Proof of Twin Prime Conjecture

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Author’s Biography

The author of this research paper is K.H.K. Geerasee Wijesuriya. And this proof of twin prime conjecture is completely K.H.K. Geerasee Wijesuriya's proof.

Geerasee she studied before at Faculty of Science, University of Colombo Sri Lanka. And she graduated with BSc (Hons) in Physics and Mathematics from the University of Colombo, Sri Lanka in 2014. And in March 2018, she completed her first Doctorate Degree in Physics with first class recognition. Now she is following her second PhD in Astrophysics with Belarusian National Technical University.

Geerasee has been invited by several Astronomy/Physics institutions and organizations worldwide, asking to get involve with them. Also, She has received several invitations from some private researchers around the world asking to contribute to their researches. She worked as Mathematics tutor/Instructor at Mathematics department, Faculty of Engineering, University of Moratuwa, Sri Lanka. Furthermore she has achieved several other scientific achievements already.

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I would be thankful to my parents who gave me the strength to go forward with mathematics and Physics knowledge and achieve my scientific goals.

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Abstract

Twin prime numbers are two prime numbers which have the difference of 2 exactly. In other words, twin primes is a pair of prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair. Up to date there is no any valid proof/disproof for twin prime conjecture. Through this research paper, my attempt is to provide a valid proof for twin prime conjecture.

Literature Review

The question of whether there exist infinitely many twin primes has been one of the great open questions in number theory for many years. This is the content of the twin prime conjecture, which states that there are infinitely many primes $p$ such that $p + 2$ is also prime. In 1849, de Polignac made the more general conjecture that for every natural number $k$, there are infinitely many primes $p$ such that $p + 2k$ is also prime. The case $k = 1$ of de Polignac's conjecture is the twin prime conjecture.

A stronger form of the twin prime conjecture, the Hardy–Littlewood conjecture, postulates a distribution law for twin primes akin to the prime number theorem. On April 17, 2013, Yitang Zhang announced a proof that for some integer $N$ that is less than 70 million, there are infinitely many pairs of primes that differ by $N$. Zhang’s paper was accepted by Annals of Mathematics in early May 2013. Terence Tao subsequently proposed a Polymath Project collaborative effort to optimize Zhang’s bound. As of April 14, 2014, one year after Zhang’s announcement, the bound has been reduced to 246. Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that the bound has been reduced to 12 and 6, respectively. These improved bounds were discovered using a different approach that was simpler than Zhang’s and was discovered independently by James Maynard and Terence Tao.
Assumption

Let's assume that there are finitely many twin prime numbers. Therefore we proceed by considering that there are finitely many twin prime numbers. Then let the highest twin prime numbers are \( P_{n-1} \) and \( (P_{n-1} + 2) \). Then for all prime numbers \( P_n \) greater than \( P_{n-1} \), \( (P_n - 2) \) is not a prime number.

Methodology

With this mathematical proof, I use the contradiction method to prove the twin prime conjecture. Let \( P_n \) is an arbitrary prime number greater than \( P_{n-1} \) (because there are infinite number of prime numbers). Then according to our consideration, \( (P_n - 2) \) is not a prime number (And \( P_n \) is a prime such that \( (P_n - 2) \) is divisible by \( x_3 \). To see the meaning of \( x_3 \) and \( P_3 \), please refer the below content). To see the existance of prime \( P_n \) (greater than \( P_{n-1} \)) such that \( (P_n - 2) \) is divisible by \( x_3 \), please refer the ‘Proof’ below. Since \( P_n > 2 \) and since \( P_n \) is a prime number and since \( P_n \) is an odd number, for all prime numbers \( P_i \):

\[
P_i \quad ( < \frac{P_n}{2} ) : \quad P_n / P_i = r_1
\]

Thus \( P_n = P_i * r_1 \).................(01.0)

Where \( r_1 \) is a rational number (which is not a natural number)

But according to our consideration, \( (P_n - 2) \) is not a prime number. Also since \( P_n \) is a prime number greater than 2, \( (P_n - 2) \) is an odd number.

