Integrability Analysis of a Generalized Truly Nonlinear Oscillator Equation

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The integrability of a general class of Liénard type equations is investigated through equation transformation theory. In this way it is shown that such a class of Liénard equations can generate a generalization of some interesting truly nonlinear oscillator equations like the cube and fifth root differential equations. It has then become possible to compute the exact and general solution to the generalized truly nonlinear oscillator equation. Under an appropriate choice of initial conditions, exact and explicit solution has been obtained in terms of Jacobi elliptic functions.

1 Introduction

The Liénard equation

$$\ddot{x}(t) + f(x) = 0 \tag{1.1}$$

where $f(x)$ is a nonlinear function of x has received a high importance in the theory of differential equations since it may exhibit periodic solutions which constitute a highly desired class of differential equation solutions in physics, when the independent variable t is time. The class of equations (1.1) includes as illustrative examples several celebrated equations like the Bratu differential equation, the polynomial differential equations such as the cubic, cubic-quintic and quintic Duffing oscillator equations. The equations of type (1.1) are also of great interest as they appear in traveling wave solution reduction of nonlinear partial differential equations [1, 2]. Although the equations of type (1.1) are widely studied, the integrability of (1.1) in terms of exact and explicit general periodic

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solutions remains to be solved when $f(x)$ is a function of fractional power of x. A well known special case of such equations, that is

$$\ddot{x}(t) + x^{1/3} = 0 \tag{1.2}$$

called cube-root oscillator equation has been studied in several mathematical works [3]. Equation (1.2) belongs to the class of truly nonlinear oscillator equations intensively investigated by [3]. One may also consider the truly nonlinear equation [3]

$$\ddot{x}(t) + x^{5/3} = 0 \tag{1.3}$$

However the generalization

$$\ddot{x}(t) + bx^{1/3} + dx^{5/3} = 0 \tag{1.4}$$

is not investigated in the literature to our best knowledge, that is its integrability analysis in terms of exact and general periodic solution is to be established. To that end an extension of the theory of nonlinear differential equations introduced by [4] is carried out (section 2). So that it becomes possible to show that equation (1.4) belongs to a general class of Liénard type equations with fractional power nonlinearities for which exact and general periodic solutions can be computed (section 3). Finally a conclusion is given for the work.

2 Extended Theory

Let us consider the following general second order linear differential equation

$$y''(\tau) + a^2 y(\tau) = c \tag{2.1}$$

where, prime means differentiation with respect to τ, a and c are arbitrary parameters. Let us consider also the generalized Sundman transformation.

$$y^m(\tau) = F(t,x), \quad d\tau = G(t,x)dt, \quad G(t,x)\frac{\partial F(t,x)}{\partial x} \neq 0 \tag{2.2}$$

with

$$F(t,x) = \int g(x)^l \mathrm{d}x, \quad G(t,x) = e^{\gamma\varphi(x)}dt$$

where l, $m \neq 0$, and γ are arbitrary parameters, $g(x) \neq 0$ and $\varphi(x)$ are arbitrary functions of x. It is important to notice that, taking $m = 1$, we obtain the theory proposed by [5]. Thus, introducing equation (2.2) into equation (2.1) and performing some mathematical operations, we get

$$\ddot{x} + \left[l\frac{g'(x)}{g(x)} + \left(\frac{1}{m} - 1\right)\frac{g(x)^l}{\int g(x)^l \mathrm{d}x} - \gamma\varphi'(x)\right]\dot{x}^2 \tag{2.3}$$

$$+a^2 m\frac{\int g(x)^l \mathrm{d}x}{g(x)^l}e^{2\gamma\varphi(x)} - mc\frac{\left(\int g(x)^l \mathrm{d}x\right)^{1-\frac{1}{m}}}{g^l(x)}e^{2\gamma\varphi(x)} = 0$$

