mathbf{x}]$$

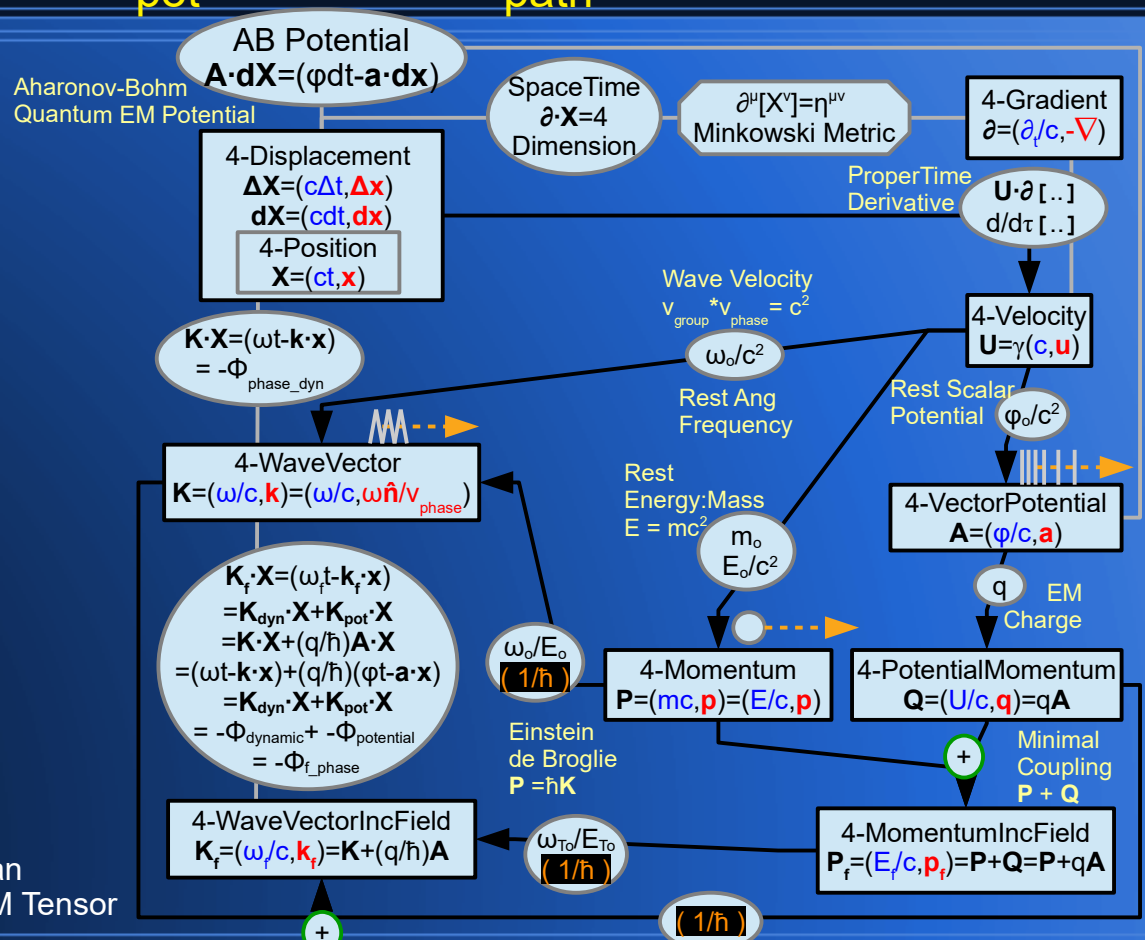
$$\Delta\Phi_{pot} = -(q/\hbar) \int_{path} (\varphi dt - \mathbf{a} \cdot d\mathbf{x})$$

Note that both the Electric and Magnetic effects come out by using the 4-Vector notation.

Electric AB effect: $\Delta\Phi_{pot_Elec} = -(q/\hbar) \int_{path} (\varphi dt)$

Magnetic AB effect: $\Delta\Phi_{pot_Mag} = + (q/\hbar) \int_{path} (\mathbf{a} \cdot d\mathbf{x})$

Proves that the 4-Vector Potential \mathbf{A} is more fundamental than \mathbf{e} and \mathbf{b} fields, which are just components of the Faraday EM Tensor



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SRQM 4-Vector Study:

The QM Josephson Junction Effect = SuperCurrent EM 4-Vector Potential $\mathbf{A} = -(\hbar/q)\partial[\Delta\Phi_{pot}]$

A Tensor Study of Physical 4-Vectors

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Josephson Effect

The EM 4-Vector Potential gives the Aharonov-Bohm Effect.

$$\text{Phase } \Phi_{pot} = -(q/\hbar)\mathbf{A} \cdot \mathbf{X} = -\mathbf{K}_{pot} \cdot \mathbf{X}$$

Rearrange the equation a bit:

$$-(\hbar/q)\Delta\Phi_{pot} = \mathbf{A} \cdot \Delta\mathbf{X}$$

$$\mathbf{A} \cdot \Delta\mathbf{X} = -(\hbar/q)\Delta\Phi_{pot}$$

$$d/d\tau[\mathbf{A} \cdot \Delta\mathbf{X}] = d/d\tau[-(\hbar/q)\Delta\Phi_{pot}] = d/d\tau[\mathbf{A}] \cdot \Delta\mathbf{X} + \mathbf{A} \cdot d/d\tau[\Delta\mathbf{X}] = d/d\tau[\mathbf{A}] \cdot \Delta\mathbf{X} + \mathbf{A} \cdot \mathbf{U}$$

Assume that $(d/d\tau[\mathbf{A}] \cdot \Delta\mathbf{X} \sim 0)$

$$[\mathbf{A} \cdot \mathbf{U}] = d/d\tau[-(\hbar/q)\Delta\Phi_{pot}]$$

$$[\mathbf{U} \cdot \mathbf{A}] = (\mathbf{U} \cdot \partial)[-(\hbar/q)\Delta\Phi_{pot}]$$

$$[\mathbf{A}] = -(\hbar/q)(\partial)[\Delta\Phi_{pot}]$$

$$\mathbf{A} = -(\hbar/q)\partial[\Delta\Phi_{pot}]$$

$$(\varphi/c, \mathbf{a}) = -(\hbar/q)(\partial_t/c, -\nabla)[\Delta\Phi_{pot}]$$

Which explains Josephson Effect criteria :

$\Delta\mathbf{X} \sim 0$: small gap

$d/d\tau[\mathbf{A}] \sim 0$: "critical current" & no voltage

$d/d\tau[\mathbf{A}] \cdot \Delta\mathbf{X} \sim$ orthogonal: ??

$$\mathbf{A} = (\hbar/q)\mathbf{K}; \mathbf{K} = (\omega/c, \mathbf{k}) = (q/\hbar)\mathbf{A} = (q/\hbar)(\varphi/c, \mathbf{a})$$

Take the temporal part:

