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Cubic curves and cubic surfaces from contact points in conformal geometric algebra*

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Abstract. This work explains how to extend standard conformal geometric algebra of the Euclidean plane in a novel way to describe cubic curves in the Euclidean plane from nine contact points or from the ten coefficients of their implicit equations. As algebraic framework serves the Clifford algebra $Cl(9,7)$ over the real sixteen dimensional vector space $\mathbb{R}^{9,7}$. These cubic curves can be intersected using the outer product based meet operation of geometric algebra. An analogous approach is explained for the description and operation with cubic surfaces in three Euclidean dimensions, using as framework $Cl(19,16)$.

Keywords: Clifford algebra · conformal geometric algebra · cubic curves · cubic surfaces · intersections

1 Introduction

Cubic curves in the Euclidean plane have historically already been studied by Isaac Newton. In the context of geometric algebra [6], especially conformal geometric algebra, triple conformal geometric algebra (TCGA) $Cl(9,3)$ of the Euclidean plane provides a frame work for representing cubics and some higher order algebraic curves [7], albeit with the disadvantage of not being able to intersect two cubic curves with each other using the outer product. Yet, TCGA has the advantage of low dimensions, the underlying vector space only has dimension twelve, and modern PCs can easily compute with Clifford algebras over such vector spaces. Another advantage is that the intuitive and efficient quaternion like versor form of geometric transformations is available in TCGA.

Another line of development was the representation of conic curves in the plane by C. Perwass in Chapter 4.5 of [16] using the extended conformal geometric algebra $Cl(5,3)$, we call conic CGA. Including transformation versors for rotation, translation and scaling, this has been worked out in more detail in [13], and with a further simplified set of transformation versors in [11]. Again, nowadays computer algebra systems like Maple [1], Mathematica [2] and Matlab [17], etc., can easily work with $Cl(5,3)$. We

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now present the algebra of an implementation of cubic curves constructed from nine contact points, or alternatively from the ten coefficients of their implicit equation in the extended conformal geometric algebra $Cl(9,7)$, named *cubic CGA*, over the real vector space $\mathbb{R}^{9,7}$. This is currently at the limit what an implementation like [17] can compute. The expectation is, that optimization will very soon progress, and enable us to apply [17] to this problem, and we even anticipate that other implementations using e.g. the optimization framework GAALOP [8] or even [4] or [14], may indeed already be able to reasonable compute in the $Cl(9,7)$ framework. The aim of the present paper is not to present a full framework, including a complete software implementation, but rather to outline the algebraic framework, and thus enable other researchers to work with cubic curves in cubic CGA, including the general intersection computation.

The paper is structured as follows. Section 2 outlines the algebraic setup of the extended CGA for cubic curves. Section 3 then proceeds to introduce the notion of cubic points (points in two dimensions, that are extended to include quadratic and cubic coordinate monomials as vector coefficients). This includes explanations on how standard CGA of two dimensional Euclidean space and conic CGA are embedded in cubic CGA. Section 4 explains how to construct cubic curve representing blades either from nine contact points, or from the ten coefficients of the implicit cubic curve equation in two dimensions. Section 5 introduces the way of computing intersections of cubic curves, utilizing the outer product of blades. Furthermore, Section 6 extends the approach to the cubic surface CGA $Cl(19,16)$ for describing in an analogous way to the previous sections the construction of cubic surfaces in three Euclidean dimensions, either from 19 surface points, or from the twenty coefficients of the implicit equation of cubic surfaces in three dimensions. The work concludes with Section 7, followed by references.

2 Cubic conformal geometric algebra

We use the following notation: Lower-case bold letters denote basis blades and multivectors (vector or multivector $\mathbf{a}$). Italic lower-case letters refer to multivector components ($a_1, x, y^2, \cdots$). For example, a_i is the i^{th} coordinate of the (multi)vector $\mathbf{a}$. The superscript star used in $\mathbf{x}^*$ represents the dualization of the multivector $\mathbf{x}$. Finally, subscript ε on $\mathbf{x}_\varepsilon$ refers to the two-dimensional Euclidean vector associated to the vector $\mathbf{x}$ of CCGA.

Note that when used in geometric algebra the inner product, contractions and the outer product have priority over the full geometric product. For instance, $\mathbf{a} \wedge \mathbf{b}\mathbf{I} = (\mathbf{a} \wedge \mathbf{b})\mathbf{I}$.

The inner products between the basis vectors of $Cl(9,7)$ are defined in Table 1.

The transformation from the common diagonal metric basis to that of Table 1 can be defined as follows: for $1 \leq i, j \leq 7$,

$$\mathbf{e}_{oi} = \frac{1}{\sqrt{2}}(\mathbf{e}_{-i} - \mathbf{e}_{+i}), \qquad \mathbf{e}_{\infty i} = \frac{1}{\sqrt{2}}(\mathbf{e}_{+i} + \mathbf{e}_{-i}). \tag{1}$$

We further define for later use another pair of null vectors

$$\mathbf{e}_\infty = \tfrac{1}{2}(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2}), \qquad \mathbf{e}_o = \mathbf{e}_{o1} + \mathbf{e}_{o2}. \tag{2}$$

