Via Geometric (Clifford) Algebra: Equation for Line of Intersection of Two Planes

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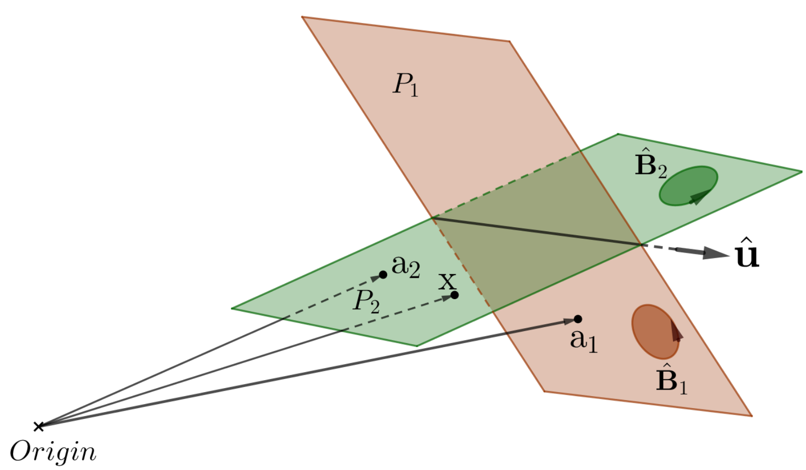
Abstract

As a high-school-level example of solving a problem via Geometric Algebra (GA), we show how to derive an equation for the line of intersection between two given planes. The solution method that we use emphasizes GA's capabilities for expressing and manipulating projections and rotations of vectors.



"Find the equation, in the form $\mathbf{z} = \mathbf{z}_0 + \lambda \hat{\mathbf{u}}$*, of the line of intersection between the planes* $P_1 : \ (\mathbf{x} - \mathbf{a}_1) \wedge \hat{\mathbf{B}}_1$ *and* $P_2 : \ (\mathbf{x} - \mathbf{a}_2) \wedge \hat{\mathbf{B}}_2$*."*

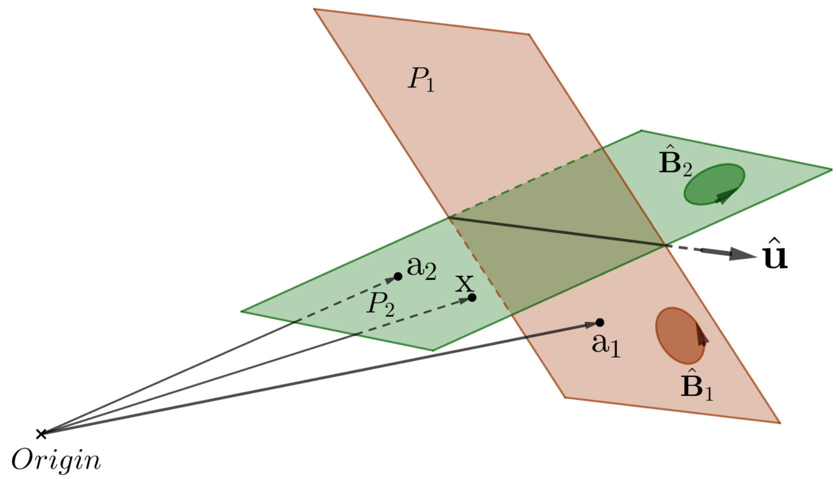


Figure 1: Plane P_1 consists of the endpoints of all those vectors $\mathbf{x}$ that satisfy the condition expressed by the equation $(\mathbf{x} - \mathbf{a}_1) \wedge \hat{\mathbf{B}}_1 = 0$. Plane P_2 consists of the endpoints of all those vectors $\mathbf{x}$ that satisfy the condition expressed by the equation $(\mathbf{x} - \mathbf{a}_2) \wedge \hat{\mathbf{B}}_2 = 0$.

1 Introduction

The line of intersection of two planes is an important element of many mathematical and physical problems. Here, we'll develop an equation for such a line using Geometric-Algebra (GA) concepts that are discussed in greater detail in Refs. [1] and [2].