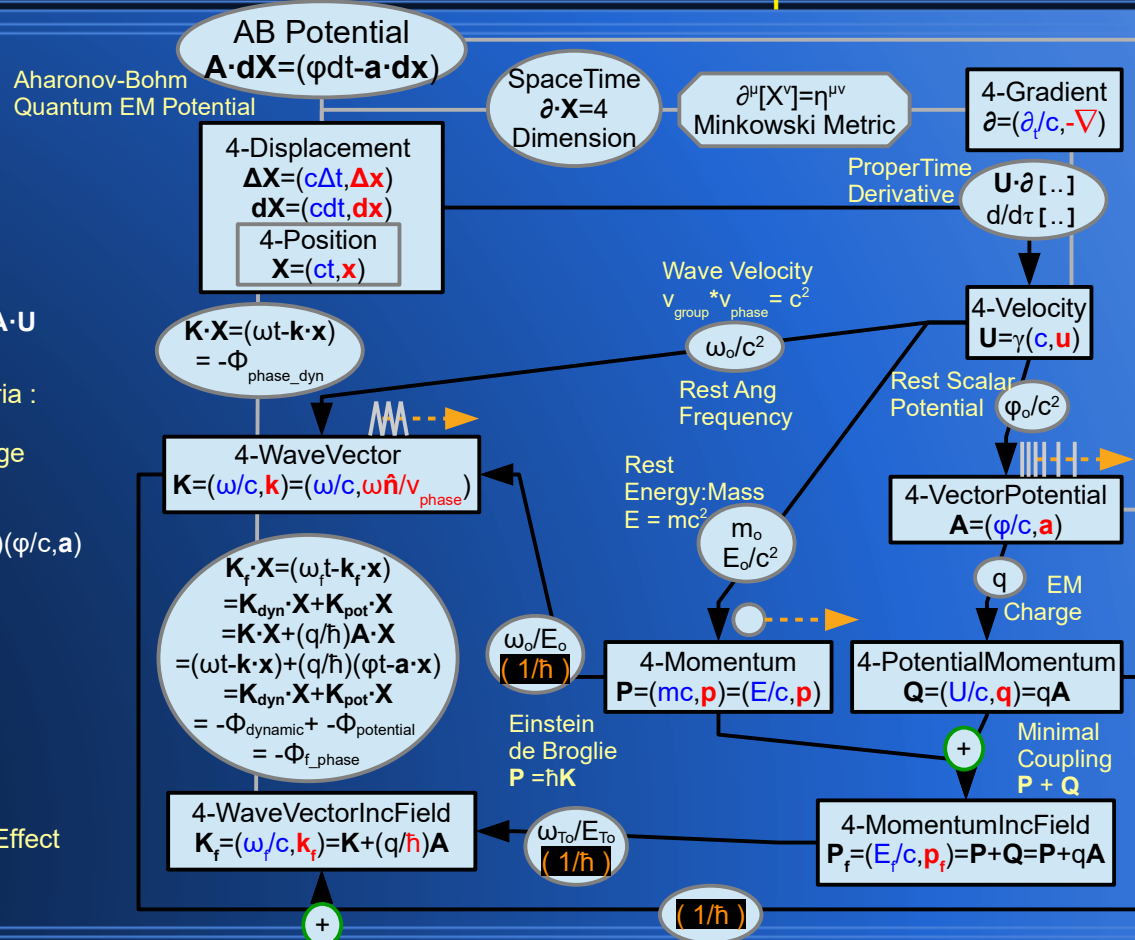
$$\text{EM Scalar Potential } \varphi = -(\hbar/q)(\partial_t)[\Delta\Phi_{pot}]; \omega = (q/\hbar)\varphi$$

If the charge (q) is a Cooper-electron-pair: $\{q = -2e\}$

$$\text{Voltage } V(t) = \varphi(t) = (\hbar/2e)(\partial/\partial t)[\Delta\Phi_{pot}]; \text{ AngFreq } \omega = -2eV/\hbar$$

This is the superconducting phase evolution equation of the Josephson Effect

$(\hbar/2e)$ is defined to be the Magnetic Flux Quantum Φ_0 .



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SRQM Symmetries: Hamilton-Jacobi vs Relativistic Action Josephson vs Aharonov-Bohm Differential (4-Vector) vs Integral (4-Scalar)

Differential Formats : 4-Vectors

Notice the Symmetry:

Integral Formats : 4-Scalars

SR Hamilton-Jacobi Equation

$$\mathbf{P}_T = -\partial[\Delta S_{\text{action}}]$$

$$\mathbf{P} + q\mathbf{A} = -\partial[\Delta S_{\text{action}}]$$

$$\mathbf{P} + q\mathbf{A} = -\partial[\hbar\Delta\Phi_{\text{phase}}]$$

$$\mathbf{P} + q\mathbf{A} = -\partial[\hbar\Delta\Phi_{\text{phase,dyn}} + (\hbar)\Delta\Phi_{\text{phase,pot}}]$$

SR Action Equation

$$\Delta S_{\text{action}} = -\int_{\text{path}} \mathbf{P}_T \cdot d\mathbf{X}$$

$$\Delta S_{\text{action}} = -\int_{\text{path}} (\mathbf{P} + q\mathbf{A}) \cdot d\mathbf{X}$$

$$\hbar\Delta\Phi_{\text{phase}} = -\int_{\text{path}} (\mathbf{P} + q\mathbf{A}) \cdot d\mathbf{X}$$

$$\hbar\Delta\Phi_{\text{phase,dyn}} + \hbar\Delta\Phi_{\text{phase,pot}} = -\int_{\text{path}} (\mathbf{P} + q\mathbf{A}) \cdot d\mathbf{X}$$

Inverse

Inverse

Inverse

Inverse

4-TotMom Conservation

$$\mathbf{P}_T = (\mathbf{P} + \mathbf{Q}) = (\mathbf{P} + q\mathbf{A})$$

Minimal Coupling

$$\mathbf{P} = (\mathbf{P}_T - q\mathbf{A}) = (\mathbf{P}_T - \mathbf{Q})$$

Dynamic Part

4-Momentum

$$\mathbf{P} = -\partial[\Delta S_{\text{act,dync}}]$$

$$-\partial[\hbar\Delta\Phi_{\text{phase,dynamic}}]$$

Dynamic Part

Action (free part)

$$\Delta S_{\text{act,dyn}} = \hbar\Delta\Phi_{\text{phase,dynamic}}$$

$$= -\int_{\text{path}} (\mathbf{P}) \cdot d\mathbf{X}$$

4-TotMom Conservation

$$\mathbf{P}_T = (\mathbf{P} + \mathbf{Q}) = (\mathbf{P} + q\mathbf{A})$$

Minimal Coupling

$$\mathbf{P} = (\mathbf{P}_T - q\mathbf{A}) = (\mathbf{P}_T - \mathbf{Q})$$

Potential Part

Action (potential part)

$$\Delta S_{\text{act,pot}} = \hbar\Delta\Phi_{\text{phase,potential}} = -\int_{\text{path}} (q\mathbf{A}) \cdot d\mathbf{X} = -\int_{\text{path}} (\mathbf{Q}) \cdot d\mathbf{X}$$

Potential Part

4-PotentialMomentum

$$\mathbf{Q} = q\mathbf{A} = -\partial[\Delta S_{\text{act,pot}}]$$

$$-\partial[\hbar\Delta\Phi_{\text{phase,potential}}]$$

Josephson Junction Relation

$$\mathbf{A} = -(\hbar/q)\partial[\Delta\Phi_{\text{potential}}]$$

$$= -(1/q)\partial[\Delta S_{\text{act,pot}}]$$

$$= \mathbf{Q}/q$$

Technically, the standard Josephson Junction uses just the temporal part $\{ \mathbf{A} = (\varphi/c, \mathbf{a}) \}$ & Cooper-pair-electrons $\{ q = -2e \}$ giving $V(t) = \varphi = (\hbar/2e)\partial/\partial t[\Delta\Phi_{\text{pot}}]$. There should be a spatial part as well.