Table 1. Inner product between cubic CGA basis vectors.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{o1}$	$\mathbf{e}_{\infty1}$	$\mathbf{e}_{o2}$	$\mathbf{e}_{\infty2}$	$\mathbf{e}_{o3}$	$\mathbf{e}_{\infty3}$	$\mathbf{e}_{o4}$	$\mathbf{e}_{\infty4}$	$\mathbf{e}_{o5}$	$\mathbf{e}_{\infty5}$	$\mathbf{e}_{o6}$	$\mathbf{e}_{\infty6}$	$\mathbf{e}_{o7}$	$\mathbf{e}_{\infty7}$
$\mathbf{e}_1$	1	0	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_2$	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{o1}$	·	·	0	−1	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{\infty1}$	·	·	−1	0	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{o2}$	·	·	·	·	0	−1	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{\infty2}$	·	·	·	·	−1	0	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{o3}$	·	·	·	·	·	·	0	−1	·	·	·	·	·	·	·	·
$\mathbf{e}_{\infty3}$	·	·	·	·	·	·	−1	0	·	·	·	·	·	·	·	·
$\mathbf{e}_{o4}$	·	·	·	·	·	·	·	·	0	−1	·	·	·	·	·	·
$\mathbf{e}_{\infty4}$	·	·	·	·	·	·	·	·	−1	0	·	·	·	·	·	·
$\mathbf{e}_{o5}$	·	·	·	·	·	·	·	·	·	·	0	−1	·	·	·	·
$\mathbf{e}_{\infty5}$	·	·	·	·	·	·	·	·	·	·	−1	0	·	·	·	·
$\mathbf{e}_{o6}$	·	·	·	·	·	·	·	·	·	·	·	·	0	−1	·	·
$\mathbf{e}_{\infty6}$	·	·	·	·	·	·	·	·	·	·	·	·	−1	0	·	·
$\mathbf{e}_{o7}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	0	−1
$\mathbf{e}_{\infty7}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−1	0

Inner products lead to

$$\mathbf{e}_{\infty i}\cdot\mathbf{e}_{oi} = -1, \qquad \mathbf{e}_{\infty}\cdot\mathbf{e}_{o} = -1, \qquad \mathbf{e}_o^2 = \mathbf{e}_\infty^2 = 0, \tag{3}$$

$$\mathbf{e}_{\infty k}\cdot\mathbf{e}_{o} = -1 \quad (k=1,2), \quad \mathbf{e}_{\infty l}\cdot\mathbf{e}_{o} = 0 \quad (l=3,4,5,6,7), \quad \mathbf{e}_{\infty i}\cdot\mathbf{e}_{\infty} = 0, \tag{4}$$

We further define the bivectors E_i, E, as

$$E_i = \mathbf{e}_{\infty i}\wedge\mathbf{e}_{oi} = \mathbf{e}_{+i}\mathbf{e}_{-i}, \qquad E = \mathbf{e}_\infty\wedge\mathbf{e}_o, \tag{5}$$

and obtain the following products

$$E_i^2 = 1, \qquad E_iE_j = E_jE_i, \tag{6}$$

$$\mathbf{e}_{oi}E_i = -E_i\mathbf{e}_{oi} = -\mathbf{e}_{oi}, \qquad \mathbf{e}_{\infty i}E_i = -E_i\mathbf{e}_{\infty i} = \mathbf{e}_{\infty i}, \tag{7}$$

$$\mathbf{e}_{oj}E_i \overset{i\neq i}{=} E_i\mathbf{e}_{oj}, \qquad \mathbf{e}_{\infty j}E_i \overset{i\neq i}{=} E_i\mathbf{e}_{\infty j}, \tag{8}$$

$$E^2 = 1, \quad \mathbf{e}_oE = -E\mathbf{e}_o = -\mathbf{e}_o, \quad \mathbf{e}_\infty E = -E\mathbf{e}_\infty = \mathbf{e}_\infty. \tag{9}$$

We further define the following blades

$$\mathbf{I}_{\infty12} = \mathbf{e}_{\infty1}\mathbf{e}_{\infty2}, \quad \mathbf{I}_{\infty c} = \mathbf{e}_{\infty4}\mathbf{e}_{\infty5}\mathbf{e}_{\infty6}\mathbf{e}_{\infty7}, \quad \mathbf{I}_{\infty b} = \mathbf{e}_{\infty3}\mathbf{I}_{\infty c}, \quad \mathbf{I}_\infty = \mathbf{I}_{\infty12}\mathbf{I}_{\infty b}, \tag{10}$$

$$\mathbf{I}_{o12} = \mathbf{e}_{o1}\mathbf{e}_{o2}, \quad \mathbf{I}_{oc} = \mathbf{e}_{o4}\mathbf{e}_{o5}\mathbf{e}_{o6}\mathbf{e}_{o7}, \quad \mathbf{I}_{ob} = \mathbf{e}_{o3}\mathbf{I}_{oc}, \quad \mathbf{I}_o = \mathbf{I}_{o12}\mathbf{I}_{ob}, \tag{11}$$

$$\mathbf{I}_{\infty o} = \mathbf{I}_\infty\wedge\mathbf{I}_o = -E_1E_2E_3E_4E_5E_6E_7, \tag{12}$$

$$\mathbf{e}_{o12}^{\triangleright} = \mathbf{e}_{o1}-\mathbf{e}_{o2}, \quad \mathbf{I}_o^{\triangleright} = \mathbf{e}_{o12}^{\triangleright}\mathbf{I}_{ob}, \quad \mathbf{e}_{\infty12}^{\triangleright} = \mathbf{e}_{\infty1}-\mathbf{e}_{\infty2}, \quad \mathbf{I}_\infty^{\triangleright} = \mathbf{e}_{\infty12}^{\triangleright}\mathbf{I}_{\infty b}, \tag{13}$$