2 Problem Statement

> *In Fig. 1,'Given the planes* $(\mathbf{x} - \mathbf{a}_1) \wedge \hat{\mathbf{B}}_1 = 0$ *and* $(\mathbf{x} - \mathbf{a}_2) \wedge \hat{\mathbf{B}}_2 = 0$, *derive an equation for their line of intersection, in the parametric form* $\mathbf{z} = \mathbf{z}_0 + \lambda \hat{\mathbf{u}}$.

3 Solution

To define the line of intersection, we will find the line's direction ($\hat{\mathbf{u}}$) and one point along the line. As that point, we will choose the one closest to the origin. We will call the vector from the origin to that point $\mathbf{z}_0$.

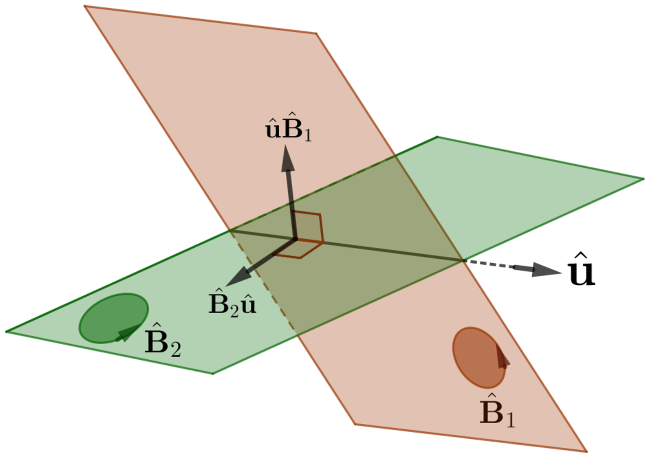


Figure 2: The vector $\hat{\mathbf{u}}\hat{\mathbf{B}}_1$ is parallel to the bivector $\hat{\mathbf{B}}_1$, and the vector $\hat{\mathbf{B}}_2\hat{\mathbf{u}}$ is parallel to the bivector $\hat{\mathbf{B}}_2$. Both vectors are 90° rotations of the vector $\hat{\mathbf{u}}$, but are not necessarily perpendicular to each other. Instead, the angle between them is equal to the angle between the two planes.

3.1 Finding $\hat{\mathbf{u}}$

The vector $\hat{\mathbf{u}}$ can be found by several routes, but a simple one is to consider the two vectors $\hat{\mathbf{u}}\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2\hat{\mathbf{u}}$ (Fig. 2). The vector $\hat{\mathbf{u}}$ that we seek is the dual of the unit bivector $\hat{\mathbf{Q}}$ (Fig. 3) that contains $\hat{\mathbf{u}}\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2\hat{\mathbf{u}}$. We find that bivector by simplifying the $\left(\hat{\mathbf{u}}\hat{\mathbf{B}}_2\right) \wedge \left(\hat{\mathbf{B}}_1\hat{\mathbf{u}}\right)$. How? We begin by recognizing that because $\hat{\mathbf{u}}$ is parallel to $\hat{\mathbf{B}}_1$, $\hat{\mathbf{u}}\hat{\mathbf{B}}_1 = -\hat{\mathbf{B}}_1\hat{\mathbf{u}}$. Similarly, $\hat{\mathbf{B}}_2\hat{\mathbf{u}} = -\hat{\mathbf{u}}\hat{\mathbf{B}}_2$. Thus,

$$\begin{aligned}\left(\hat{\mathbf{u}}\hat{\mathbf{B}}_1\right) \wedge \left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right) &= \langle\left(\hat{\mathbf{u}}\hat{\mathbf{B}}_1\right)\left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right)\rangle_2 \\ &= \langle\left(-\hat{\mathbf{B}}_1\hat{\mathbf{u}}\right)\left(-\hat{\mathbf{u}}\hat{\mathbf{B}}_2\right)\rangle_2 \\ &= \langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2.\end{aligned}$$

The corresponding unit bivector (which we have called $\hat{\mathbf{Q}}$) is $\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2/\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|$. Therefore,

$$\begin{aligned}\hat{\mathbf{u}} &= \hat{\mathbf{Q}}I_3^{-1} \\ &= \left[\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2/\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|\right] I_3^{-1}.\end{aligned} \tag{3.1}$$

3.2 Finding $\mathbf{z}_0$

The vector $\mathbf{z}_0$ that we seek lies in a plane that passes through the origin, and is parallel to $\mathbf{Q}$ (Fig. 4). Vectors $\hat{\mathbf{u}}\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2\hat{\mathbf{u}}$ are also parallel to said plane. We will find $\mathbf{z}_0$ by solving the triangle whose vertices are the endpoint of said vector, and the endpoints of the projections, upon that same plane, of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ (Figs. 5, 6).