Aharonov-Bohm Relation

$$\Delta\Phi_{\text{potential}} = -(q/\hbar)\int_{\text{path}} \mathbf{A} \cdot d\mathbf{X}$$

$$= -(1/\hbar)\int_{\text{path}} \mathbf{Q} \cdot d\mathbf{X}$$

$$= \Delta S_{\text{act,pot}}/\hbar$$

SR 4-Tensor

(2,0)-Tensor $T^{\mu\nu}$

(1,1)-Tensor T^{μ}_{ν} or T_{μ}^{ν}

(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector

(1,0)-Tensor $V^{\mu} = \mathbf{V} = (v^0, \mathbf{v})$

SR 4-CoVector

(0,1)-Tensor $V_{\mu} = (v_0, -\mathbf{v})$

SR 4-Scalar

(0,0)-Tensor S

Lorentz Scalar

Existing SR Rules
Quantum Principles

SRQM 4-Vector Study: Einstein-de Broglie The (\hbar) Connection

A Tensor Study of Physical 4-Vectors

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John B. Wilson

The \hbar Connection

$\mathbf{P} = \hbar\mathbf{K}$: Basic Einstein-de Broglie

$$\mathbf{P} + \mathbf{Q} = \mathbf{P} + \mathbf{Q}$$

$$\mathbf{P} + \mathbf{Q} = \hbar\mathbf{K}_{\text{dyn}} + \hbar\mathbf{K}_{\text{pot}}$$

$$\mathbf{P} + \mathbf{Q} = \hbar(\mathbf{K}_{\text{dyn}} + \mathbf{K}_{\text{pot}})$$

$$\text{Sum over } n \text{ particles: } \mathbf{P}_T = \sum_n (\mathbf{P} + \mathbf{Q}), \mathbf{K}_T = \sum_n (\mathbf{K}_{\text{dyn}} + \mathbf{K}_{\text{pot}})$$

$$\mathbf{P}_T = \hbar\mathbf{K}_T$$

$$\mathbf{P}_T \cdot \mathbf{X} = \hbar\mathbf{K}_T \cdot \mathbf{X}$$

$$-(\mathbf{P}_T \cdot \mathbf{X}) = \hbar(\mathbf{K}_T \cdot \mathbf{X})$$

$$-S_{\text{action}} = -\hbar\Phi_{\text{phase}}$$

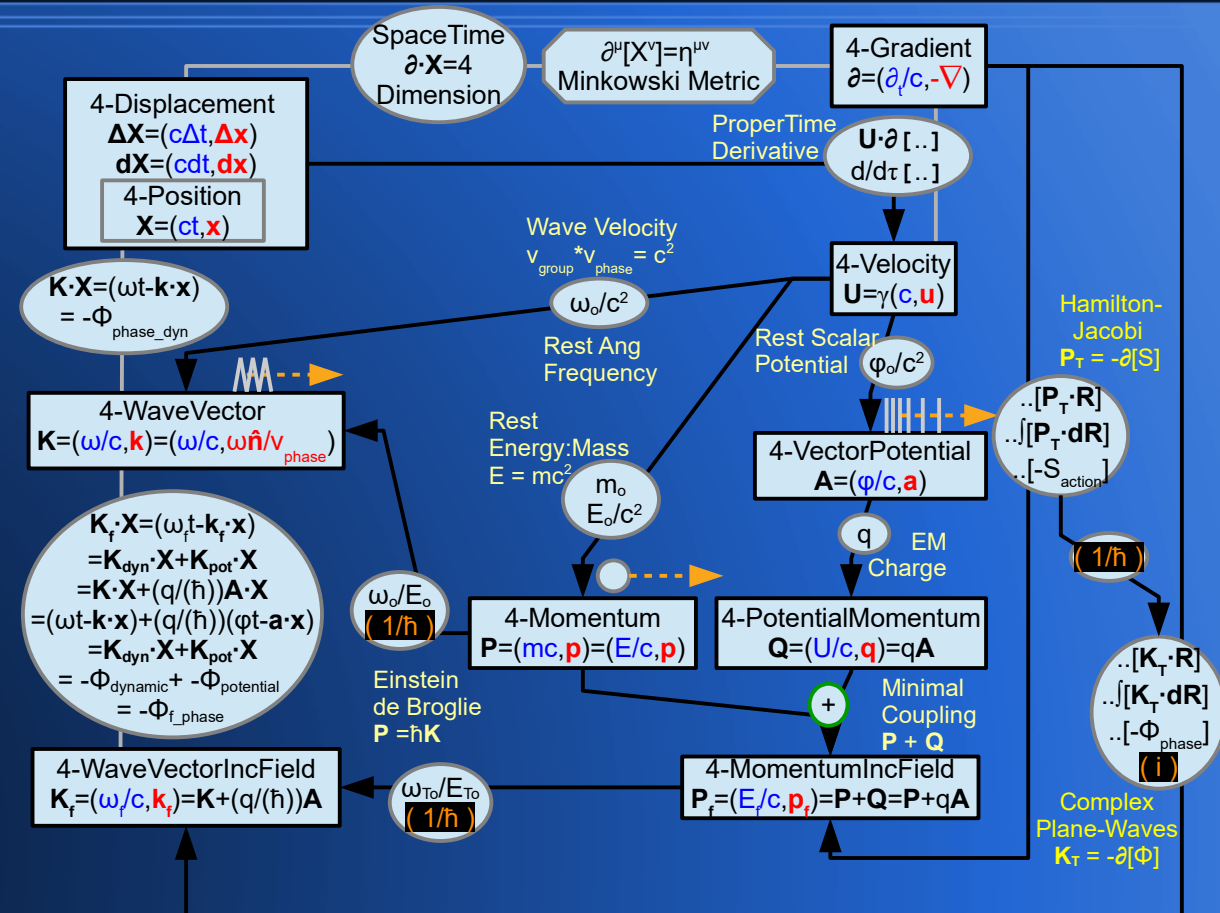
$$S_{\text{action}} = \hbar\Phi_{\text{phase}}$$

$$-\partial[S_{\text{action}}] = -\hbar\partial[\Phi_{\text{phase}}]$$

$$\mathbf{P}_T = \hbar\mathbf{K}_T$$

$$\{\text{SR Hamilton-Jacobi}\} = \hbar\{\text{QM Complex Plane-Waves}\}$$

The SR Hamilton-Jacobi Equation, and the QM idea of Complex Plane-Waves, are related by a simple constant (\hbar) relation.



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(0,0)-Tensor S
Lorentz Scalar

Existing SR Rules
Quantum Principles

$$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T^{\mu}_{\mu} = T$$

$$\mathbf{V} \cdot \mathbf{V} = V^{\mu}\eta_{\mu\nu}V^{\nu} = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$$

SRQM 4-Vector Study: Dimensionless Physical Objects

A Tensor Study of Physical 4-Vectors

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Dimensionless Physical Objects

There are a number of dimensionless physical objects in SR that can be constructed from Physical 4-Vectors. Most are 4-Scalars, but there are few 4-Vector and 4-Tensors.

$\partial \cdot X = 4$: SpaceTime Dimension
 $\partial^\mu [X^\nu] = \eta^{\mu\nu}$: The SR Minkowski Metric

$T \cdot T = 1$: Lorentz Scalar "Magnitude" of the 4-UnitTemporal
 $T \cdot S = 0$: Lorentz Scalar of 4-UnitTemporal with 4-UnitSpatial
 $S \cdot S = -1$: Lorentz Scalar "Magnitude" of the 4-UnitSpatial

$K \cdot X = (\omega t - \mathbf{k} \cdot \mathbf{x}) = -\Phi_{\text{phase_dyn}}$: Phase of an SR Wave used in SRQM wave functions $\psi = a^* e^{\Lambda} \cdot (K \cdot X)$