Thus for some prime number \( P_1 \) ( \( < \lfloor (P_n - 2) / 2 \rfloor \) ); \( (P_n - 2) / P_1 = x_1 \). Where we choose \( P_1 \) such that \( x_1 \) is a natural number. But since previously chose \( P_1 \) is any arbitrary prime number less than \( (P_n / 2) \); now we consider \( P_1 = P_i \)

Then \( (P_n - 2) = P_1 * x_1 \).................(02) and \( P_n = P_1 * r_1 \).................(01)

Let \( P_N \) is a prime number greater than \( P_{n-1} \) (which is not \( P_n \)). Then according to our assumption, \( (P_N + 2) \) is not a prime number. Here \( P_N \) is a prime number such that \( (P_N + 2) \) is dividing by prime number \( P_2 \) ........................(1.1)
*** Here we should consider a prime number $P_N$ such that $P_3 \neq x_3$ and $x_3$ is not divisible by $P_3$; whenever $(P_N - 2) = P_3 \cdot x_3$. See the below content in the 'Proof' to see the verification of the existence of prime number $P_N$ such that $P_3 \neq x_3$ and $x_3$ is not divisible by $P_3$; whenever $(P_N - 2) = P_3 \cdot x_3$.

Thus $(P_N + 2) = P_2 \cdot x_2$ for some $x_2$ natural number. Because there are infinitely many prime numbers. Since $P_N$ is a prime number, for some $r_2$ (rational number which is not a natural number): $P_N / r_2 = P_2$.

Thus $(P_N + 2) = P_2 \cdot x_2$ .........................(03) and $P_N = r_2 \cdot P_2$ .....................(04)

$x_1$ and $x_2$ are natural numbers and $P_1$ and $P_2$ are prime numbers.

Since $P_N$ is a prime number, $(P_N - 2)$ is also not a prime number (Since $P_N - 2 > P_{n-1}$)

Then for some prime $P_3$, $(P_N - 2) / P_3 = x_3$. Here we should considered prime number $P_N$ such that $P_3 \neq x_3$ and $x_3$ is not divisible by $P_3$; whenever $(P_N - 2) = P_3 \cdot x_3$. See the below content in the 'Proof' to see the verification of the existence of prime number $P_N$ such that $P_3 \neq x_3$ and $x_3$ is not divisible by $P_3$; whenever $(P_N - 2) = P_3 \cdot x_3$

$(P_N - 2) = P_3 \cdot x_3$ .........................(05)

By (04) and (05): $P_3 \cdot x_3 = P_2 \cdot r_2 - 2$ .........................(06)

But according to the below induction method proof which is in the "Proof" below, there exists the prime $P_n$ such that $(P_n - 2)$ divides by $x_3$.

But $(P_N + 2), (P_N - 2)$ both are odd numbers. Thus $(P_N + 2) = (P_n - 2) + 2l$ for some $l$ integer number.........(06)’

Then $(P_N - 2) = (P_n - 2) + 2l - 4 = P_n + 2l - 6 = P_n + 2 \cdot (l - 3)$ ..........(6.1)’

Since $(P_N - 2)$ is divisible by $P_3$, [ $P_n + 2 \cdot (l - 3)$ ] is divisible by $P_3$. ..........(6.1)

And we know that $(P_N + 2) = (P_n - 2) + 2l \Rightarrow P_N = P_n + 2l - 4$ .........................(*)

Thus by (*): $P_1, r_1 + 2l - 4 = r_2 \cdot P_2$. Thus by (06): $P_3 \cdot x_3 + 2 = P_1, r_1 + 2l - 4$

Thus $P_3 \cdot x_3 - 2l + 6 = P_1, r_1 = P_n$ .......................(6.1.0)
Thus $P_n \cdot x_3 + 2. (l - 3) = P_n + 4. (l - 3) = P_n + 2P_n - 4 - 2P_n = 2P_n - 4 - P_n$ (by (6.1)’)

Thus $P_n \cdot x_3 + 2. (l - 3) = 2P_n - 4 - P_n = P_n''$.

Thus $P_n \cdot x_3 + 2. (l - 3) = P_n''$ ......................(7)

Thus $P_n \cdot x_3 + 2. l = 6 + 2. P_n \cdot x_3 - P_n$

$P_n^*(x_3 + 1)+(2. l - P_n) = (6 - P_n) + 2. P_n^* x_3$; Where $M$ is an integer...........(8)

But we chose $M$ such that $(P_n - 6 - M) = x_3 \cdot m_1$ for some integer $m_1$ ($m_1$ is divisible by $x_3$). But we choose an integer $F$ (not divisible by $x_3$) such that $(P_n + F)$ is not divisible by $x_3$ and we chose integer $F$ and $m_1$ such that $(F + x_3 \cdot m_1 + P_n)$ divisible by $x_3$. Because $(x_3 \cdot m_1)$ has the order of $x_3$ as at least 2, since $m_1$ is divisible by $x_3$.