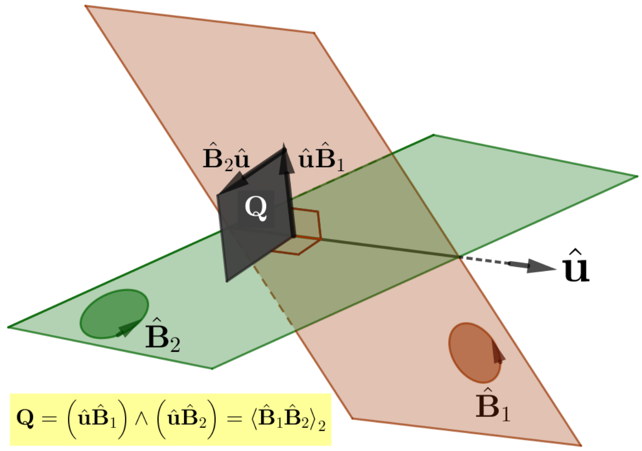


Figure 3: The bivector $\mathbf{Q}$ contains (and is the outer product of))the vectors $\hat{\mathbf{u}}\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2\hat{\mathbf{u}}$. Although those two vectors are unit vectors, $\mathbf{Q}$ is not necessarily a unit bivector, because the two vectors are not necessarily perpendicular to each other.

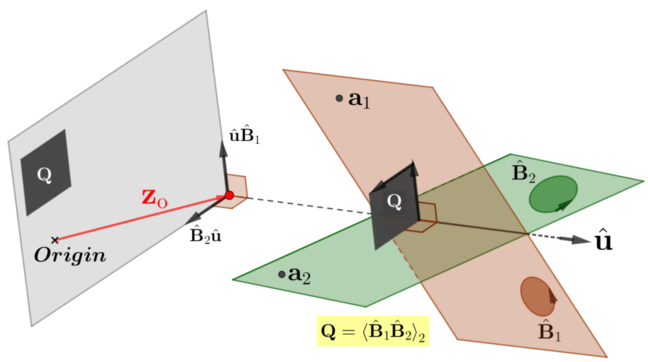


Figure 4: The vector $\mathbf{z}_0$ that we seek lies in a plane that passes through the origin, and is parallel to $\mathbf{Q}$. Vectors $\hat{\mathbf{u}}\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2\hat{\mathbf{u}}$ are also parallel to said plane.

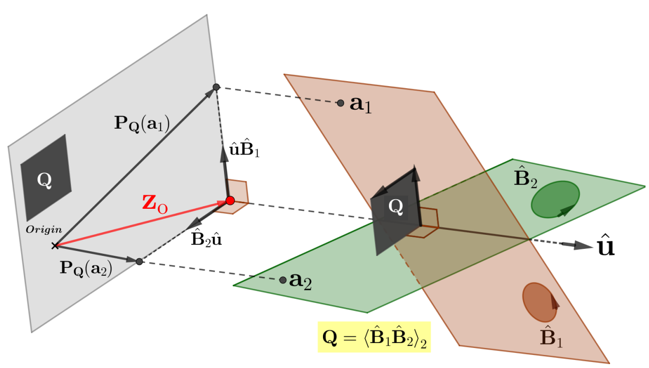


Figure 5: To Fig. 4, we have added $\mathbf{P}_\mathbf{Q}(\mathbf{a}_1)$ (the projection of vector $\mathbf{a}_1$) and $\mathbf{P}_\mathbf{Q}(\mathbf{a}_2)$ (the projection of vector $\mathbf{a}_2$).

