The universal profinitization of a topological space

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To a topological space X we attach in two equivalent ways a profinite space X' and a continuous map $\phi : X \to X'$ such that, for any continuous map $f : X \to Y$, where Y is a profinite space, there is a unique continuous map $f' : X' \to Y$ such that $f' \circ \phi = f$.

Let X be a topological space.

Let X' be a profinite space. Say that a map $\phi : X \to X'$ is a ***universal profinitization*** of X, or, more concisely, that $\phi : X \to X'$ has *Property (P)*, if ϕ is continuous and if for any continuous map $f : X \to Y$, where Y is a profinite space, there is a unique continuous map $f' : X' \to Y$ such that $f' \circ \phi = f$:

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, dashed, "f'"] \\ & Y. & \end{tikzcd}$$

Such a pair (X', ϕ), if it exists, is unique in an obvious sense. More precisely, a simple and standard argument shows that, if $\phi' : X \to X''$ also has Property (P), then

• there is a unique continuous map $u : X' \to X''$ such that $u \circ \phi = \phi'$:

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X' \arrow[rr, "u"'] & & X'', \end{tikzcd}$$

• there is a unique continuous map $v : X'' \to X'$ such that $v \circ \phi' = \phi$:

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X' & & X''. \arrow[ll, "v"] \end{tikzcd}$$

- u and v are inverse homeomorphisms.

The purpose of this text is to prove that such a pair *does* exist, and to define it in two (necessarily equivalent) ways.

The first method is to use the Stone-Čech compactification $\beta(X)$ of X, and to set $X' := \beta(X)/\sim$ where $x_1 \sim x_2$ if and only if x_1 and x_2 are in the same connected component. The only thing to verify is that $\beta(X)/\sim$ is a profinite space. To prove this, the only slightly delicate point is to check that $\beta(X)/\sim$ is Hausdorff, but this follows from Corollary 5.7.11 in [1].

The sequel is dedicated to the second method, which can be described as follows:

Let $A(X)$ be the boolean algebra formed by the clopen subsets of X, let X' be the set of all boolean algebra morphisms from $A(X)$ to $\mathbf{2} := \{0, 1\}$, and define $\phi : X \to X'$ by

$$(\phi(x))(U) = 1 \iff x \in U$$

for all point x of X and all clopen subset U of X. By Proposition 4.1.3 and Lemma 4.1.8 in [1], there is unique topology τ on X' such that the subsets

$$O_U := \{x' \in X' \mid x'(U) = 0\}$$

with U clopen in X, form a basis for τ. We equip X' with this topology τ. By Proposition 4.1.11 in [1] the space X' is profinite. Our main result is that this works, that is

Theorem. *The map $\phi : X \to X'$ defined above has Property (P).*

To prove the theorem we first claim

(a) ϕ is continuous.

Proof. We easily check that $\phi^{-1}(O_U) = X \setminus U$ for all clopen subset U of X. □

We claim

(b) $\phi(X)$ is dense in X'.

Proof. Using Lemma 4.1.8 in [1] again we see that any nonempty open subset of X' contains some O_U with $U \neq X$ (U clopen). As noted above, if x is in $X \setminus U$, then $\phi(x)$ is in O_U. □

Say that a topological space Y has *Property (Q)* if Y is profinite and if, for any continuous map $f : X \to Y$, there is a unique continuous map $f' : X' \to Y$ such that $f' \circ \phi = f$. Our task is to show that all profinite spaces have Property (Q).

We claim

(c) If $Y = \lim_i Y_i$ with Y_i profinite for all i and if Y_i has Property (Q) for all i, then Y has Property (Q).

Proof. This is clear. □

In view of (c), it suffices to verify that all *finite* discrete topological spaces have Property (Q).

Say that a topological space Y has *Property (Q')* if Y is profinite and if, for any continuous map $f : X \to Y$, there is *at least one* continuous map $f' : X' \to Y$ such that $f' \circ \phi = f$.

In view of (b),

(d) Properties (Q) and (Q') are equivalent.

Above we denoted by **2** the set $\{0, 1\}$ viewed a boolean algebra. We also denote by **2** the set $\{0, 1\}$ when it is viewed a discrete topological space.

We claim

(e) **2** has Property (Q').

Proof. Given a continuous map $f : X \to \mathbf{2}$ we define $f' : X' \to \mathbf{2}$ by $f'(x') = x'(f^{-1}(1))$. This implies $f'^{-1}(0) = O_{f^{-1}(1)}$. By Lemma 4.1.8 in [1], this is a clopen subset of X'. This shows that f' is continuous. Finally we have

$$f'(\phi(x)) = (\phi(x))(f^{-1}(1)) = f(x). \quad \square$$

We claim

(f) $\mathbf{2}^n$ has Property (Q) for all $n \in \mathbb{N}$.

Proof. This follows from (c), (d) and (e). □

We claim

(g) If Y has Property (Q') and Y_0 is a closed subspace of Y, then Y_0 has Property (Q').

Proof. Let $i : Y_0 \hookrightarrow Y$ be the inclusion, let $f : X \to Y_0$ be continuous, set $g := i \circ f$ and let $g' : X' \to Y$ be a continuous map satisfying $g' \circ \phi = g$. We must show that

g induces a map $f' : X' \to Y_0$:

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "f"] \arrow[ddr, "g"'] & & X' \arrow[dl, dashed, "f'"'] \arrow[ddl, "g'"] \\
& Y_0 \arrow[d, hook, "i"'] & \\
& Y. &
\end{tikzcd}$$

In other words we must show $g'(X') \subset Y_0$, that is $g'^{-1}(Y_0) = X'$. But this follows from the facts that $g'^{-1}(Y_0)$ is a closed subspace of X' containing $\phi(X)$, and that $\phi(X)$ is dense in X' by (b). □

We claim

(h) All finite discrete topological spaces have Property (Q).

Proof. This follows from (d), (f) and (g). □

As already indicated, the Theorem follows from (c) and (h). □

Reference.

[1] Borceux, F., & Janelidze, G. (2001). **Galois Theories** (Cambridge Studies in Advanced Mathematics). Cambridge: Cambridge University Press.
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Tex file available at

https://tinyurl.com/y5ef4y73 and https://tinyurl.com/yynts2eg

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