Note that we defined the 7-blades $\mathbf{I}_\infty$ and $\mathbf{I}_o$ as the products of all infinity null vectors, respectively of all origin null vectors. As we will see later in this model, the 6-blades $\mathbf{I}_\infty^{\triangleright}$ and $\mathbf{I}_o^{\triangleright}$ are frequently used, e.g. to work with the embedding of standard CGA in cubic CGA (Section 3), for the dual vector representation of cubics (Section 4), and for intersection computations (Section 5). The blades $\mathbf{I}_\infty$ and $\mathbf{I}_\infty^\triangleright$ are directly related by (18). A similar relationship exists between $\mathbf{I}_o$ and $\mathbf{I}_o^\triangleright$. As a consequence of the blade definitions we have

$$\mathbf{e}_{\infty 12}^\triangleright \cdot \mathbf{e}_{o12}^\triangleright = -2, \tag{14}$$

$$\mathbf{I}_{\infty c} \cdot \mathbf{I}_{oc} = \mathbf{I}_{oc} \cdot \mathbf{I}_{\infty c} = \mathbf{I}_{\infty c} \rfloor \mathbf{I}_{oc} = \mathbf{I}_{\infty c} \lfloor \mathbf{I}_{oc} = 1, \tag{15}$$

$$\mathbf{I}_\infty^\triangleright \cdot \mathbf{I}_o^\triangleright = \mathbf{I}_o^\triangleright \cdot \mathbf{I}_\infty^\triangleright = \mathbf{I}_\infty^\triangleright \rfloor \mathbf{I}_o^\triangleright = \mathbf{I}_\infty^\triangleright \lfloor \mathbf{I}_o^\triangleright = -2. \tag{16}$$

We have the following outer product relationships

$$\begin{aligned}\mathbf{I}_{\infty 12} &= -\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 12}^\triangleright = -\mathbf{e}_{\infty 2} \wedge \mathbf{e}_{\infty 12}^\triangleright = -\mathbf{e}_\infty \wedge \mathbf{e}_{\infty 12}^\triangleright \\ &= -\mathbf{e}_{\infty 1}\, \mathbf{e}_{\infty 12}^\triangleright = -\mathbf{e}_{\infty 2}\, \mathbf{e}_{\infty 12}^\triangleright = -\mathbf{e}_\infty\, \mathbf{e}_{\infty 12}^\triangleright,\end{aligned} \tag{17}$$

which further lead to

$$\begin{aligned}\mathbf{I}_\infty &= -\mathbf{e}_{\infty 1} \wedge \mathbf{I}_\infty^\triangleright = -\mathbf{e}_{\infty 2} \wedge \mathbf{I}_\infty^\triangleright = -\mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright \\ &= -\mathbf{e}_{\infty 1}\, \mathbf{I}_\infty^\triangleright = -\mathbf{e}_{\infty 2}\, \mathbf{I}_\infty^\triangleright = -\mathbf{e}_\infty\, \mathbf{I}_\infty^\triangleright,\end{aligned} \tag{18}$$

Similar to [11] we obtain for products with the simple 6-vector $\mathbf{I}_\infty^\triangleright$ that

$$\{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \wedge \mathbf{I}_\infty^\triangleright = \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}\, \mathbf{I}_\infty^\triangleright = \mathbf{I}_\infty^\triangleright \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}, \tag{19}$$

We define the pseudo-scalar $\mathbf{I}_\varepsilon$ in $\mathbb{R}^2$:

$$\mathbf{I}_\varepsilon = \mathbf{e}_1 \mathbf{e}_2, \quad \mathbf{I}_\varepsilon^2 = -1, \quad \mathbf{I}_\varepsilon^{-1} = -\mathbf{I}_\varepsilon. \tag{20}$$

The full pseudo-scalar $\mathbf{I}$ and its inverse $\mathbf{I}^{-1}$ (used for dualization) are:

$$\mathbf{I} = \mathbf{I}_\varepsilon \mathbf{I}_{\infty o} = -\mathbf{I}_\varepsilon E_1 E_2 E_3 E_4 E_5 E_6 E_7, \quad \mathbf{I}^2 = -1, \quad \mathbf{I}^{-1} = -\mathbf{I}. \tag{21}$$

The dual of a multivector indicates division by the pseudo-scalar, e.g., $\mathbf{a}^* = -\mathbf{a}\mathbf{I}$, $\mathbf{a} = \mathbf{a}^*\mathbf{I}$. From eq. (1.19) in [10], we have the useful duality between outer and inner products of non-scalar blades A, B in geometric algebra:

$$(A \wedge B)^* = A \cdot B^*, \quad A \wedge (B^*) = (A \cdot B)^* \;\Leftrightarrow\; A \wedge (B\mathbf{I}) = (A \cdot B)\mathbf{I}, \tag{22}$$

which indicates that

$$A \wedge B = 0 \;\Leftrightarrow\; A \cdot B^* = 0, \qquad A \cdot B = 0 \;\Leftrightarrow\; A \wedge B^* = 0. \tag{23}$$

3 Point in cubic CGA

The point $\mathbf{x}$ of cubic CGA corresponding to the Euclidean point $\mathbf{x}_\varepsilon = x\mathbf{e}_1 + y\mathbf{e}_2 \in \mathbb{R}^2$, is defined as[3]

$$\mathbf{x} = \mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2 \mathbf{e}_{\infty 1} + y^2 \mathbf{e}_{\infty 2}) + xy\mathbf{e}_{\infty 3} + x^3 \mathbf{e}_{\infty 5} + x^2 y \mathbf{e}_{\infty 5} + xy^2 \mathbf{e}_{\infty 6} + y^3 \mathbf{e}_{\infty 7} + \mathbf{e}_o. \tag{24}$$

[3] The use of the factor one half in $\frac{1}{2}(x^2 \mathbf{e}_{\infty 1} + y^2 \mathbf{e}_{\infty 2})$ is taken over from the point definition in standard CGA [6], and has importance in preserving the inner product to distance relationship of CGA in (26).