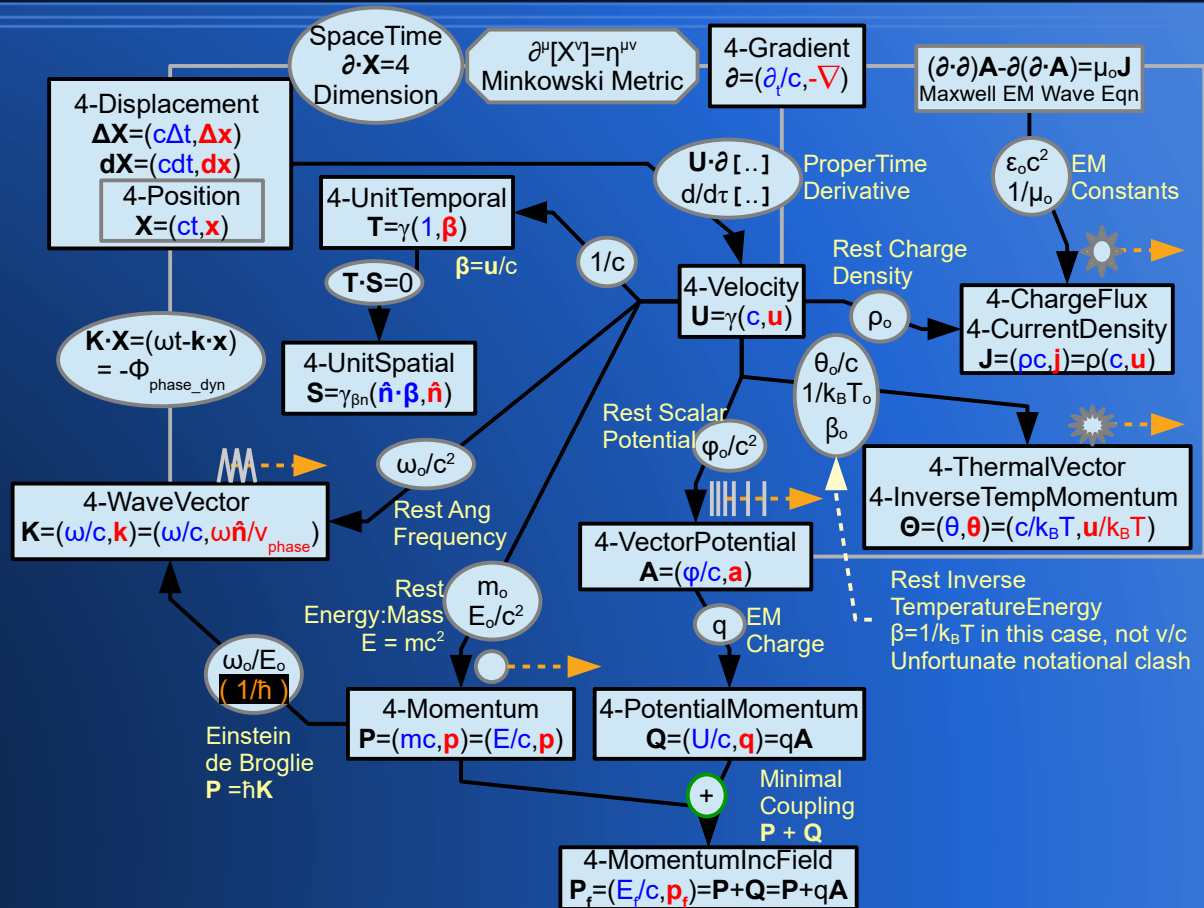
$(P \cdot \Theta) = (E_0 / k_B T_0)$: 4-Momentum with 4-InvThermalMomentum used in statistical mechanics particle distributions
 $F(\text{state}) \sim e^{\Lambda} \cdot (P \cdot \Theta) = e^{\Lambda} \cdot (E_0 / k_B T_0)$

$\alpha = (1/4\pi\epsilon_0)(e^2/\hbar c) = (\mu_0/4\pi)(ce^2/\hbar)$: Fine Structure Constant constructed from Lorentz 4-Scalars, which are themselves constructed from 4-Vectors via the Lorentz Scalar Product.
 ex. $\hbar = (P \cdot X) / (K \cdot X)$; $q = (Q \cdot X) / (A \cdot X) \rightarrow e$ for electron; $c = (T \cdot U)$
 $\mu_0 = \{(\partial \cdot \partial) [A] \cdot X\} / (J \cdot X)$ when $(\partial \cdot A) = 0$

$\{\gamma^\mu\}$: Dirac Gamma Matrix ("4-Vector")

$\{\sigma^\mu\}$: Pauli Spin Matrix ("4-Vector")

Components are matrices of numbers, not just numbers



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 (0,0)-Tensor S
 Lorentz Scalar

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$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM: QM Axioms Unnecessary

QM Principles emerge from SR

*QM is derivable from SR plus a few empirical facts – the “QM Axioms” aren't necessary
These properties are either empirically measured or are emergent from SR properties...*

3 “QM Axioms” are really just empirical constant relations between purely SR 4-Vectors:

Particle-Wave Duality [$\mathbf{P} = \hbar\mathbf{K}$]

Unitary Evolution [$\partial = (-i)\mathbf{K}$]

Operator Formalism [$(\partial) = -i\mathbf{K}$]

2 “QM Axioms” are just the result of the Klein-Gordon Equation being a linear wave PDE:

Hilbert Space Representation ($\langle \text{bra} |, | \text{ket} \rangle$, wavefunctions, etc.) & The Principle of Superposition

3 “QM Axioms” are a property of the Minkowski Metric and the empirical fact of Operator Formalism

The Canonical Commutation Relation

The Heisenberg Uncertainty Principle (time-like-separated measurement exchange)

The Pauli Exclusion Principle (space-like-separated particle exchange)

1 “QM Axiom” only holds in the NRQM case

The Born QM Probability Interpretation – Not applicable to RQM, use Conservation of Worldlines instead

1 “QM Axiom” is really just another level of limiting cases, just like SR → CM in limit of low velocity

The QM Correspondence Principle (QM → CM in limit of $\{\nabla^2[\phi] \ll (\nabla[\phi])^2\}$)

SRQM Interpretation: Relational QM & EPR

The SRQM interpretation fits fairly well with Carlo Rovelli's Relational QM interpretation:

Relational QM treats the state of a quantum system as being observer-dependent, that is, the QM State is the relation between the observer and the system. This is inspired by the key idea behind Special Relativity, that the details of an observation depend on the reference frame of the observer.

All systems are quantum systems: no artificial Copenhagen dichotomy between classical/macroscopic/conscious objects and quantum objects.

The QM States reflect the observers' information about a quantum system.

Wave function “collapse” is informational – not physical. A particle always knows its complete properties. An observer has at best only partial information about the particle's properties.

No Spooky Action at a Distance. When a measurement is done locally on an entangled system, it is only the partial information about the distant entangled state that “changes/becomes-available-instantaneously”. There is no superluminal signal. Measuring/physically-changing the local particle does not physically change the distant particle.

ex. Place two identical-except-for-color marbles into a box, close lid, and shake. Without looking, pick one marble at random and place it into another box. Send that box very far away. After receiving signal of the far box arrival at a distant point, open the near box and look at the marble. You now instantaneously know the far marble's color as well. The information did not come by signal. You already had the possibilities (partial knowledge). Looking at the near marble color simply reduced the partial knowledge of both marble's color to complete knowledge of both marbles' color. No signal was required, superluminal or otherwise.

ex. The quantum version of the same experiment uses the spin of entangled particles. When measured on the same axis, one will always be spin-up, the other will be spin-down. It is conceptually analogous. Entanglement is only about correlations of system that interacted in the past and are determined by conservation laws.

SRQM Interpretation: Interpretation of EPR-Bell Experiment

A Tensor Study
of Physical 4-Vectors

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Einstein and Bohr can both be “right” about EPR:

Per Einstein: The QM State measured is not a “complete” description, just one observer's point-of-view.

Per Bohr: The QM State measured is a “complete” description, it's all that a single observer can get.

The point is that many observers can all see the “same” system, but see different facets of it. But a single measurement is the maximal information that a single observer can get without re-interacting with the system, which of course changes the system in general. Remember, the Heisenberg Uncertainty comes from non-zero commutation properties which *require separate measurement arrangements*. The properties of a particle are always there. Properties define particles. We as observers simply have only partial information about them.

