Proof of Twin Prime Conjecture

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Author’s Biography

The author of this research paper is K.H.K. Geerasee Wijesuriya. And this proof of twin prime conjecture is completely K.H.K. Geerasee Wijesuriya's proof.

Geerasee she studied before at Faculty of Science, University of Colombo Sri Lanka. And she graduated with BSc (Hons) in Physics and Mathematics from the University of Colombo, Sri Lanka in 2014. And in March 2018, she completed her first Doctorate Degree in Physics with first class recognition. Now she is following her second PhD in Astrophysics with Belarusian National Technical University.

Geerasee has been invited by several Astronomy/Physics institutions and organizations worldwide, asking to get involve with them. Also, She has received several invitations from some private researchers around the world asking to contribute to their researches. She worked as Mathematics tutor/Instructor at Mathematics department, Faculty of Engineering, University of Moratuwa, Sri Lanka. Now she is a research scientist in Physics as her career. Furthermore she has achieved several other scientific achievements already.

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Abstract

Twin prime numbers are two prime numbers which have the difference of 2 exactly. In other words, twin primes is a pair of prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair. Up to date there is no any valid proof/disproof for twin prime conjecture. Through this research paper, my attempt is to provide a valid proof for twin prime conjecture.

Literature Review

The question of whether there exist infinitely many twin primes has been one of the great open questions in number theory for many years. This is the content of the twin prime conjecture, which states that there are infinitely many primes $p$ such that $p + 2$ is also prime. In 1849, de Polignac made the more general conjecture that for every natural number $k$, there are infinitely many primes $p$ such that $p + 2k$ is also prime. The case $k = 1$ of de Polignac's conjecture is the twin prime conjecture.

A stronger form of the twin prime conjecture, the Hardy–Littlewood conjecture, postulates a distribution law for twin primes akin to the prime number theorem. On April 17, 2013, Yitang Zhang announced a proof that for some integer $N$ that is less than 70 million, there are infinitely many pairs of primes that differ by $N$. Zhang's paper was accepted by Annals of Mathematics in early May 2013. Terence Tao subsequently proposed a Polymath Project collaborative effort to optimize Zhang's bound. As of April 14, 2014, one year after Zhang's announcement, the bound has been reduced to 246. Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that the bound has been reduced to 12 and 6, respectively. These improved bounds were discovered using a different approach that was simpler than Zhang's and was discovered independently by James Maynard and Terence Tao.
Assumption

Let's assume that there are finitely many twin prime numbers……………...(1.0)

Therefore we proceed by considering that there are finitely many twin prime numbers. Then let the highest twin prime numbers are \( P_{n-1} \) and \((P_{n-1} + 2)\). Then for all prime numbers \( P_N \) greater than \((P_{n-1} + 2)\), \((P_N + 2)\) is not a prime number.

Methodology

With this mathematical proof, I use the contradiction method to prove that there are infinitely many twin prime numbers.

Let \( P_n \) is an odd number greater than 1. But let \( P_3 \) is divisible by \( x_3 \). But \( x_3^2 \) does not divide \( P_3 \). And let \( P_3 \) is not divisible by \( x_3 \). We choose \( P_n \) such that \( P_n = (M + 4) – (D.P_3/x_3) \); for some integer \( D \neq 0 \). Where \( D \) is not divisible by \( x_3 \).

To see the meaning of \( P_3, x_3 \) and \( M \), please refer the below content.

Let \( P_N \) is an arbitrary prime number greater than \((P_{n-1} + 2)\). Because there are infinitely many prime numbers. And here \((P_N - 2) > (P_{n-1} + 2)\). Thus \((P_N - 2)\) is not a prime number.

And here \((P_N + 2) > (P_{n-1} + 2)\). Then according to our assumption, \((P_N + 2)\) is also not a prime number. Here \( P_N \) is a prime number such that \((P_N + 2)\) is dividing by prime number \( P_2 \).