Putting now

$$l\frac{g'(x)}{g(x)} + \left(\frac{1}{m} - 1\right)\frac{g(x)^l}{\int g(x)^l \mathrm{d}x} - \gamma\varphi'(x) = 0, \tag{2.4}$$

that is

$$\varphi(x) = \frac{1}{\gamma}\ln g(x)^l \left(\int g(x)^l \mathrm{d}x\right)^{\frac{1}{m}-1} \tag{2.5}$$

equation (2.3) becomes

$$\ddot{x} + a^2 m g(x)^l \left(\int g(x)^l \mathrm{d}x\right)^{\frac{2}{m}-1} - cmg(x)^l \left(\int g(x)^l \mathrm{d}x\right)^{\frac{1}{m}-1} = 0 \tag{2.6}$$

Introducing $g(x) = x$, into equation(2.6), yields

$$\ddot{x} + \frac{ma^2}{(l+1)^{\frac{2-m}{m}}} x^{\frac{2l+2}{m}-1} - \frac{mc}{(l+1)^{\frac{1}{m}-1}} x^{\frac{l+1}{m}-1} = 0 \tag{2.7}$$

Let $l = m$. Then equation (2.7) reduces to

$$\ddot{x} + \frac{la^2}{(l+1)^{\frac{2-l}{l}}} x^{\frac{l+2}{l}} - \frac{lc}{(l+1)^{\frac{1-l}{l}}} x^{\frac{1}{l}} = 0 \tag{2.8}$$

Equation (2.8) represents the general class of Liénard equations with fractional power nonlinearities for an integer l, $l \neq 2$ or $l \neq 1$. We investigate in the following, the exact and explicit general solution of this class of Liénard equations and that of the truly nonlinear oscillator equation (1.4). Now, the general solution of equation (2.8) may be explicitly and exactly computed under the nonlocal transformation (2.2), such that one may deduce the exact solution to the truly nonlinear oscillator equations (1.4). The use of equations (2.2) leads to

$$y(\tau) = \left(\frac{1}{l+1} x^{l+1}\right)^{\frac{1}{l}}, \quad d\tau = \left(\frac{1}{(l+1)^{(1-l)/l}} x^{1/l}\right) dt, \quad l \neq -1 \tag{2.9}$$

such that the desired exact and explicit general solution to (2.8) takes the form

$$x(t) = (l+1)^{\frac{1}{l+1}} \left(A_0 \cos(a\phi(t) + \alpha) + \frac{c}{a^2}\right)^{\frac{l}{l+1}} \tag{2.10}$$

where, α is an arbitrary constant and the function $\tau = \phi(t)$ satisfies to

$$\int \frac{d\phi(t)}{(A_0 \cos(a\phi(t) + \alpha) + \frac{c}{a^2})^{\frac{1}{l+1}}} = \frac{a}{2}(l+1)^{\frac{l}{l+1}}(t + K) \tag{2.11}$$

K is an integration constant and

$$y(\tau) = \frac{c}{a^2} + A_0 \cos(a\tau + \alpha) \tag{2.12}$$

is the solution to (2.1).

Imposing $A_0 = \frac{c}{a^2}$, equation (2.11) becomes

$$\frac{d\phi(t)}{(1 + \cos(a\phi(t) + \alpha))^{\frac{1}{l+1}}} = A_0^{\frac{1}{l+1}} (l+1)^{\frac{l}{l+1}} dt \tag{2.13}$$

that is

$$\frac{d\phi(t)}{(\cos^2(\frac{a\phi(t)+\alpha}{2}))^{\frac{1}{l+1}}} = (2A_0)^{\frac{1}{l+1}}(l+1)^{\frac{l}{l+1}}dt \tag{2.14}$$

such that

$$\int \frac{d\theta}{(\cos^2(\theta))^{\frac{1}{l+1}}} = (2A_0)^{\frac{1}{l+1}}\frac{a}{2}(l+1)^{\frac{l}{l+1}}(t+K) \tag{2.15}$$