Relativistic QM, being derived from SR, should be local – The low-velocity limit to QM may give unexpected anomalous results if taken out of context, or out of the applicable validity range, such as with velocity addition $v_{12} = v_1 + v_2$, where the correct formula should be the relativistic velocity composition $v_{12} = (v_1 + v_2) / [1 + v_1 v_2 / c^2]$

These ideas lead to the conclusion that the wavefunction is just one observer's state of information about a physical system, not the state of the physical system itself. The “collapse” of the wavefunction is simply the change in an observer's information about a system brought about by a measurement or, in the case of EPR, an inference about the physical state.

EPR doesn't break Heisenberg because measurements are made on different particles. The happy fact is that those particles interacted and became correlated in the causal past. The EPR-Bell experiments prove that it is possible to maintain those correlations over long distances. It does not prove superluminal signaling

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

SRQM Interpretation: Range-of-Validity Facts & Fallacies

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We should not be surprised by the “quantum” probabilities being correct instead of “classical” in the EPR and Bell Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

Examples

*The limit of $\hbar \rightarrow 0$ {Fallacy}:

\hbar is a Lorentz Scalar Invariant and Fundamental Physical Constant. It never becomes 0. {Fact}

*The classical commutator being zero $[p^k, x^j] = 0$ {Fallacy}:

$[P^\mu, X^\nu] = i\hbar\eta^{\mu\nu}$; $[p^k, x^j] = -i\hbar\delta^{kj}$; $[p^0, x^0] = [E/c, ct] = [E, t] = i\hbar$; Again, it never becomes 0 {Fact}

*Using Maxwell-Boltzmann (distinguishable) statistics for counting probabilities of (indistinguishable) quantum states {Fallacy}:

Must use Fermi-Dirac statistics for Fermions: Spin=(n+1/2); Bose-Einstein statistics for Bosons: Spin=(n) {Fact}

*Using sums of classical probabilities on quantum states {Fallacy}:

Must use sums of quantum probability-amplitudes {Fact}

*Ignoring phase cross-terms and interference effects in calculations {Fallacy}:

Quantum systems and entanglement require phase cross-terms {Fact}

*Assuming that one can simultaneously “measure” non-commuting properties at a single spacetime event {Fallacy}:

Particle properties always exist. However, non-commuting ones require separate measurement arrangements to get information about the properties.

The required measurement arrangements on a single particle/worldline are at best sequential events, where the temporal order plays a role; {Fact}

However, EPR allows one to “infer (not measure)” the other property of a particle by the separate measurement of an entangled partner. {Fact}

This does not break Heisenberg Uncertainty, which is about the order of operations (measurement events) on a single worldline. {Fact}

In the entangled case, both/all of the entangled partners share common past-causal entanglement events, typically due to a conservation law. {Fact}

Information is not transmitted at FTL. The particles simply carried their normal respective “correlated” properties (no hidden variables) with them. {Fact}

*Assuming that QM is a generalization of CM, or that classical probabilities apply to QM {Fallacy}:

CM is a limiting-case of QM for when changes in a system by a few quanta have a negligible effect on the whole/overall system. {Fact}

SRQM Interpretation: Quantum Information

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{from Wikipedia}

No-Communication Theorem/No-Signaling:

A no-go theorem from quantum information theory which states that, during measurement of an entangled quantum state, it is not possible for one observer, by making a measurement of a subsystem of the total state, to communicate information to another observer. The theorem shows that quantum correlations do not lead to what could be referred to as "spooky communication at a distance". **SRQM: There is no FTL signaling.**

No-Teleportation Theorem:

The no-teleportation theorem stems from the Heisenberg uncertainty principle and the EPR paradox: although a qubit $|\psi\rangle$ can be imagined to be a specific direction on the Bloch sphere, that direction cannot be measured precisely, for the general case $|\psi\rangle$. The no-teleportation theorem is implied by the no-cloning theorem.

SRQM: Ket states are informational, not physical.

No-Cloning Theorem:

In physics, the no-cloning theorem states that it is impossible to create an identical copy of an arbitrary unknown quantum state. This no-go theorem of quantum mechanics proves the impossibility of a simple perfect non-disturbing measurement scheme. The no-cloning theorem is normally stated and proven for pure states; the no-broadcast theorem generalizes this result to mixed states. **SRQM: Measurements are arrangements of particles that interact with a subject particle.**

No-Broadcast Theorem:

Since quantum states cannot be copied in general, they cannot be broadcast. Here, the word "broadcast" is used in the sense of conveying the state to two or more recipients. For multiple recipients to each receive the state, there must be, in some sense, a way of duplicating the state. The no-broadcast theorem generalizes the no-cloning theorem for mixed states. The no-cloning theorem says that it is impossible to create two copies of an unknown state given a single copy of the state.

SRQM: Conservation of worldlines.

No-Deleting Theorem:

In physics, the no-deleting theorem of quantum information theory is a no-go theorem which states that, in general, given two copies of some arbitrary quantum state, it is impossible to delete one of the copies. It is a time-reversed dual to the no-cloning theorem, which states that arbitrary states cannot be copied.

SRQM: Conservation of worldlines.

No-Hiding Theorem:

the no-hiding theorem is the ultimate proof of the conservation of quantum information. The importance of the no-hiding theorem is that it proves the conservation of wave function in quantum theory.

SRQM: Conservation of worldlines. RQM wavefunctions are Lorentz Scalars (spin=0), Spinors (spin=1/2), 4-Vectors (spin=1), all of which are Lorentz Invariant.

SRQM Interpretation: Quantum Information

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We should not be surprised by the “quantum” probabilities being correct instead of “classical” probabilities in the EPR/Bell-Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity. {from Wikipedia}

Quantum information (qubits) differs strongly from classical information, epitomized by the bit, in many striking and unfamiliar ways. Among these are the following:

A unit of quantum information is the qubit. Unlike classical digital states (which are discrete), a qubit is continuous-valued, describable by a direction on the Bloch sphere. Despite being continuously valued in this way, a qubit is the smallest possible unit of quantum information, as despite the qubit state being continuously-valued, it is impossible to measure the value precisely.

A qubit cannot be (wholly) converted into classical bits; that is, it cannot be "read". This is the no-teleportation theorem.

Despite the awkwardly-named no-teleportation theorem, qubits can be moved from one physical particle to another, by means of quantum teleportation. That is, qubits can be transported, independently of the underlying physical particle. **SRQM: Ket states are informational, not physical.**

An arbitrary qubit can neither be copied, nor destroyed. This is the content of the no cloning theorem and the no-deleting theorem. **SRQM: Conservation of worldlines.**

Although a single qubit can be transported from place to place (e.g. via quantum teleportation), it cannot be delivered to multiple recipients; this is the no-broadcast theorem, and is essentially implied by the no-cloning theorem. **SRQM: Conservation of worldlines.**

Qubits can be changed, by applying linear transformations or quantum gates to them, to alter their state. While classical gates correspond to the familiar operations of Boolean logic, quantum gates are physical unitary operators that in the case of qubits correspond to rotations of the Bloch sphere.

Due to the volatility of quantum systems and the impossibility of copying states, the storing of quantum information is much more difficult than storing classical information. Nevertheless, with the use of quantum error correction quantum information can still be reliably stored in principle. The existence of quantum error correcting codes has also led to the possibility of fault tolerant quantum computation.

Classical bits can be encoded into and subsequently retrieved from configurations of qubits, through the use of quantum gates. By itself, a single qubit can convey no more than one bit of accessible classical information about its preparation. This is Holevo's theorem. However, in superdense coding a sender, by acting on one of two entangled qubits, can convey two bits of accessible information about their joint state to a receiver.

Quantum information can be moved about, in a quantum channel, analogous to the concept of a classical communications channel. Quantum messages have a finite size, measured in qubits; quantum channels have a finite channel capacity, measured in qubits per second.