And we chose $P_n$ such that $(P_n - 2)$ divides by $x_3$..............(9) But here $P_n \neq x_3$.

Thus $(P_n - 6 - M) = x_3 \cdot m_1$ and $(P_n - 2) = x_3 \cdot m_0$ for some integer $m_1$ and $m_0$.

But $(P_n - 2)$ divides by $x_3$. Thus according to our choice, $[(P_n - 2)$ - $(P_n - 2)]$ divides by $x_3$. i.e. $(P_n - P_n)$ divides by $x_3$. Thus $P_n - P_n = x_3 \cdot m_2$; for some integer $m_2$.

Thus according to our choice: $[(P_n - 6 - M) = x_3 \cdot m_1$] and $[(P_n - P_n) = x_3 \cdot m_2]$ for some integer $m_1$ and integer $m_2$.

But $(P_n - 6 - M) = x_3 \cdot m_1$ and $(P_n - 2) = x_3 \cdot m_0$ for some integer $m_1$ and integer $m_0$.

Thus $(4 + M) = x_3 \cdot m_3$; $m_3 = (m_0 - m_1)$ for an integer $m_3$ .................(10)

By (*): $P_n - P_n + 4 = 2l$

Thus $2l - P_3 + M = (P_n - P_n) + 4 - P_3 + M = x_3 \cdot m_2 + (M + 4 - P_3) = x_3 \cdot r'$ (since $[M + 4]$ is divisible by $x_3$ and since $P_3$ is a prime number which is not equal to $x_3$). And here $r'$ is not an integer. But $r'$ is a rational number.

Then $2l - P_3 + M - F = (P_n - P_n) + 4 - P_3 + M - F = x_3 \cdot m_2 + (M + 4 - P_3 - F) = x_3 \cdot r''$; $r''$ is not an integer. Because $(P_3 + F)$ is not divisible by $x_3$.

Thus by (8): $(x_3 + 1) \cdot P_3 + x_3 \cdot r'' = - x_3 \cdot m_1 - F + 2. P_3 \cdot x_3$ .................(11)
But we chose \( F \) and \( m_1 \) such that \( (x_3 \cdot m_1 + P_3 + F) \) is divisible by \( x_3 \). Because \( (x_3 \cdot m_1) \) has order of \( x_3 \) as 2, since \( m_1 \) is divisible by \( x_3 \). Let \( (x_3 \cdot m_1 + P_3 + F) = x_3 \cdot m_4 \); where \( m_4 \) is an integer.

Thus by (11): \( P_3, x_3 + [x_3, r''] = 2, P_3, x_3 - x_3, m_4 \) where \( m_4 \) is an integer.

Then \( r'' = P_3 - m_4 \) ............(12)

But \( m_4 \) and \( P_3 \) both are integers. But \( r'' \) is not an integer. Thus by (12), we have a contradiction. Therefore the only possibility is: our assumption is false. Therefore there are infinitely many Twin Prime Numbers.

**Proof**

Now let’s prove that there exists infinite number of Prime numbers \( P_n \) such that \( (P_n - 2) = x_3 \cdot m_0 \) for an integer \( m_0 \).

Let’s consider the statement \( Q(n) : [P(n) - 2] / x_3 = x(n) \); where \( P(n) \) is the \( n \)th prime number which obeys \( [P(n) - 2] = x_3 \cdot x(n) \). And \( x(n) \) is the \( n \)th integer which is in the form of \( [P(n) - 2] / x_3 \).

For \( n = 1 \), L.H.S. of \( Q(1) = [2 - 2] / x_3 = 0 \). But for \( x(n) = 0 \) (which is an integer), R.H.S. of \( Q(1) \): 0. Thus for \( n = 1 \), the result holds.

Now assume for \( n = s \), the result \( Q(s) \) holds. Then \( [P_s - 2] / x_3 = x(s) = \text{natural number} \), where \( x(s) = \text{integer number} \).

Here we must considered \( n = s \) part as below.