Note that basically each quadratic and cubic coordinate monomial is assigned to a different infinity null vector. Standard CGA $Cl(3,1)$ has only one infinity null vector, which means that only objects of constant curvature (flat or round) can be described. Already, the fundamental approach for the description of conics by Perwass [16] in $Cl(5,3)$ needed to assign each of the quadratic monomials x^2, y^2 and xy to individual, linearly independent (mutually orthogonal) null vector dimensions. Our point definition (24) is the consequent continuation of this approach. Note further that the five null vectors $\mathbf{e}_{o3}, \mathbf{e}_{o4}, \mathbf{e}_{o5}, \mathbf{e}_{o6}, \mathbf{e}_{o7}$ are not present in the definition of the point. This is chiefly to keep the convenient properties of the CGA points, namely, the inner product between two points is identical with the squared distance between them. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points, their inner product is

$$\begin{aligned}
&\mathbf{x}_1 \cdot \mathbf{x}_2 = \\
&(\mathbf{x}_{1\varepsilon} + \tfrac{1}{2}x_1^2\mathbf{e}_{\infty 1} + \tfrac{1}{2}y_1^2\mathbf{e}_{\infty 2} + x_1y_1\mathbf{e}_{\infty 3} + x_1^3\mathbf{e}_{\infty 4} + x_1^2y_1\mathbf{e}_{\infty 5} + x_1y_1^2\mathbf{e}_{\infty 6} + y_1^3\mathbf{e}_{\infty 7} + \mathbf{e}_o) \qquad (25)\\
&\quad\cdot (\mathbf{x}_{2\varepsilon} + \tfrac{1}{2}x_2^2\mathbf{e}_{\infty 1} + \tfrac{1}{2}y_2^2\mathbf{e}_{\infty 2} + x_2y_2\mathbf{e}_{\infty 3} + + x_2^3\mathbf{e}_{\infty 4} + x_2^2y_2\mathbf{e}_{\infty 5} + x_2y_2^2\mathbf{e}_{\infty 6} + y_2^3\mathbf{e}_{\infty 7} + \mathbf{e}_o).
\end{aligned}$$

from which together with Table 1, it follows that

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_{1\varepsilon} \cdot \mathbf{x}_{2\varepsilon} - \tfrac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2) = -\tfrac{1}{2}(\mathbf{x}_{1\varepsilon} - \mathbf{x}_{2\varepsilon})^2. \qquad (26)$$

We see that the inner product is equivalent to the minus half of the squared Euclidean distance between $\mathbf{x}_1$ and $\mathbf{x}_2$.

By wedging a cubic CGA point with the blade $\mathbf{I}_{\infty c}$ we obtain

$$\begin{aligned}
\mathbf{x} \wedge \mathbf{I}_{\infty c} &= \big(\mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2}) + xy\mathbf{e}_{\infty 3} + \mathbf{e}_o\big) \wedge \mathbf{I}_{\infty c} && (27)\\
&= \mathbf{x}_{\text{conic}} \wedge \mathbf{I}_{\infty c} = \mathbf{x}_{\text{conic}}\mathbf{I}_{\infty c}, && (28)
\end{aligned}$$

that is all four third power terms in the coordinates x^3, x^2y, xy^2, y^3 drop out, and what remains is identical to the conic point definition $\mathbf{x}_{\text{conic}}$ in conic CGA $Cl(5,3)$ generated by the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{o1}, \mathbf{e}_{\infty 1}, \mathbf{e}_{o2}, \mathbf{e}_{\infty 2}, \mathbf{e}_{o3}, \mathbf{e}_{\infty 3}\}$ in [11]. The conic point can be obtained explicitly by

$$\mathbf{x}_{\text{conic}} = \mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2}) + xy\mathbf{e}_{\infty 3} + \mathbf{e}_o = (\mathbf{x} \wedge \mathbf{I}_{\infty c}) \rfloor \mathbf{I}_{oc}. \qquad (29)$$

A consequence of this embedding of conic CGA in cubic CGA is, that all results of conic CGA are perfectly valid in cubic CGA.