Thus \((P_N + 2) = P_2 * x_2\) for some \( x_2 \) natural number. Since \( P_N \) is a prime number, for some \( r_2 \) (rational number which is not a natural number): \( P_N / r_2 = P_2 \).

Thus \((P_N + 2) = P_2 * x_2\) …………………..(02) and \( P_N = r_2 * P_2 \) …………………..(03)

\( x_2 \) is a natural number and \( P_2 \) is a prime number.

Since \( P_N \) is a prime number , \((P_N - 2)\) is also not a prime number ( Since \( P_N - 2 > P_{n-1} + 2 \))

Then for some integer \( P_3 \) greater than 1 such that \((P_N - 2) / P_3 = x_3 \); where \( x_3 \) is an integer greater than 1. But here we considered that \( x_3 \mid P_3 \).
\((P_n - 2) = P_3 \times x_3 \) ..............................(04)

But \((P_n + 2), P_n\) both are odd numbers. Thus \((P_n + 2) = P_n + 2.l\); for some \(l\) integer (where \(l \neq 0\)) .................................(05)

Then \((P_n - 2) = P_n + 2.l - 4 = P_n + 2 \times (l - 2) \) ..........(6.1)’

And we know that \((P_n + 2) = P_n + 2.l \rightarrow P_N = P_n + 2.l - 2 \) .........................(*)

Thus by (*): \(P_n + 2.l - 2 = P_N\). Thus by (04) and (*): \(P_3 \times x_3 + 2 = P_n + 2.l - 2\)

Thus \(P_3 \times x_3 - 2.l + 4 = P_n \) ......................(6.1.0)

Thus \(P_3 \times x_3 + 2. (l - 2) = P_n + 4. (l - 2) = P_n + 2.P_N - 4 - 2.P_n = 2.P_N - 4 - P_n\) (by (6.1)’)

Thus \(P_3 \times x_3 + 2. (l - 2) = 2.P_N - 4 - P_n = P_n''\)

Thus \(P_3 \times x_3 + 2. (l - 2) = P_n'' = 2. P_3 \times x_3 - P_n \) .........................(7)

Thus \(P_3 \times x_3 + 2.l = 4 + 2. P_3 \times x_3 - P_n\)

\(P_3 \times x_3 + (2.l + M) = (4 + M - P_n) + 2. P_3 \times x_3\); Where \(M\) is an integer \((M \neq 0)\)

\((2.l + M) = (4 + M - P_n) + P_3 \times x_3\); Where \(M\) is an integer \(\neq 0\)...........(8)

But we chose \(M\) such that \((M + 4)\) is divisible by \(x_3\). But let \((M + 4)\) is not divisible by \(P_3\).

But we know that \(P_3\) is divisible by \(x_3\). But \(x_3^2\) does not divide \(P_3\). And we know that \((P_3 \times x_3)\) is divisible by \(x_3\). And we know that \(P_n\) is not divisible by \(x_3\).............(8.1).

Thus by (8): \(x_3\) does not divide \((2.l + M)\). Since \(P_3\) is divisible by \(x_3\), \(P_3\) does not divide \((2.l + M)\) ...............(i)

But \(P_N\) is an arbitrary prime greater than \((P_n-1 + 2)\). Then let \((P_N + A_1)\) and \(P_N\) are two arbitrary consecutive primes greater than \((P_n-1 + 2)\).

Here since \(P_N > (P_n-1 + 2)\) and since \((P_N + A_1) > (P_n-1 + 2)\), \(A_1 \neq (+/-) 2\). Because for any two arbitrary consecutive primes greater than \((P_n-1 + 2)\), the difference between those consecutive primes is greater than 2 (since the greatest twin primes are \(P_n-1\) and \([P_n-1 + 2]\)).