Let \( C_s \) is a positive real number \( C_s = [ - B + P_s + C_s - 2 + x_3, k' ] / P_s > 0 \) for all \( s > (L - 2) \), \( h_s < P_s \cdot C_s \) (since the only existing \( s > (L - 2) \) is \( (L - 1) \)); " for all \( s > (L - 2) \) means \( s = (L - 1) \)’’. Where \( k' \) is an integer number. Here the chosen \( k' \) integer number is responsible for \( h_s < P_s \cdot C_s \) for all \( s > (L - 2) \) and \( k' \) is responsible for \( C_{L-1} > 0 \). That means here the value of \( k' \) is responsible to say : " \( C_s \) is existing such that \( h_s < P_s \cdot C_s \), for \( s = (L - 1) \)". Here \( h_j = b_j \) for all \( j < (L - 1) = s \). And where \( \Sigma b_j = B \) for \( j < (L - 1) = s \). Then for \( C_s, h_s = P_s \cdot C_s - C_s \); here \( s \equiv L - 1 \). *** the meaning of ‘j’ is the order number and \( h_j \) is the prime gap between \( P_{j+1} \) and \( P_j \), please refer the below content and the 2nd reference.
But \( s \equiv (L - 1) \). But here we chose \( C_{L-1} \) such that \( h_{L-1} = P_{L-1} * C_{L-1} - C_{L-1} \)

But \( h_{L-1} = P_{L-1} * C_{L-1} - C_{L-1} = (P_s - B - 2 + x_3 .k') \). Where \( k' \) is an integer number.

Then let’s show for \( n = s + 1 \), \( Q(s+1) \) holds. We denote \( P(s+1) = P_L \)

But we know \( [P_s - 2] / x_3 = x(s) \) …………………(13)

Now let’s use the 2nd reference to proceed further.

By 2nd reference, \( P_L = 2 + \sum_{j=1}^{L-1} h_j \) …………………(i)

But we know already that for \( L_{-1} > 0 \), \( h_{L-1} < P_{L-1} * C_{L-1} \) for \( L - 1 = s \).

Here \( s \equiv (L - 1) \)

(*** refer the 2nd reference below)

Then we already know that for some \( C_{L-1} \) positive number, \( h_{L-1} = P_{L-1} * C_{L-1} - C_{L-1} \).

But \( h_{L-1} = P_{L-1} * C_{L-1} - C_{L-1} \) for \( (L - 1) \equiv s \)

We know already that \( C_{L-1} = [P_s - B + C_{L-1} - 2 + x_3 .k'] / P_{L-1} > 0 \).

And \( h_{L-1} = P_{L-1} * C_{L-1} - C_{L-1} = (-B + P_s - 2 + x_3 .k') \). Where \( k' \) is an integer number. We know already that the chosen \( k' \) integer number is responsible for \( C_{L-1} > 0 \).

We know that \( h_j = b_j \) for all \( j < (L - 1) \). Where \( b_j \) is a natural number. Also we know that \( \Sigma b_j = B \) for \( j < L - 1 \).

Thus by (i): \( P_L = 2 + P_s + x_3.k' - B - 2 + B = x_3 .k' + P_s \)

Thus \( (P_L - 2) = (P_s - 2) + x_3.k' \) …………………(14)

But \( (P_s - 2) \) is divisible by \( x_3 \) and \( (P_s - 2) / x_3 = x(s) \) according to (13). Thus \( (P_L - 2) \) is divisible by \( x_3 \) according to (14), since \( x_3.k' \) is divisible by \( x_3 \).

Thus \( (P_L - 2) \) is divisible by \( x_3 \). i.e. \( [P(s+1) - 2] \) is divisible by \( x_3 \).

Thus for \( n = s + 1 \) ( = L) , the result \( Q(n+1) \) holds. Thus by mathematical induction method:
There exists infinite number of prime numbers $P_L$ such that $(P_L - 2) = x_3.m'_0$ for some integer number $m'_0$.

Thus there exists $P_n$ prime (where we consider that as prime numbers greater than $P_{n-1}$) such that $(P_n - 2)$ is divisible by $x_3$ and $(P_n - 2) = x_3.m_0$ for some integer number $m_0$, whenever $P_3 \neq x_3$.