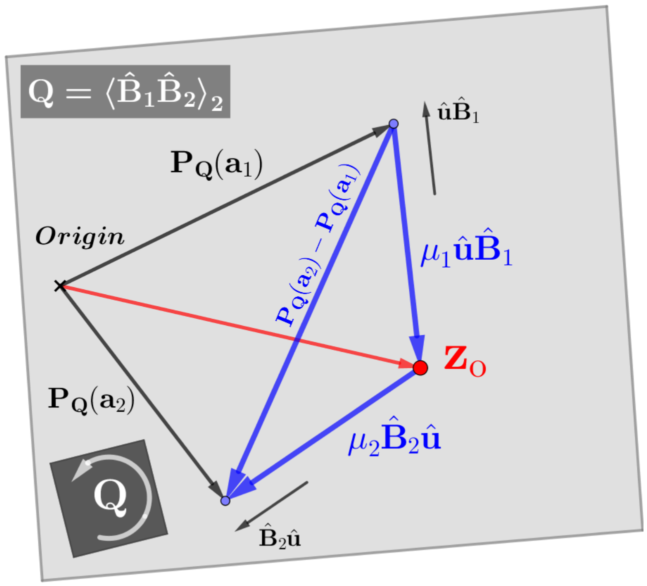


Figure 6: Looking downward (i.e., in the direction $-\hat{\mathbf{u}}$) upon the gray plane shown in Figs. 4 and 5. To find $\mathbf{z}_0$, we will begin by solving the blue triangle for the vector $\mu_1\hat{\mathbf{u}}\hat{\mathbf{B}}_1$. Then, $\mathbf{z}_0 = \mathbf{P_Q}\left(\mathbf{a}_1\right) + \mu_1\hat{\mathbf{u}}\hat{\mathbf{B}}_1$.

From Fig. 6,

$$\mu_1\hat{\mathbf{u}}\hat{\mathbf{B}}_1 + \mu_2\hat{\mathbf{B}}_2\hat{\mathbf{u}} = \mathbf{P_Q}\left(\mathbf{a}_2\right) - \mathbf{P_Q}\left(\mathbf{a}_1\right).$$

To solve for μ_1, we'll eliminate μ_2 by taking the outer product of both sides with the vector $\hat{\mathbf{B}}_2\hat{\mathbf{u}}$:

$$\begin{aligned}\left(\mu_1\hat{\mathbf{u}}\hat{\mathbf{B}}_1 + \mu_2\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right) \wedge \left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right) &= \left[\mathbf{P_Q}\left(\mathbf{a}_2\right) - \mathbf{P_Q}\left(\mathbf{a}_1\right)\right] \wedge \left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right) \\ \mu_1\left(\hat{\mathbf{u}}\hat{\mathbf{B}}_1\right) \wedge \left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right) &= \left[\mathbf{P_Q}\left(\mathbf{a}_2\right) - \mathbf{P_Q}\left(\mathbf{a}_1\right)\right] \wedge \left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right).\end{aligned}$$

Next, we use properties of projections and the outer product to solve for μ_1:

$$\begin{aligned}\mu_1\left[\left\langle\left(\hat{\mathbf{u}}\hat{\mathbf{B}}_1\right)\left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right)\right\rangle_2\right] &= \left[\mathbf{P_Q}\left(\mathbf{a}_2 - \mathbf{a}_1\right)\right] \wedge \left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right) \\ \mu_1\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2 &= \left\langle\underbrace{\left\{\left[\left(\mathbf{a}_2 - \mathbf{a}_1\right)\cdot\hat{\mathbf{Q}}\right]\hat{\mathbf{Q}}^{-1}\right\}}_{=\mathbf{P_Q}(\mathbf{a}_2-\mathbf{a}_1)}\left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right)\right\rangle_2 \\ \mu_1\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2 &= -\left\langle\left\{\left[\left(\mathbf{a}_2 - \mathbf{a}_1\right)\cdot\hat{\mathbf{Q}}\right]\hat{\mathbf{Q}}\right\}\left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right)\right\rangle_2 \\ \mu_1\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\left[\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2^{-1}\right] &= \left\langle\left\{\left[\left(\mathbf{a}_2 - \mathbf{a}_1\right)\cdot\hat{\mathbf{Q}}\right]\hat{\mathbf{Q}}\right\}\left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right)\right\rangle_2\left[\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\right]/\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|^2 \\ \mu_1 &= \left\langle\underbrace{\left\langle\left(\mathbf{a}_2 - \mathbf{a}_1\right)\hat{\mathbf{Q}}\right\rangle_1}_{=(\mathbf{a}_2-\mathbf{a}_1)\cdot\hat{\mathbf{Q}}}\hat{\mathbf{Q}}\left(\hat{\mathbf{B}}_2\hat{\mathbf{u}}\right)\right\rangle_2\left[\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\right]/\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|^2 \\ &= \left\langle\left\langle\left(\mathbf{a}_2 - \mathbf{a}_1\right)\hat{\mathbf{Q}}\right\rangle_1\hat{\mathbf{Q}}\left(-\hat{\mathbf{u}}\hat{\mathbf{B}}_2\right)\right\rangle_2\left[\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\right]/\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|^2 \\ &= \left\langle\left\langle\left(\mathbf{a}_2 - \mathbf{a}_1\right)\hat{\mathbf{Q}}\right\rangle_1\left(-\hat{\mathbf{Q}}\hat{\mathbf{u}}\right)\hat{\mathbf{B}}_2\right\rangle_2\left[\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\right]/\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|^2.\end{aligned}$$