But \(A_1 \neq 2. (x_3 - 1)\). But here \([P_3\mid (A_1 - 2)]\). Since \(A_1 \neq - 2\), there exists an odd number \(P_3\) greater than 1 such that \([P_3\mid (A_1 - 2)]\). Refer the ‘Proof’ below to see the existence of a
prime number \( (P_N + A_1) \) such that \( [P_3] \ (A_1 - 2) \). And refer ‘Proof 1’ to see the existence of an integer \( (P_N - 2) \) such that \( (P_N - 2) = P_3.x_3 \) such that \( P_3 \) is divisible by \( x_3 \). But \( x_3^2 \) does not divide \( P_3 \).

But we know that \( (P_N + A_1) > (P_{n-1} + 2) \). Thus here \( A_1 \neq (+/-) 2 \), since there are finite number of twin primes according to our assumption. BUT REMEMBER THAT \( P_N \) AND \( (P_N + A_1) \) ARE CONSECUTIVE PRIMES greater than \( (P_{n-1} + 2) \).

\[
\text{Here} \ (P_N - 2) = P_3.x_3 \quad \text{and} \quad (P_N + A_1) = P = \text{Prime. That means} \quad P_3.x_3 + (A_1 + 2) = P \\
\text{But (A_1 - 2) is divisible by P_3} \quad \text{. Thus (A_1 + 2) is not divisible by P_3} \quad \text{. Because P_3 does not divide 4.} \\
\text{But since P_3 * x_3 is divisible by P_3, P is not divisible by P_3.} \\
\text{But (A_1 - 2) is divisible by P_3 and since (x_3 | P_3), x_3 | (A_1 - 2). Thus (A_1 + 2) is not divisible by x_3. Because x_3 does not divide 4 since x_3 is an odd number (since (P_N – 2) = P_3.x_3).} \\
\text{But since P_3 * x_3 is divisible by x_3, P is not divisible by x_3.} \\
\text{But P = P_3.x_3 + A_1 + 2 \neq P_3.x_3 + 2.(x_3 - 1) + 2 = P_3.x_3 + 2.x_3 = x_3.(P_3 + 2) \quad \text{. Thus P \neq x_3.(P_3 + 2).} \\
\text{Therefore according to above steps, we can write P_3.x_3 + (A_1 + 2) = P as a prime} \}
\]

But \( (2.l + M) = P_N - P_n + 2 + M = (P_N + A_1) + (M + 2 - A_1 - P_n) \)………………(9)

By (8.1): \( x_3 | (M + 4) \). But \( [P_3] (A_1 - 2) \). ……..(10)

But since \( [P_3] (P_N - 2) \) and since P_3 does not divide \( (A_1 + 2) \), \{ (A_1 + 2) + (P_N - 2) \} does not divide by P_3 . i.e. P (= \( P_N + A_1 \) ) does not divide by P_3. Thus our choice of \( A_1 \) such that \( [P_3] (A_1 - 2) \) is okay.

But \( [P_3] (P_N - 2) \) and \([P_3] (A_1 - 2)\]. Thus \( (P_N - 2) = P_3.x_3 \) and \( (A_1 - 2) = P_3.B_2 \) ; where \( x_3 \) and \( B_2 \) are integers and each of them not equals to 0. Thus \( (P_N + A_1 - 4) = P_3.x_3 + P_3.B_2 = (P - 4) \)
i.e \( P_3 | (P - 4) \)………………..(11)

Let’s consider \( M \) integer such that \( M = P - C \) ; for some integer ‘C’ \( \neq 0 \) …………………..(12).

But \( x_3 | (M + 4) \) and \( P_3 | (P - 4) \) by (8.1) and (11).

By (12): \( P = (M + C) \). Thus \( [M + (M + C)] - 4 = P_3 \). P_L…………..(13)
Where \( P_L = \left( \frac{P - 4}{P_3} \right) \) = integer, but not equals to 0.