Minkowski still applies in local GR

QM is a local phenomenon

The QM Schrodinger Equation is not fundamental. It is just the low-energy limiting-case of the RQM Klein-Gordon Equation. All of the standard QM Axioms are shown to be empirically measured constants or emergent properties of SR. It is a bad approach to start with NRQM as an axiomatic starting point and try to generalize it to RQM, in the same way that one cannot start with CM and derive SR. Since QM *can* be derived from SR, this partially explains the difficulty of uniting QM with GR: QM is not a “separate formalism” outside of SR that can be used to “quantize” just anything...

Strictly speaking, the use of the Minkowski space to describe physical systems over finite distances applies only in the SR limit of systems without significant gravitation. In the case of significant gravitation, SpaceTime becomes curved and one must abandon SR in favor of the full theory of GR.

Nevertheless, even in such cases, based on the GR Equivalence Principle, Minkowski space is still a good description in a local region surrounding any point (barring gravitational singularities). More abstractly, we say that in the presence of gravity, SpaceTime is described by a curved 4-dimensional manifold for which the tangent space to any point is a 4-dimensional Minkowski Space. Thus, the structure of Minkowski Space is still essential in the description of GR.

So, even in GR, at the local level things are considered to be Minkowskian:
i.e. SR → QM “lives inside the surface” of this local SpaceTime, GR curves the surface.

SRQM Interpretation: Main Result

QM is derivable from SR!

Hopefully, this interpretation will shed light on why Quantum Gravity has been so elusive. Basically, QM rules of “quantization” don’t apply to GR. They are a manifestation-of/derivation-from SR. Relativity *is* the “Theory of Measurement” that QM has been looking for.

This would explain why no one has been able to produce a successful theory of Quantum Gravity, and why there have been no violations of Lorentz Invariance nor of the Equivalence Principle.

If quantum effects “live” in Minkowski SpaceTime with SR, then GR curvature effects are at a level above the RQM description, and two levels above standard QM. SR+QM are “in” SpaceTime, GR is the “shape” of SpaceTime...

Thus, this treatise explains the following:

- Why GR works so well in it’s realm of applicability {large scale systems}.
- Why QM works so well in it’s realm of applicability {micro scale systems and certain macroscopic systems}.
i.e. The tangent space to any point in GR curvature is locally Minkowskian, and thus QM is typically found in small local volumes...
- Why RQM explains more stuff than QM without SR {because QM is just the low-velocity limiting-case of RQM}.
- Why all attempts to “quantize gravity” have failed {essentially, everyone has been trying to put the cart (QM) before the horse (GR)}.
- Why all attempts to modify GR keep conflicting with experimental data {because GR is apparently fundamental}.
- Why QM works perfectly well with SR as RQM but not with GR {because QM is derivable from SR, hence a manifestation of SR rules}.
- How Minkowski Space, 4-Vectors, and Lorentz Invariants play vital roles in RQM, and give the SRQM Interpretation of Quantum Mechanics.

SRQM:

SR → QM Interpretation Simplified

A Tensor Study
of Physical 4-VectorsSciRealm.org
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http://scirealm.org/SRQM.pdfSRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR,
although technically SR is itself the low-curvature limiting-case of GR

{c,τ,m₀,ħ,i}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors:

Related by these SR Lorentz Invariants

4-Position	$\mathbf{R} = (ct, \mathbf{r})$	= <Event>	$(\mathbf{R} \cdot \mathbf{R}) = (c\tau)^2$
4-Velocity	$\mathbf{U} = \gamma(\mathbf{c}, \mathbf{u})$	= $(\mathbf{U} \cdot \partial)\mathbf{R} = (d/d\tau)\mathbf{R} = d\mathbf{R}/d\tau$	$(\mathbf{U} \cdot \mathbf{U}) = (c)^2$
4-Momentum	$\mathbf{P} = (E/c, \mathbf{p})$	= $m_0\mathbf{U}$	$(\mathbf{P} \cdot \mathbf{P}) = (m_0c)^2$
4-WaveVector	$\mathbf{K} = (\omega/c, \mathbf{k})$	= \mathbf{P}/\hbar	$(\mathbf{K} \cdot \mathbf{K}) = (m_0c/\hbar)^2$
4-Gradient	$\partial = (\partial_t/c, -\nabla)$	= $-i\mathbf{K}$	$(\partial \cdot \partial) = -(m_0c/\hbar)^2 = KG \text{ Eqn} \rightarrow RQM \rightarrow QM$

|v| << c

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn,
and thence to QM via the low-velocity limit { |v| << c }, giving the Schrödinger Eqn.

The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

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SRQM Diagram:

Special Relativity → Quantum Mechanics RoadMap of SR→QM

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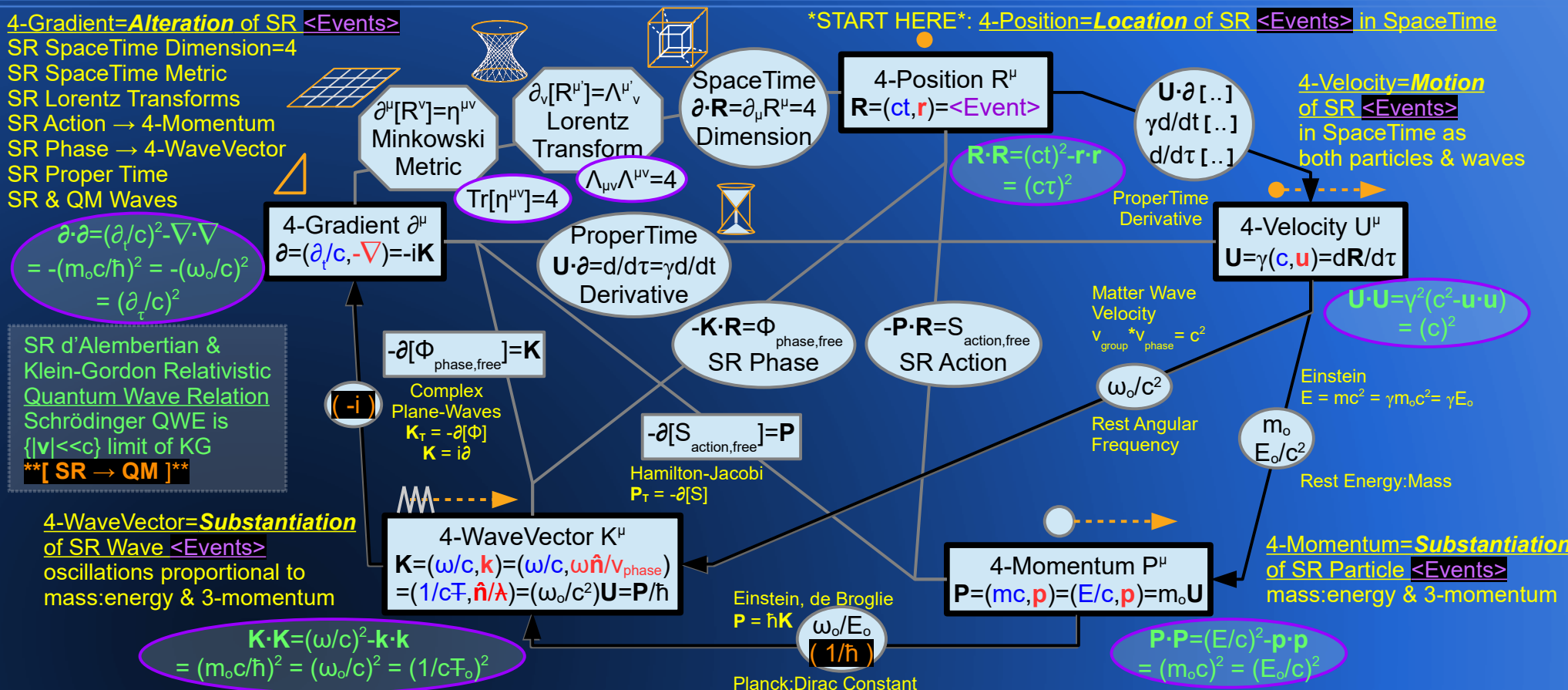
http://scirealm.org/SRQM.pdf