Furthermore, by wedging a cubic CGA point with the blade $\mathbf{I}_\infty^{\triangleright}$ we obtain

$$\begin{aligned}
\mathbf{x} \wedge \mathbf{I}_\infty^{\triangleright} &= \big(\mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2}) + \mathbf{e}_o\big) \wedge \mathbf{I}_\infty^{\triangleright} = \big(\mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2 + y^2)\mathbf{e}_\infty + \mathbf{e}_o\big) \wedge \mathbf{I}_\infty^{\triangleright}\\
&= \big(\mathbf{x}_\varepsilon + \tfrac{1}{2}\mathbf{x}_\varepsilon^2\mathbf{e}_\infty + \mathbf{e}_o\big) \wedge \mathbf{I}_\infty^{\triangleright} = \mathbf{x}_{\text{C}} \wedge \mathbf{I}_\infty^{\triangleright} = \mathbf{x}_{\text{C}}\mathbf{I}_\infty^{\triangleright}, \qquad (30)
\end{aligned}$$

that is all four third power terms in the coordinates x^3, x^2y, xy^2, y^3 and the mixed second order term xy drop out, and what remains is identical to the standard CGA point definition $\mathbf{x}_{\text{C}}$ in standard CGA $Cl(3,1)$ of the Euclidean plane generated by the basis

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_o, \mathbf{e}_\infty\}$. The standard CGA point $\mathbf{x}_\mathrm{C}$ can be obtained explicitly[4] by

$$\mathbf{x}_\mathrm{C} = \mathbf{x}_\varepsilon + \tfrac{1}{2}\mathbf{x}_\varepsilon^2\mathbf{e}_\infty + \mathbf{e}_o = -\tfrac{1}{2}(\mathbf{x}\wedge \mathbf{I}_\infty^\triangleright)\rfloor \mathbf{I}_o^\triangleright. \tag{31}$$

We thus see, that standard CGA of the Euclidean plane in $Cl(3,1)$ is fully embedded in cubic CGA $Cl(9,7)$. The sequence of embedding is standard CGA in conic CGA in cubic CGA.

4 Cubic curve

This section describes how cubic CGA handles plane cubic curves. A cubic curve in $\mathbb{R}^2$ is formulated as

$$F(x,y) = \mathrm{a}x^3 + \mathrm{b}x^2y + \mathrm{c}xy^2 + \mathrm{d}y^3 + \mathrm{e}x^2 + \mathrm{f}y^2 + \mathrm{g}xy + \mathrm{h}x + \mathrm{i}y + \mathrm{j} = 0. \tag{32}$$

We note, that *the set of cubic curves has a natural structure of a projective space* $\mathbb{P}^9$ [15]. The first way to represent a cubic curve in cubic CGA is constructive by wedging nine contact points together as follows

$$\mathbf{q} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_9. \tag{33}$$

The multivector $\mathbf{q}$ corresponds to the primal form of a cubic curve in cubic CGA, with grade nine and essentially ten components, with ten coefficients a, b, ..., j. If we further wedge[5] the 9-blade $\mathbf{q}$ with the 6-blade $\mathbf{I}_o^\triangleright$, we obtain a 15-blade

$$\begin{aligned}\mathbf{q}\wedge\mathbf{I}_0^\triangleright &= \big(-(2\mathrm{e}\mathbf{e}_{o1} + 2\mathrm{f}\mathbf{e}_{o2} + \mathrm{g}\mathbf{e}_{o3} + \mathrm{a}\mathbf{e}_{o4} + \mathrm{b}\mathbf{e}_{o5} + \mathrm{c}\mathbf{e}_{o6} + \mathrm{d}\mathbf{e}_{o7}) + \mathrm{h}\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2 - \mathrm{j}\mathbf{e}_\infty\big)\,\mathbf{I} \\ &= (\mathbf{q}\wedge\mathbf{I}_0^\triangleright)^*\,\mathbf{I},\end{aligned} \tag{34}$$

The expression for the dual 1-vector $(\mathbf{q}\wedge\mathbf{I}_o^\triangleright)^*$ is therefore simply

$$(\mathbf{q}\wedge\mathbf{I}_o^\triangleright)^* = -(2\mathrm{e}\mathbf{e}_{o1} + 2\mathrm{f}\mathbf{e}_{o2} + \mathrm{g}\mathbf{e}_{o3} + \mathrm{a}\mathbf{e}_{o4} + \mathrm{b}\mathbf{e}_{o5} + \mathrm{c}\mathbf{e}_{o6} + \mathrm{d}\mathbf{e}_{o7}) + \mathrm{h}\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2 - \mathrm{j}\mathbf{e}_\infty. \tag{35}$$

Proposition 41 *A point* $\mathbf{x}$ *lies on the cubic curve* $\mathbf{q}$ *if and only* $\mathbf{x}\wedge\mathbf{q}\wedge\mathbf{I}_o^\triangleright = 0$.

Proof.

$$\begin{aligned}\mathbf{x}\wedge(\mathbf{q}\wedge\mathbf{I}_o^\triangleright) &= \mathbf{x}\wedge((\mathbf{q}\wedge\mathbf{I}_o^\triangleright)^*\mathbf{I}) = \mathbf{x}\cdot(\mathbf{q}\wedge\mathbf{I}_o^\triangleright)^*\,\mathbf{I} \\ &= \mathbf{x}\cdot\big(-(2\mathrm{e}\mathbf{e}_{o1} + 2\mathrm{f}\mathbf{e}_{o2} + \mathrm{g}\mathbf{e}_{o3} + \mathrm{a}\mathbf{e}_{o4} + \mathrm{b}\mathbf{e}_{o5} + \mathrm{c}\mathbf{e}_{o6} + \mathrm{d}\mathbf{e}_{o7}) \\ &\qquad + \mathrm{h}\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2 - \mathrm{j}\mathbf{e}_\infty\big)\,\mathbf{I} \\ &= (\mathrm{a}x^3 + \mathrm{b}x^2y + \mathrm{c}xy^2 + \mathrm{d}y^3 + \mathrm{e}x^2 + \mathrm{f}y^2 + \mathrm{g}xy + \mathrm{h}x + \mathrm{i}y + \mathrm{j})\,\mathbf{I}.\end{aligned} \tag{36}$$

This corresponds to the formula (32) representing a general cubic curve.