Then \( (P_L / x_3) = \left( \frac{P - 4}{(x_3 P_3)} \right) = (P - 4) / (P_N - 2) \)

\[
= \left[ (P_N - 2) + (A_1 - 2) \right] / (P_N - 2) = 1 + \left[ (A_1 - 2) / (P_N - 2) \right] = 1 + \left[ (A_1 - P_N + P_N - 2) / (P_N - 2) \right]
\]

\[
= 2 + \left[ (A_1 - P_N) / (P_N - 2) \right]. \text{Since } A_1 \neq (+/-) 2, \left[ (A_1 - P_N) / (P_N - 2) \right] \text{ is not an integer.}
\]

Thus \( (P_L / x_3) \) is not an integer. Thus here \( x_3 \) does not divide \( P_L \)…………………..(13.1)

But \( (M + 4) / x_3 = P_Q = \text{ integer, but not equals to 0.} \)

Thus by (13): \( (x_3 . P_Q - 4 + C) \) - 4 = \( P_3 . P_L \)

Thus \( C - 4 = \left( \frac{(P_3 . P_L + 4) - x_3 . P_Q}{x_3 . P_Q} \right) \) ……………………..(14)

By (09): \( (2. l + M) = (P_N + A_1) + (M + 2 - A_1 - P_n) = P + (M + 2 - A_1 - P_n) \)

\[
= P + P - C + 2 - A_1 - P_n = 2. P - C - P_3 . B_2 - P_n \quad \text{(15)}
\]

By (14): \( C = \left( \frac{(P_3 . P_L + 8) - x_3 . P_Q}{x_3 . P_Q} \right) \). Then \( 2.P - C - P_n = 2.P + x_3 . P_Q - (P_3 . P_L + 8) - P_n \)

\[
\]

\[
= P_3 . P_L + P_3 . [x_3 . (P_Q / P_3) - (P_n / P_3)] \quad \text{(16)}
\]

But we chose \( P_n \) such that \( (x_3 . P_n) = x_3 . (M + 4) - D . P_3 \); for some integer \( D \) (But we choose \( D \) such that \( x^2 \) \( | (P_L . x_3 + D) \)). Where \( D \) is not divisible by \( x_3 \) and \( D \neq 0 \).

Then \( P_n = (M + 4) - (D . P_3 / x_3) \). Then \( x_3 . (P_Q / P_3) - (P_n / P_3) = (D / x_3) \)

Then by (16): \( 2.P - C - P_n = P_3 . (P_L + (D / x_3)) = P_3 . x_3 \left[ (P_L / x_3) + (D / x^2) \right] \); where \( (P_L / x_3) \)

and \( (D / x^2) \) are not integers (by 13.1). But we choose \( D \) such that \( x^2 \) \( | (P_L . x_3 + D) \).

Then \( 2.P - C - P_n = P_3 . x_3 \left[ (P_L . x_3 + D) / x^3 \right] = P_3 . x_3 . \text{D'} \); where \( \text{D'} = (P_L . x_3 + D) / x^2 \)

= integer , but not equals to 0.

Then \( P_3 | (2.P - C - P_n) \)…………………………(17)

Thus by (15), (17): \( P_3 | (2. l + M) \)……………………..(18)

Thus by (i): \( P_3 \) does not divide \( (2. l + M) \)……………………………..(19)
Thus by (18) and (19): We have a contradiction………………..(20)

Therefore the only possibility is: our assumption (1.0) is false. Therefore there are infinitely many Twin Prime Numbers.

Proof

Let’s prove that \( [ P_3 \mid (A_1 - 2) ] \) when there exist consecutive prime numbers \( P_N \) and \( (P_N + A_1) \) which both are greater than \( (P_{n-1} + 2) \) in this ‘Proof’ as below.