$\hat{\mathbf{Q}}^{-1} = \hat{\mathbf{Q}}$, and $\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2^{-1} = -\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2/\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|^2/$

Now, we recall that $\hat{\mathbf{u}} = \hat{\mathbf{Q}} I_3^{-1}$. Therefore, $-\hat{\mathbf{Q}}\hat{\mathbf{u}} = -\hat{\mathbf{Q}}\hat{\mathbf{Q}} I_3^{-1} = I_3^{-1}$, and

$$\mu_1 = \langle\langle(\mathbf{a}_2 - \mathbf{a}_1)\,\hat{\mathbf{Q}}\rangle_1 I_3^{-1}\hat{\mathbf{B}}_2\rangle_2 \langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2 / \|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|^2. \tag{3.2}$$

Our purpose in finding μ_1 was to then proceed to find $\mu_1\hat{\mathbf{u}}\hat{\mathbf{B}}_1$. We'll do that now.

$$\begin{aligned}\mu_1\hat{\mathbf{u}}\hat{\mathbf{B}}_1 &= \underbrace{\left\{\left[\frac{\langle\langle(\mathbf{a}_2 - \mathbf{a}_1)\,\hat{\mathbf{Q}}\rangle_1 I_3^{-1}\hat{\mathbf{B}}_2\rangle_2}{\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|^2}\right]\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\right\}}_{=\mu_1}\underbrace{\left\{\left[\frac{\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2}{\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|}\right] I_3^{-1}\right\}}_{=\hat{\mathbf{u}}}\hat{\mathbf{B}}_1 \\ &= -\left\{\langle\langle(\mathbf{a}_2 - \mathbf{a}_1)\,\hat{\mathbf{Q}}\rangle_1 I_3^{-1}\hat{\mathbf{B}}_2\rangle_2 I_3^{-1}\hat{\mathbf{B}}_1\right\} / \|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|.\end{aligned}$$

To finish, we recognize that $I_3^{-1} = -I_3$, so that

$$\mu_1\hat{\mathbf{u}}\hat{\mathbf{B}}_1 = -\left\{\langle\langle(\mathbf{a}_2 - \mathbf{a}_1)\,\hat{\mathbf{Q}}\rangle_1 I_3\hat{\mathbf{B}}_2\rangle_2 I_3\hat{\mathbf{B}}_1\right\} / \|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|.$$

Then,

$$\begin{aligned}\mathbf{z}_0 &= \mathbf{P}_Q\left(\mathbf{a}_2 - gabolda_1\right) + \mu_1\hat{\mathbf{u}}\hat{\mathbf{B}}_1 \\ &= -\langle(\mathbf{a}_2 - \mathbf{a}_1)\,\hat{\mathbf{Q}}\rangle_1\hat{\mathbf{Q}} - \frac{\langle\langle(\mathbf{a}_2 - \mathbf{a}_1)\,\hat{\mathbf{Q}}\rangle_1 I_3\hat{\mathbf{B}}_2\rangle_2 I_3\hat{\mathbf{B}}_1}{\|\langle\hat{\mathbf{B}}_1\hat{\mathbf{B}}_2\rangle_2\|}.\end{aligned} \tag{3.3}$$

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[2] J. Smith, 2016, "Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra", http://vixra.org/abs/1610.0054.