[4] The operation $(\mathbf{x}\wedge\mathbf{I}_\infty^\triangleright)\rfloor\mathbf{I}_o^\triangleright$ combining outer product and contraction is typical for projection operations in geometric algebra. For example, in $Cl(3,0)$ the projection of multivector $\mathbf{a}$ onto a blade $\mathbf{b}$ is given by $(\mathbf{a}\wedge\mathbf{b})\rfloor\mathbf{b}^{-1}$. Since $\mathbf{I}_\infty^\triangleright$ is a product of null vectors and has no inverse, the projection operation is completed by contracting with $\mathbf{I}_o^\triangleright$ from the right, see (16).

[5] This is a strategy similarly employed by Perwass for conics [16] and in [11], and for quadrics in [3,12]. Treating the outer product of contact points (33) as the actual algebraic representation of the geometric object in question, was essential for the formulation of rotations, translations and scaling by means of versors in [12]. We intuitively expect that this may turn out to be similar in the current cubic CGA $Cl(9,7)$.

The dualization of the primal cubic $\mathbf{q}$ wedged with $\mathbf{I}_o^{\triangleright}$ leads to the 1-vector dual form $(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*$ of (35).

Corollary 42 *A point* $\mathbf{x}$ *lies on the cubic curve defined by* $\mathbf{q}$ *if and only if* $\mathbf{x}\cdot(\mathbf{q}\wedge\mathbf{I}_0^{\triangleright})^* = 0$.

The ten coefficients $\{a,\dots,j\}$ of the cubic equation (32) can be easily extracted from the cubic curve 9-blade $\mathbf{q}$ of (33) by computing the following scalar products with vector $(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*$ as

$$\begin{aligned}
&a = (\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty 4}, \quad b = (\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty 5}, \quad c = (\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty 6},\\
&d = (\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty 7}, \quad e = \tfrac{1}{2}(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty 1}, \quad f = \tfrac{1}{2}(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty 2},\\
&g = (\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty 3}, \quad h = (\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{1}, \quad i = (\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{2},\\
&j = (\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{o}.
\end{aligned} \tag{37}$$

Remark 43 *Based on the ten coefficients of the implicit cubic equation* (32)*, the vector* $(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*$ *of* (35)*, can easily be constructed, providing another valid dual representation of the cubic curve in cubic CGA, that can e.g. be used for intersection computations as described in the next Section.*

5 Intersections

Any number of linearly independent embedded standard CGA objects, embedded conics and cubic curves $\{\mathbf{A},\mathbf{B},\dots,\mathbf{Z}\}$, after wedging with the 6-blade $\mathbf{I}_o^{\triangleright}$, can be intersected by computing the dual of the outer product of their duals

$$(\mathbf{M}\wedge\mathbf{I}_o^{\triangleright})^* = (\mathbf{A}\wedge\mathbf{I}_o^{\triangleright})^*\wedge(\mathbf{B}\wedge\mathbf{I}_o^{\triangleright})^*\wedge\dots\wedge(\mathbf{Z}\wedge\mathbf{I}_o^{\triangleright})^*. \tag{38}$$

The approach is analogous to intersecting two circles in CGA $Cl(3,1)$ of two-dimensional Euclidean space, or two spheres in CGA $Cl(4,1)$ of three-dimensional Euclidean space. But in $Cl(3,1)$ [8] and $Cl(4,1)$ [6,10] there is only one pair of null-vectors $\mathbf{e}_o$, $\mathbf{e}_\infty$, and both are used in the point construction. In cubic CGA $Cl(9,7)$, there are seven pairs of null vectors $\mathbf{e}_{oi}$, $\mathbf{e}_{\infty i}$, $1\le i\le 7$, and only one fixed combination of origin null vectors $\mathbf{e}_o = \mathbf{e}_{o1}+\mathbf{e}_{o2}$ of (2) is actually used for the cubic point construction (24). That is essentially six origin null vector dimensions do not appear in the point construction, and therefore also not in the multivector $\mathbf{q}$ representing a cubic curve (33). To make up for that, and to maintain the computation of intersections from dual 1-vector representations, the outer product with the 6-blade $\mathbf{I}_o^{\triangleright}$ extends the multivector $\mathbf{q}$ to a 14-blade $\mathbf{q}\wedge\mathbf{I}_o^{\triangleright}$, which by dualization results in the dual 1-vector representation.

A criterion for a general point $\mathbf{x}$ to be on the intersection $\mathbf{M}$ is

$$\mathbf{x}\cdot(\mathbf{M}\wedge\mathbf{I}_o^{\triangleright})^* = 0, \qquad \mathbf{M} = -\tfrac{1}{2}\big((\mathbf{M}\wedge\mathbf{I}_o^{\triangleright})^*\mathbf{I}\big)\rfloor\mathbf{I}_\infty^{\triangleright}. \tag{39}$$

The last equation allows to extract the intersection blade $\mathbf{M}$ itself. The product with $\mathbf{I}$ reverses dualization, and the subsequent right contraction with $\mathbf{I}_\infty^{\triangleright}$ removes the factor $\mathbf{I}_o^{\triangleright}$, according to (16).