By 2nd reference: \( P_{N+1} = (P_N + A_1) = 2 + \sum_{j=1}^{N} h_j \), where \( h_j = P_{j+1} - P_j \) for all \( j \in \{1, 2, \ldots, N\} \)

Then \( A_1 - 2 = -P_N + \sum_{j=1}^{N} h_j \) ………..(21)

But by 2nd reference: for all \( \epsilon > 0 \), there is a natural number ‘m’ such that for all \( N > m; \)

\( g_N < P_N \cdot \epsilon \)

Let \( \epsilon_s \) is a positive real number \( \epsilon_s = \frac{[ -B + C_s + 2 + k']}{P_s} \), such that \( h_s < P_s \cdot \epsilon_s \) for all \( s > (N - 1) \). Let here the chosen \( \epsilon_s \) implies that \( m = (N - 1) \) (Here \( s \) is going from 1 to \( N \). Then " for all \( s > (N - 1)" \) means \( s = N \). Where \( k' \) is an integer number which is not equals to 0 and \( k' \) is divisible by \( P_3 \). Here the chosen \( k' \) integer number is responsible for \( h_s < P_s \cdot \epsilon_s \) for all \( s > (N - 1) \) (i.e. \( s = N \) ) and \( \epsilon_s > 0 \). That means here the value of \( k' \) is responsible to say " \( \epsilon_s \) is existing such that \( h_s < P_s \cdot \epsilon_s \), for \( s = N \) " . Here \( h_j = b_j \) for all \( j < N = s \). And where \( \Sigma b_j = B \) for \( j < N = s \). Then for some \( C_s \), \( h_s = P_s \cdot \epsilon_s - C_s \); here \( s = N \). *** the meaning of ‘j’ is the order number and \( h_j \) is the prime gap between \( P_{j+1} \) and \( P_j \). Please refer the below content and the 2nd reference. But here we chose \( C_N \) such that \( h_N = P_N \cdot \epsilon_N - C_N \)

But \( h_N = P_N \cdot \epsilon_N - C_N = (-B + 2 + k') \). Where \( k' \) is an integer number which is divisible by \( P_3 \). Now let’s use the 2nd reference to proceed further. By (21):

\( (A_1 - 2) = -P_N + \sum_{j=1}^{N} h_j = -P_N + (-B + 2 + k') + B = (2 - P_N) + k' \) …………..(22)

But \( P_3 \mid (P_N - 2) \). Also we chose \( k' \) as an integer which is divisible by \( P_3 \). Thus by (22):

\( [ P_3 \mid (A_1 - 2) ] \). Thus there exist consecutive prime numbers \( P_N \) and \( (P_N + A_1) \) both greater than \( (P_{n-1} + 2) \) such that \( [ P_3 \mid (A_1 - 2) ] \).
Proof 1

Let’s prove the existence of an integer \((P_N - 2) > P_{n+1} + 2\) such that \((P_N - 2) = P_3x_3\) such that \(P_3\) is divisible by \(x_3\). But \(x_3^2\) does not divide \(P_3\) as below.

By 2\(^{nd}\) reference: \(P_N = 2 + \sum_{j=1}^{N-1} g_j\), where \(g_j = P_{j+1} - P_j\) for all \(j \in \{1, 2, \ldots, N-1\}\)

Then \((P_N - 2) = \sum_{j=1}^{N-1} g_j \ldots \ldots \ldots \ldots (23)\)

But by 2\(^{nd}\) reference: for all \(\epsilon > 0\), there is a natural number ‘\(m_0\)’ such that for all \(N > m_0\);

\(g_N < P_N \cdot \epsilon\).