**Verification of existence of prime number $P_N$ (greater than $P_{n-1}$) such that $(P_N - 2) = P_3.x_3$; for the integer number $x_3$ which is not divisible by $P_3$**

Let $C_s$ is a positive real number $C_s = [ - A + C_s + (P_3).t_s ] / P_s > 0$ for all $s > (R - 2)$, $g_s < P_s * C_s$ (since the only existing $s > (R - 2)$ is $(R - 1)$, "for all $s > (R - 2)$ means $s = (R - 1)"$). Where $t_s$ is an integer number such that $t_s$ is not divisible by $P_3$. Here the chosen $t_s$ integer number is responsible for $g_s < P_s * C_s$ for all $s > (R - 2)$ and $t_s$ is responsible for $C_{R-1} > 0$. That means here the value of $t_s$ is responsible to say : "$C_s$ is existing such that $g_s < P_s * C_s$, for $s = (R-1)$ ". Here $g_j = a_j$ for all $j < (R - 1) = s$. And where $\Sigma a_j = A$ for $j < (R - 1) = s$. Then for $C_s$, $g_s = P_s * C_s - C_s$; here $s \equiv R - 1$. *** the meaning of ‘$j$’ is the order number and $g_j$ is the prime gap between $P_{j+1}$ and $P_j$.

But $s \equiv (R - 1)$. But here we chose $C_{R-1}$ such that $g_{R-1} = P_{R-1} * C_{R-1} - C_{R-1}$

But $g_{R-1} = P_{R-1} * C_{R-1} - C_{R-1} = (- A + P_3.t_s)$. Where $t_s$ is an integer number which is not divisible by $P_3$.

Now let’s use the 2nd reference to proceed further.

By 2nd reference, $P_R = 2 + \sum_{j=1}^{R-1} g_j$ .................(ii)

But we know already that for $C_{R-1} > 0$, $g_{R-1} < P_{R-1} * C_{R-1}$ for $R - 1 = s$.

Here $s \equiv (R - 1)$

(** refer the 2nd reference below)

Then we already know that for some $C_{R-1}$ positive number, $g_{R-1} = P_{R-1} * C_{R-1} - C_{R-1}$. 

But $g_{R-1} = P_{R-1} \cdot C_{R-1} - C_{R-1}$ for $(R - 1) \equiv s$

We know already that $C_{R-1} = \lfloor -A + C_{R-1} + P_{3.t} \rfloor / P_{R-1} > 0$.

And $g_{R-1} = P_{R-1} \cdot C_{R-1} - C_{R-1} = (-A + P_{3.t})$. Where $t$ is an integer number that is not divisible by $P_3$. We know already that the chosen $t$ integer number is responsible for $C_{R-1} > 0$.

We know that $g_j = a_j$ for all $j < (R - 1)$. Where $a_j$ is a natural number. Also we know that $\Sigma a_j = A$ for $j < R - 1$.

Thus by (ii): $P_R = 2 + P_{3.t} - A + A = P_{3.t} + 2$

Thus there exists prime number $P_R$ such that $(P_R - 2) = P_{3.t}$ ………..(15) where $t$ is not divisible by $P_3$.

Now put $N \equiv R$. Then we can state that $(P_N - 2) = P_{3.x_3}$ ; for some integer $x_3$ which is not divisible by $P_3$.

**Discussion**

We assumed initially that there are finitely many twin primes. After proceeding with that, I ended up with a contradiction. But to get the contradiction, I used that $P_n$ and $P_N$ as primes numbers greater than $P_{n-1}$ . Also to get the contradiction, I used the facts that $(P_n - 2)$ and $(P_N + 2)$ and $(P_N - 2)$ as non-primes. And also I have used that $x_1$, $x_2$ and $x_3$ as natural numbers (since $(P_n - 2)$, $(P_N + 2)$ and $(P_N - 2)$ are not prime numbers). Therefore to get the contradiction, I have used the facts got from our assumption. Then the only possibility is our assumption is false.

**Results**

Therefore I have used our assumption to get a contradiction finally as showed in (A.6). Therefore it is possible to conclude that our assumption is false.

**Thus there are infinitely many twin prime numbers.**
Appendix

Prime number: A natural number which divides by 1 and itself only.

Twin Prime Numbers: Two prime numbers which have the difference exactly 2.

We denote ‘i’ th prime gap $g_i = P_{i+1} - P_i$

Then according to the 2nd reference; Prime number $P_N = 2 + \sum_{j=1}^{N-1} g_j$

Also by 2nd reference: for all $\epsilon > 0$, there is a natural number ‘n’ such that for all $N-1 > n$;

$g_{N-1} < P_{N-1} \cdot \epsilon$

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