In this context the general solution (2.10) may take the form

$$x(t) = (l+1)^{\frac{1}{l+1}}\,(2A_0)^{\frac{l}{l+1}}\cos^{\frac{2l}{l+1}}(\theta) \tag{2.16}$$

where $\theta = \frac{a\phi(t)+\alpha}{2}$ satisfies to (2.15)

3 Exact solution to equation (1.4)

Setting $l = 3$ into (2.8) yields equation (1.4) when $b = 3(2)^{4/3}c$ and $d = 3(2)^{2/3}a^2$. Using now (2.16), an exact and general solution to equation (1.4) may be obtained as

$$x(t) = 2^{5/4}A_0^{3/4}\cos^{3/2}(\frac{a\phi(t)+\alpha}{2}) \tag{3.1}$$

such that $\phi(t)$ satisfies to

$$\frac{d\phi(t)}{\sqrt{\cos(\frac{a\phi(t)+\alpha}{2})}} = (2A_0)^{1/4}\frac{a}{2}2^{3/2}dt \tag{3.2}$$

The integration of the left hand side of equation (3.2) may be evaluated as

$$J = \int \frac{d\phi(t)}{\sqrt{\cos(\frac{a\phi(t)+\alpha}{2})}} \tag{3.3}$$

which is also

$$J = \frac{a}{2}\int \frac{d\theta}{\sqrt{\cos(2\theta)}} \tag{3.4}$$

where $\theta = \frac{a\phi(t)+\alpha}{2}$. According to [6]

$$J = \frac{a}{2\sqrt{2}}F\left(\alpha, \frac{1}{\sqrt{2}}\right) \tag{3.5}$$

where $\alpha = arcsin(\sqrt{2}sin(\theta))$ and $F(z,k)$, is the Jacobi elliptic integral of the first kind. Thus, one may find $\frac{a\phi(t)+\alpha}{2}$ such that

$$sin(\theta) = \frac{\sqrt{2}}{2}sn(\sqrt[4]{2ca^2}(t+K), \frac{\sqrt{2}}{2}) \tag{3.6}$$

In this context one may rewrite the solution of (1.4) in the form

$$x(t) = 2\sqrt[4]{2}(c/a^2)^{3/4} cn^3(\sqrt[4]{2ca^2}(t+K), \frac{\sqrt{2}}{2}) \tag{3.7}$$

Using the initial conditions $x(0) = x_0$, and $\dot{x}(0) = 0$, we find $K = 0$, $x_0 = 2\sqrt[4]{2}(\frac{c}{a^2})^{3/4}$. Thus, the exact solution takes the form

$$x(t) = x_0 cn^3\left(\left(\frac{4}{x_0}\right)^{\frac{1}{3}}\sqrt{c}t, \frac{\sqrt{2}}{2}\right) \tag{3.8}$$

In Figure 1, the periodic behavior of solution (3.8) is plotted with $c = 2$, $a = 0.5$ and $x_0 = 11.31$

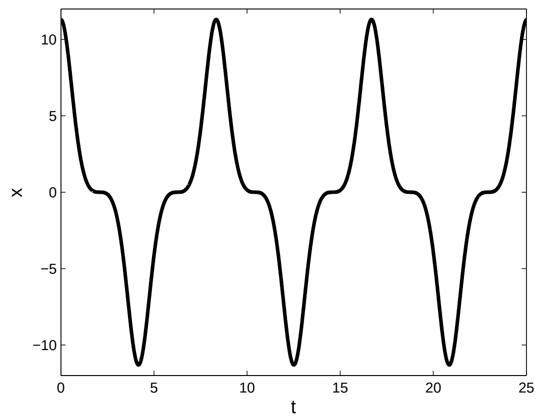

Figure 1: A typical periodic solution (3.8) with $a = 0.5$, $c = 2$ and $x_0 = 11.31$.

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