A Tensor Study
of Physical 4-Vectors

4-Gradient=**Alteration** of SR <Events>

SR SpaceTime Dimension=4
SR SpaceTime Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves

START HERE: 4-Position=**Location** of SR <Events> in SpaceTime



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(2,0)-Tensor $T^{\mu\nu}$
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(0,2)-Tensor $T_{\mu\nu}$

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SRQM Diagram:

Special Relativity → Quantum Mechanics RoadMap of SR→QM (EM Potential)

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SR → QM

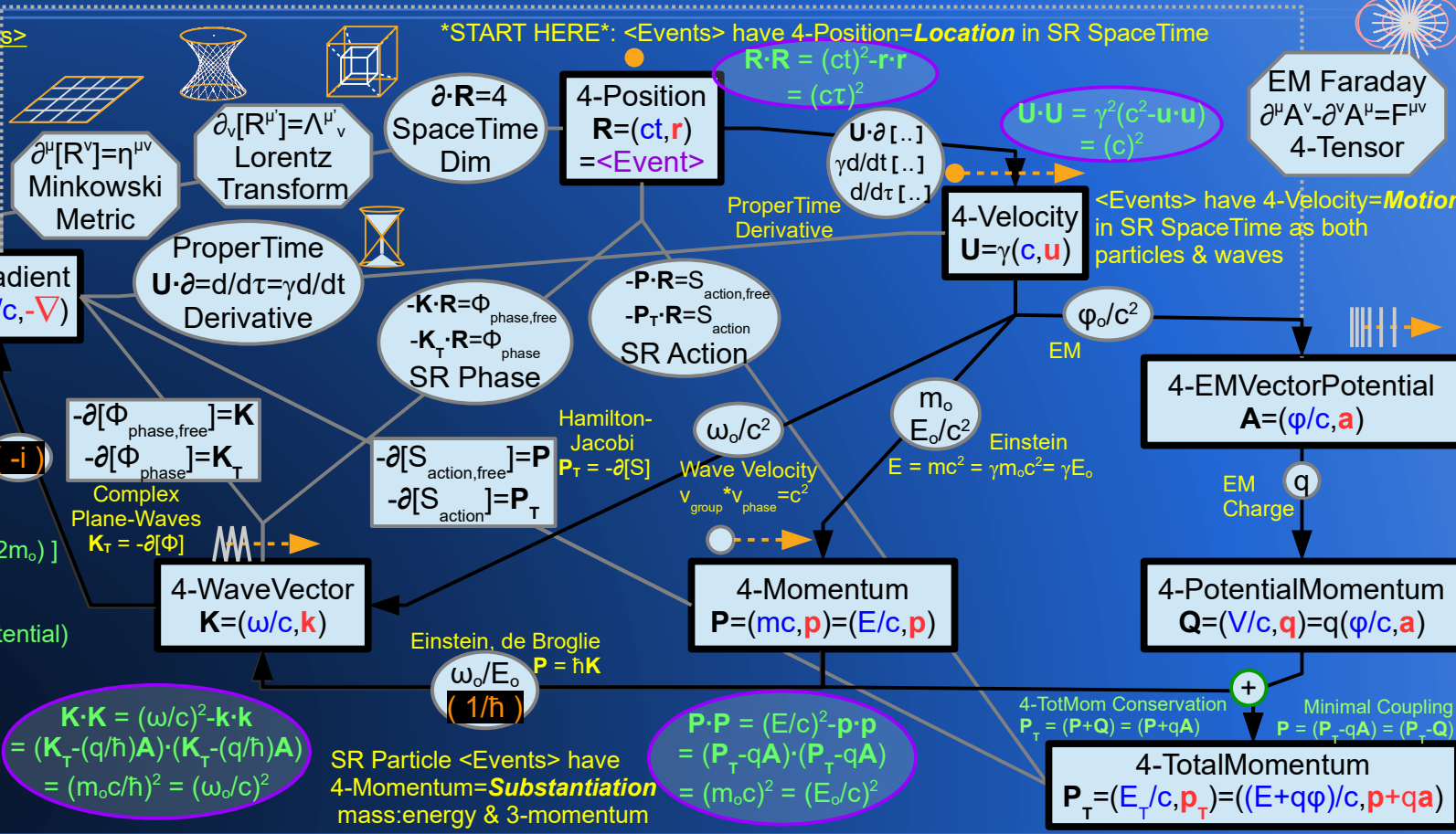
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 SR SpaceTime Metric
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 SR Proper Time
 SR & QM Waves

SR → RQM Klein-Gordon
 Relativistic Quantum
 Particle in EM Potential
 d'Alembertian Wave Equation

$\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla$
 $= (\partial_t + (iq/\hbar)\mathbf{A}) \cdot (\partial_t + (iq/\hbar)\mathbf{A})$
 $= -(\omega_0/c)^2 = -(m_0c/\hbar)^2$
 $= (\partial_t/c)^2$

Limit: $\{ |v| \ll c \}$
 $(i\hbar\partial_{tT}) \sim [q\phi + (m_0c^2) + (i\hbar\nabla_T + qa)^2/(2m_0)]$
 $(i\hbar\partial_{tT}) \sim [V + (i\hbar\nabla_T + qa)^2/(2m_0)]$
 with potential $V = q\phi + (m_0c^2)$
 =Schrödinger QM Equation (EM potential)
 [SR → QM]



SR Wave <Events> have 4-WaveVector=**Substantiation** oscillations proportional to mass:energy & 3-momentum

SR Particle <Events> have 4-Momentum=**Substantiation** mass:energy & 3-momentum

SR 4-Tensor (2,0)-Tensor $T^{\mu\nu}$ (1,1)-Tensor T^μ_ν or T_μ^ν (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector (1,0)-Tensor $V^\mu = \mathbf{V} = (v^0, \mathbf{v})$ SR 4-CoVector (0,1)-Tensor $V_\mu = (v_0, -\mathbf{v})$