6 Cubic surfaces

Though it may still be far beyond today's computing power, cubic surfaces are a classical subject in mathematics, and they indeed can be represented in a cubic surface CGA (CSCGA) $Cl(3+16,16) = Cl(19,16)$ over the real vector space $\mathbb{R}^{19,16}$. We are not able to state all the details here, but will try to outline the algebraic approach.

The vector basis of $\mathbb{R}^{19,16}$ consists of the three Euclidean orthonormal vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, their product forming the Euclidean three dimensional unit pseudoscalar

$$\mathbf{I}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \quad \mathbf{I}_3^2 = -1, \quad \mathbf{I}_3^{-1} = -\mathbf{I}_3; \tag{40}$$

the remaining vectors form 16 pairs of null vectors, $\{\mathbf{e}_{\infty i}, \mathbf{e}_{oi}\}$, $1 \leq i \leq 16$,

$$\mathbf{e}_{\infty i}^2 = \mathbf{e}_{oi}^2 = 0, \quad \mathbf{e}_{\infty i} \cdot \mathbf{e}_{oi} = -1, \quad E_i = \mathbf{e}_{\infty i} \wedge \mathbf{e}_{oi}, \quad E_i^2 = 1. \tag{41}$$

We also define

$$\mathbf{e}_\infty = \tfrac{1}{3}(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3}), \qquad \mathbf{e}_o = \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}. \tag{42}$$

We further define the following blades

$$\mathbf{I}_{\infty a}^{\triangleright} = (\mathbf{e}_{\infty 1} - \mathbf{e}_{\infty 2}) \wedge (\mathbf{e}_{\infty 2} - \mathbf{e}_{\infty 3}), \quad \mathbf{I}_{\infty b} = \mathbf{e}_{\infty 4}\mathbf{e}_{\infty 5}\mathbf{e}_{\infty 6}, \tag{43}$$

$$\mathbf{I}_{\infty c} = \mathbf{e}_{\infty 7}\mathbf{e}_{\infty 8}\mathbf{e}_{\infty 9}\mathbf{e}_{\infty 10}\mathbf{e}_{\infty 11}\mathbf{e}_{\infty 12}\mathbf{e}_{\infty 13}\mathbf{e}_{\infty 14}\mathbf{e}_{\infty 15}\mathbf{e}_{\infty 16}, \quad \mathbf{I}_{\infty}^{\triangleright} = \mathbf{I}_{\infty a}^{\triangleright}\mathbf{I}_{\infty b}\mathbf{I}_{\infty c}, \tag{44}$$

$$\mathbf{I}_{oa}^{\triangleright} = (\mathbf{e}_{o1} - \mathbf{e}_{o2}) \wedge (\mathbf{e}_{o2} - \mathbf{e}_{o3}), \quad \mathbf{I}_{ob} = \mathbf{e}_{o4}\mathbf{e}_{o5}\mathbf{e}_{o6}, \tag{45}$$

$$\mathbf{I}_{oc} = \mathbf{e}_{o7}\mathbf{e}_{o8}\mathbf{e}_{o9}\mathbf{e}_{o10}\mathbf{e}_{o11}\mathbf{e}_{o12}\mathbf{e}_{o13}\mathbf{e}_{o14}\mathbf{e}_{o15}\mathbf{e}_{o16}, \quad \mathbf{I}_{o}^{\triangleright} = \mathbf{I}_{oa}^{\triangleright}\mathbf{I}_{ob}\mathbf{I}_{oc}, \tag{46}$$

Inner products yield

$$\mathbf{I}_{\infty a}^{\triangleright} \cdot \mathbf{I}_{oa}^{\triangleright} = -3, \quad \mathbf{I}_{\infty}^{\triangleright} \cdot \mathbf{I}_{o}^{\triangleright} = +3, \quad \mathbf{I}_{\infty c} \cdot \mathbf{I}_{oc} = -1. \tag{47}$$

In this setting a cubic surface point in three dimensions is defined from its position in three dimensional Euclidean space $\mathbf{x}_\varepsilon = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ as an extension to the CGA point or to the quadric QCGA point [3]

$$\begin{aligned}\mathbf{x} =& \mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2} + z^2\mathbf{e}_{\infty 3}) + xy\mathbf{e}_{\infty 4} + xz\mathbf{e}_{\infty 5} + yz\mathbf{e}_{\infty 6} \\ &+ x^3\mathbf{e}_{\infty 7} + x^2y\mathbf{e}_{\infty 8} + x^2z\mathbf{e}_{\infty 9} + xy^2\mathbf{e}_{\infty 10} + xyz\mathbf{e}_{\infty 11} + xz^2\mathbf{e}_{\infty 12} \\ &+ y^3\mathbf{e}_{\infty 13} + y^2z\mathbf{e}_{\infty 14} + yz^2\mathbf{e}_{\infty 15} + z^3\mathbf{e}_{\infty 16} + \mathbf{e}_o.\end{aligned} \tag{48}$$

Note, that the vectors $\{\mathbf{e}_{o4}, \ldots, \mathbf{e}_{o16}\}$ are not used in the above point definition in order to preserve the CGA property that

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \tfrac{1}{2}(\mathbf{x}_{1\varepsilon} - \mathbf{x}_{2\varepsilon})^2. \tag{49}$$

