

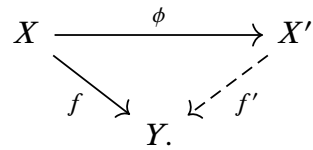
# The universal profinitization of a topological space

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To a topological space  $X$  we attach in two equivalent ways a profinite space  $X'$  and a continuous map  $\phi : X \rightarrow X'$  such that, for any continuous map  $f : X \rightarrow Y$ , where  $Y$  is a profinite space, there is a unique continuous map  $f' : X' \rightarrow Y$  such that  $f' \circ \phi = f$ .

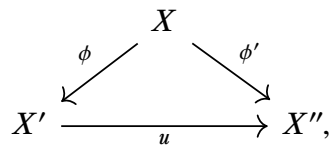
Let  $X$  be a topological space.

Let  $X'$  be a profinite space. Say that a map  $\phi : X \rightarrow X'$  is a *universal profinitization* of  $X$ , or, more concisely, that  $\phi : X \rightarrow X'$  has *Property (P)*, if  $\phi$  is continuous and if for any continuous map  $f : X \rightarrow Y$ , where  $Y$  is a profinite space, there is a unique continuous map  $f' : X' \rightarrow Y$  such that  $f' \circ \phi = f$ :

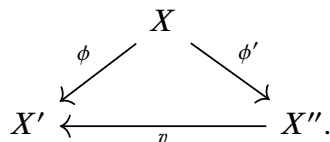


Such a pair  $(X', \phi)$ , if it exists, is unique in an obvious sense. More precisely, a simple and standard argument shows that, if  $\phi' : X \rightarrow X''$  also has Property (P), then

- there is a unique continuous map  $u : X' \rightarrow X''$  such that  $u \circ \phi = \phi'$ :



- there is a unique continuous map  $v : X'' \rightarrow X'$  such that  $v \circ \phi' = \phi$ :



- $u$  and  $v$  are inverse homeomorphisms.

The purpose of this text is to prove that such a pair *does* exist, and to define it in two (necessarily equivalent) ways.

The first method is to use the Stone-Ćech compactification  $\beta(X)$  of  $X$ , and to set  $X' := \beta(X)/\sim$  where  $x_1 \sim x_2$  if and only if  $x_1$  and  $x_2$  are in the same connected component. The only thing to verify is that  $\beta(X)/\sim$  is a profinite space. To prove this, the only slightly delicate point is to check that  $\beta(X)/\sim$  is Hausdorff, but this follows from Corollary 5.7.11 in [1].

The sequel is dedicated to the second method, which can be described as follows:

Let  $A(X)$  be the boolean algebra formed by the clopen subsets of  $X$ , let  $X'$  be the set of all boolean algebra morphisms from  $A(X)$  to  $\mathbf{2} := \{0, 1\}$ , and define  $\phi : X \rightarrow X'$  by

$$(\phi(x))(U) = 1 \iff x \in U$$

for all point  $x$  of  $X$  and all clopen subset  $U$  of  $X$ . By Proposition 4.1.3 and Lemma 4.1.8 in [1], there is unique topology  $\tau$  on  $X'$  such that the subsets

$$O_U := \{x' \in X' \mid x'(U) = 0\}$$

with  $U$  clopen in  $X$ , form a basis for  $\tau$ . We equip  $X'$  with this topology  $\tau$ . By Proposition 4.1.11 in [1] the space  $X'$  is profinite. Our main result is that this works, that is

**Theorem.** *The map  $\phi : X \rightarrow X'$  defined above has Property (P).*

To prove the theorem we first claim

- (a)  $\phi$  is continuous.

*Proof.* We easily check that  $\phi^{-1}(O_U) = X \setminus U$  for all clopen subset  $U$  of  $X$ .  $\square$

We claim

- (b)  $\phi(X)$  is dense in  $X'$ .