Let \(\epsilon_s\) is a positive real number \(\epsilon_s = \left\lfloor \frac{-A + x_3^2}{P_s} \right\rfloor\) for all \(s > (N - 2)\). Let here the chosen \(\epsilon_s\) implies that \(m_0 = (N - 2)\) (Here \(s\) is going from 1 to \(N-1\). Then " for all \(s > (N - 2)"\) means \(s = (N - 1)\). Where \(k_1\) is an integer number which is not divisible by \(x_3\). Here the chosen \(k_1\) integer number \((\neq 0)\) is responsible for \(g_s < P_s \cdot \epsilon_s\) for all \(s > (N - 2)\) (i.e. \(s = N - 1\) ) and \(\epsilon_s > 0\). That means here the value of \(k_1\) is responsible to say " \(\epsilon_s\) is existing such that \(g_s < P_s \cdot \epsilon_s\), for \(s = N - 1\) " . Here \(g_j = a_j\) for all \(j < (N - 1) = s\). And where \(\sum a_j = A\) for \(j < (N - 1) = s\). Then for some \(C_s\), \(g_s = P_s \cdot C_s - C_s\); here \(s = (N - 1)\). *** the meaning of ‘\(j\)’ is the order number and \(g_j\) is the prime gap between \(P_{j+1}\) and \(P_j\). Please refer the below content and the 2\(^{nd}\) reference. But here we chose \(C_{N-1}\) such that \(g_{N-1} = P_{N-1} \cdot \epsilon_{N-1} - C_{N-1}\)

But \(g_{N-1} = (-A + x_3^2, k_1)\).

Now let’s use the 2\(^{nd}\) reference to proceed further. By (23): \((P_N - 2) = \sum_{j=1}^{N-1} g_j\)

But \(\sum_{j=1}^{N-1} g_j = A + (-A + x_3^2, k_1) = x_3^2 \cdot k_1 \ldots \ldots \ldots \ldots (24)\)

Thus by (23) and (24): \((P_N - 2) = x_3^2 \cdot k_1\); where \(k_1\) is not divisible by \(x_3\).

Then \((P_N - 2) = x_3(x_3k_1) = x_3P_3\) ; where \(P_3\) is divisible by \(x_3\). But since \(k_1\) is not divisible by \(x_3\), \(P_3\) is not divisible by \(x_3^2\).

Thus \((P_N - 2) = P_3 \cdot x_3\); where \(P_3\) is divisible by \(x_3\). But \(P_3\) is not divisible by \(x_3^2\).
Discussion

We assumed initially that there are finitely many twin primes. After proceeding with that, I ended up with a contradiction. But to get the contradiction, I used that \( P_N \) as a prime number greater than \( (P_{n-1} + 2) \). And we chose \( P_n \) odd integer \( (> 1) \) and also we chose an integer \( A_1 \) such that \( P_5 \mid (A_1 - 2) \). Also to get the contradiction, I used the facts that \( (P_N + 2) \) and \( (P_N - 2) \) as non-primes since \( P_N - 2 > (P_{n-1} + 2) \). And also I have used that \( x_2 \) and \( x_3 \) as natural numbers (since, \( (P_N + 2) \) and \( (P_N - 2) \) are not prime numbers). And also I have used the fact (to get the contradiction as in (20) ): The difference between any two consecutive prime numbers (which are greater than \( (P_{n-1} + 2) \) ) is greater than 2. Therefore to get the contradiction, I have used the facts got from our assumption (1.0). Then the only possibility is our assumption (1.0) is false.

Results

Therefore I have used our assumption (1.0) to get the contradiction finally, as showed in (20). Therefore it is possible to conclude that our assumption (1.0) is false. Thus the negation of the assumption (1.0) is true.

Thus there are infinitely many twin prime numbers.

Appendix

Prime number: A natural number which divides by 1 and itself only.

Twin Prime Numbers: Two prime numbers which have the difference exactly 2.

We denote ‘i’ th prime gap \( g_i = P_{i+1} - P_i \)

Then according to the 2\(^{nd}\) reference; Prime number \( P_N = 2 + \sum_{j=1}^{N-1} g_j \)

Also by 2\(^{nd}\) reference: for all \( \epsilon > 0 \), there is a natural number ‘n’ such that for all \( N - 1 > n \);

\( g_{N-1} < P_{N-1} \cdot \epsilon \)
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