By wedging a cubic surface point $\mathbf{x}$ with the 10-blade $\mathbf{I}_{\infty c}$, we effectively remove all third power coordinate components

$$\begin{aligned}\mathbf{x} \wedge \mathbf{I}_{\infty c} &= \big(\mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2} + z^2\mathbf{e}_{\infty 3}) + xy\mathbf{e}_{\infty 4} + xz\mathbf{e}_{\infty 5} + yz\mathbf{e}_{\infty 6} + \mathbf{e}_o\big) \wedge \mathbf{I}_{\infty c} \\ &= \mathbf{x}_{\mathrm{Q}}\mathbf{I}_{\infty c},\end{aligned} \tag{50}$$

which means to project down to the subalgebra of quadric surfaces $Cl(9,6)$ [3] with

$$\begin{aligned}\mathbf{x}_{\mathrm{Q}} &= \mathbf{x}_{\varepsilon} + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2} + z^2\mathbf{e}_{\infty 3}) + xy\mathbf{e}_{\infty 4} + xz\mathbf{e}_{\infty 5} + yz\mathbf{e}_{\infty 6} + \mathbf{e}_o \\ &= -(\mathbf{x} \wedge \mathbf{I}_{\infty c})\rfloor \mathbf{I}_{oc} \end{aligned} \tag{51}$$

By alternatively wedging a cubic surface point $\mathbf{x}$ with $\mathbf{I}_{\infty}^{\triangleright}$ we effectively project the point to a subalgebra isomorphic to standard CGA [6]

$$\mathbf{x} \wedge \mathbf{I}_{\infty}^{\triangleright} = \big(\mathbf{x}_{\varepsilon} + \tfrac{1}{2}(x^2+y^2+z^2)\mathbf{e}_{\infty} + \mathbf{e}_o\big) \wedge \mathbf{I}_{\infty}^{\triangleright} \tag{52}$$

with standard CGA point

$$\mathbf{x}_{\mathrm{C}} = \mathbf{x}_{\varepsilon} + \tfrac{1}{2}\mathbf{x}_{\varepsilon}^2\mathbf{e}_{\infty} + \mathbf{e}_o = \tfrac{1}{3}(\mathbf{x} \wedge \mathbf{I}_{\infty}^{\triangleright})\rfloor \mathbf{I}_{o}^{\triangleright}. \tag{53}$$

This means that we have standard CGA of three dimensions embedded in quadric CGA embedded in cubic surface CGA. So all the known results of standard CGA, and quadric CGA can be applied in cubic surface CGA.

Cubic surfaces in three dimensions are described by the implicit equation with 20 coefficients $a, b, \ldots t$,

$$\begin{aligned}F(x,y,z) =& ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx^3 + hx^2y + ix^2z + jxy^2 + kxyz \\ &+ lxz^2 + my^3 + ny^2z + oyz^2 + pz^3 + qx + ry + sz + t = 0.\end{aligned} \tag{54}$$

We note, that *cubic surfaces are parametrized by the points in (projective space)* $\mathbb{P}^{19}$ [5]. We can use 19 cubic surface contact points $\{\mathbf{x}_i, 1 \leq i \leq 19\}$, to form a 19-blade multivector in $Cl(19,16)$, describing a cubic surface $\mathbf{q}$ as

$$\mathbf{q} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \mathbf{x}_{19}. \tag{55}$$

Similar to the case of cubic curves we can identify points on a cubic surface as follows.

Proposition 61 *A point* $\mathbf{x}$ *lies on the three dimensional cubic surface* $\mathbf{q}$ *if and only* $\mathbf{x} \wedge \mathbf{q} \wedge \mathbf{I}_{o}^{\triangleright} = 0$.

The proof is analogous to the case of cubic curves. And we also obtain the corollary.

Corollary 62 *A point* $\mathbf{x}$ *lies on the cubic surface defined by* $\mathbf{q}$ *if and only if* $\mathbf{x} \cdot (\mathbf{q} \wedge \mathbf{I}_{o}^{\triangleright})^* = 0$.

The construction of the dual representation of a cubic surface $(\mathbf{q} \wedge \mathbf{I}_{0}^{\triangleright})^*$ is similarly possible from the 20 coefficients of the implicit equation (54). Finally intersection operations of linearly independent cubic surfaces work analogous to the description of the intersection of cubic curves given in Section 5.

7 Conclusion

This work described how to represent cubic curves by multivector blades in cubic conformal geometric algebra $Cl(9,7)$. The construction can either proceed from nine contact points or by using the ten coefficients of the implicit equation of a cubic curve in the plane. The multivector expressions obtained for cubic curves can then e.g. be used for computing the intersection of curves, using the outer product. It is found, that cubic CGA contains an embedding of conic CGA [11] and of standard CGA [6] of the Euclidean plane.

In future work, it is intended to optimize the Clifford Multivector Toolbox for Matlab [17] and GAALOP [8] further, so as to be able to compute with $Cl(9,7)$ on a standard PC. Furthermore, we intend to establish the geometric transformation versors for rotation, translation and scaling, which we expect to be somewhat more intricate than for conics in conic CGA [16,13,11]. We hope, that the current work on cubic curves in CGA will find applications, wherever cubic curves occur in computations and graphics, where they may be used for interpolation, etc.

In the last part, we described the analogous construction for the representation of cubic surfaces in three dimensions and their intersections in cubic surface CGA $Cl(19,16)$. This latter algebra may not yet be attainable to computations with current computer algebra systems, except perhaps with super computers. In this case as well research should be done for constructing versors for rotations, translations and scaling.

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