*Proof.* Using Lemma 4.1.8 in [1] again we see that any nonempty open subset of  $X'$  contains some  $O_U$  with  $U \neq X$  ( $U$  clopen). As noted above, if  $x$  is in  $X \setminus U$ , then  $\phi(x)$  is in  $O_U$ .  $\square$

Say that a topological space  $Y$  has *Property (Q)* if  $Y$  is profinite and if, for any continuous map  $f : X \rightarrow Y$ , there is a unique continuous map  $f' : X' \rightarrow Y$  such that  $f' \circ \phi = f$ . Our task is to show that all profinite spaces have Property (Q).

We claim

(c) If  $Y = \lim_i Y_i$  with  $Y_i$  profinite for all  $i$  and if  $Y_i$  has Property (Q) for all  $i$ , then  $Y$  has Property (Q).

*Proof.* This is clear.  $\square$

In view of (c), it suffices to verify that all *finite* discrete topological spaces have Property (Q).

Say that a topological space  $Y$  has *Property (Q')* if  $Y$  is profinite and if, for any continuous map  $f : X \rightarrow Y$ , there is *at least one* continuous map  $f' : X' \rightarrow Y$  such that  $f' \circ \phi = f$ .

In view of (b),

(d) Properties (Q) and (Q') are equivalent.

Above we denoted by  $\mathbf{2}$  the set  $\{0, 1\}$  viewed a boolean algebra. We also denote by  $\mathbf{2}$  the set  $\{0, 1\}$  when it is viewed a discrete topological space.

We claim

(e)  $\mathbf{2}$  has Property (Q').

*Proof.* Given a continuous map  $f : X \rightarrow \mathbf{2}$  we define  $f' : X' \rightarrow \mathbf{2}$  by  $f'(x') = x'(f^{-1}(1))$ . This implies  $f'^{-1}(0) = O_{f^{-1}(1)}$ . By Lemma 4.1.8 in [1], this is a clopen subset of  $X'$ . This shows that  $f'$  is continuous. Finally we have

$$f'(\phi(x)) = (\phi(x))(f^{-1}(1)) = f(x). \quad \square$$

We claim

(f)  $\mathbf{2}^n$  has Property (Q) for all  $n \in \mathbb{N}$ .

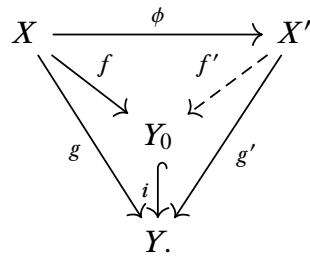
*Proof.* This follows from (c), (d) and (e).  $\square$

We claim

(g) If  $Y$  has Property (Q') and  $Y_0$  is a closed subspace of  $Y$ , then  $Y_0$  has Property (Q').

*Proof.* Let  $i : Y_0 \hookrightarrow Y$  be the inclusion, let  $f : X \rightarrow Y_0$  be continuous, set  $g := i \circ f$  and let  $g' : X' \rightarrow Y$  be a continuous map satisfying  $g' \circ \phi = g$ . We must show that

$g$  induces a map  $f' : X' \rightarrow Y_0$ :



In other words we must show  $g'(X') \subset Y_0$ , that is  $g'^{-1}(Y_0) = X'$ . But this follows from the facts that  $g'^{-1}(Y_0)$  is a closed subspace of  $X'$  containing  $\phi(X)$ , and that  $\phi(X)$  is dense in  $X'$  by (b).  $\square$

We claim

(h) All finite discrete topological spaces have Property (Q).

*Proof.* This follows from (d), (f) and (g).  $\square$

As already indicated, the Theorem follows from (c) and (h).  $\square$

#### Reference.

[1] Borceux, F., & Janelidze, G. (2001). **Galois Theories** (Cambridge Studies in Advanced Mathematics). Cambridge: Cambridge University Press.  
doi:10.1017/CBO9780511619939

Tex file available at

<https://tinyurl.com/y5ef4y73> and <https://tinyurl.com/yynts2eg>

May 3, 2019