SR 4-Scalar (0,0)-Tensor S Lorentz Scalar

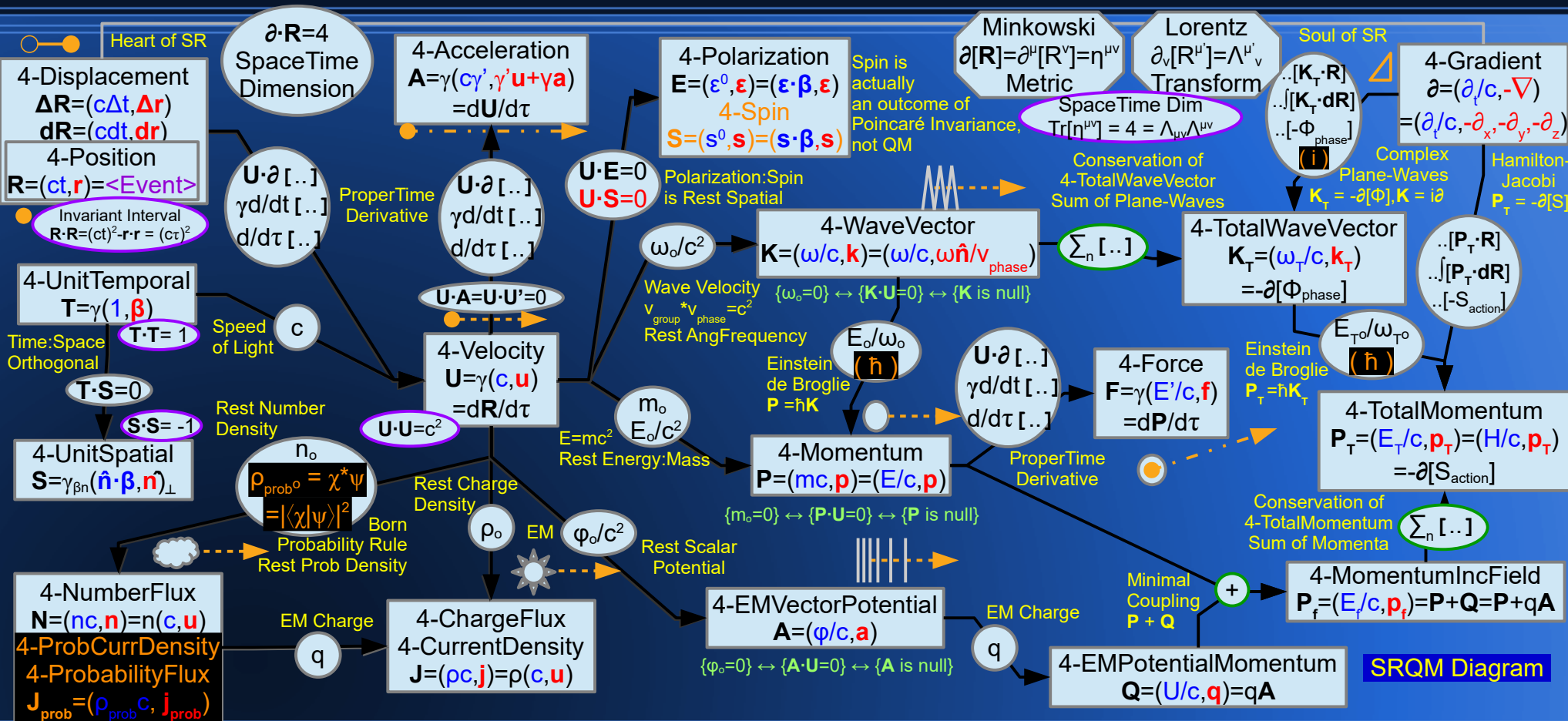
Existing SR Rules
Quantum Principles

Trace[$T^{\mu\nu}$] = $\eta_{\mu\nu} T^{\mu\nu} = T^\mu_\mu = T$
 $\mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu\nu} V^\nu = [(v^0)^2 - \mathbf{v} \cdot \mathbf{v}] = (v^0_c)^2 = \text{Lorentz Scalar}$

SRQM Diagram: SRQM 4-Vectors and Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson
SciRealm@aol.com
http://scirealm.org/SRQM.pdf



Special Relativity → Quantum Mechanics

The SRQM Interpretation: Links

A Tensor Study
of Physical 4-Vectors

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See also:

<http://scirealm.org/SRQM.html> (alt discussion)

<http://scirealm.org/SRQM-RoadMap.html> (main SRQM website)

<http://scirealm.org/4Vectors.html> (4-Vector study)

<http://scirealm.org/SRQM-Tensors.html> (Tensor & 4-Vector Calculator)

<http://scirealm.org/SciCalculator.html> (Complex-capable RPN Calculator)

or Google “SRQM”

<http://scirealm.org/SRQM.pdf> (this document)

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)

The 4-Vector SRQM Interpretation

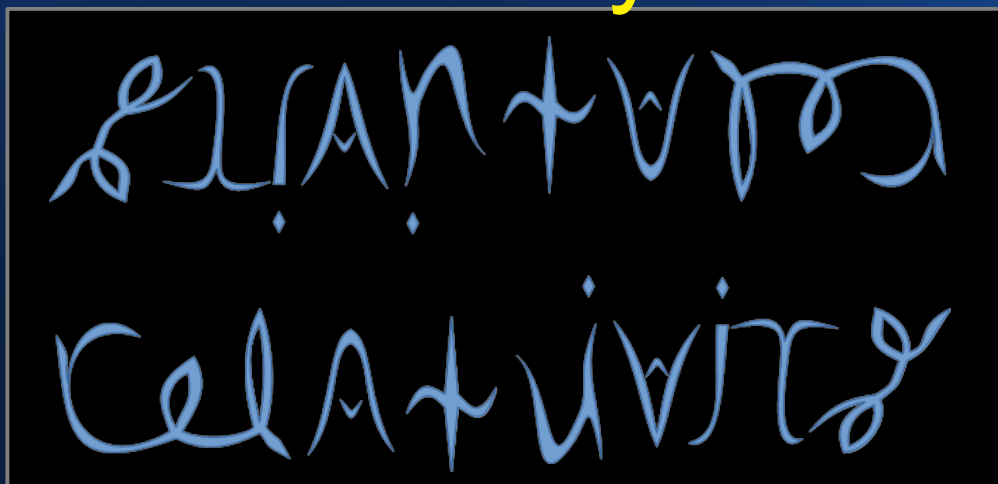
QM is derivable from SR!

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson
SciRealm@aol.com
<http://scirealm.org/SRQM.pdf>

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

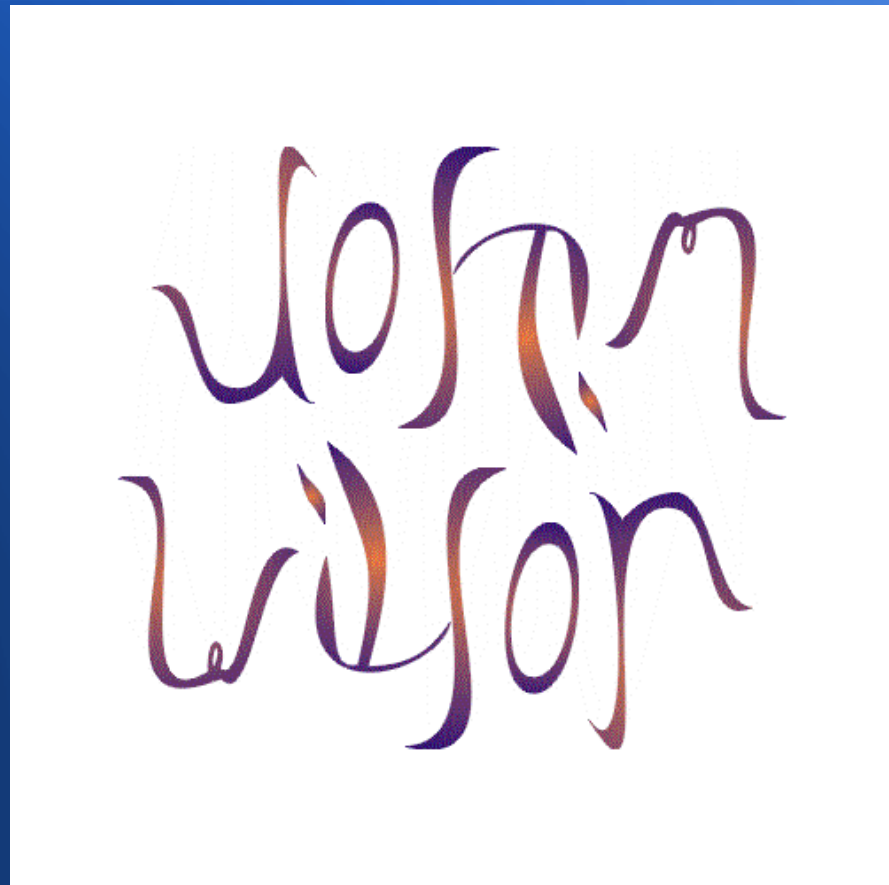
quantum relativity



SRQM = SciRealm QM?

A happy coincidence... :)

Ambigrams



SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)