A trigonometric integral

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Abstract. In this note we give a trigonometric integral.

1. Integral

Entry 1.

$$\int_0^{\pi/6}\left(\sqrt[4]{\frac{\sin((\pi/6)+x)}{\sin((\pi/6)-x)}}-\sqrt[4]{\frac{\sin((\pi/6)-x)}{\sin((\pi/6)+x)}}\right)\sin(2x)\,dx=\frac{3\pi}{8\sqrt{12+6\sqrt{3}}} \tag{1}$$

2. Remarks

Entry 2.

$$\int_0^{1/2}\left(\sqrt[4]{\frac{\sqrt{1-x^2}+\sqrt{3}x}{\sqrt{1-x^2}-\sqrt{3}x}}-\sqrt[4]{\frac{\sqrt{1-x^2}-\sqrt{3}x}{\sqrt{1-x^2}+\sqrt{3}x}}\right)x\,dx=\frac{3\pi}{16\sqrt{12+6\sqrt{3}}} \tag{2}$$

$$\int_{\sqrt{3}/2}^{1}\left(\sqrt[4]{\frac{x+\sqrt{3}\sqrt{1-x^2}}{x-\sqrt{3}\sqrt{1-x^2}}}-\sqrt[4]{\frac{x-\sqrt{3}\sqrt{1-x^2}}{x+\sqrt{3}\sqrt{1-x^2}}}\right)x\,dx=\frac{3\pi}{16\sqrt{12+6\sqrt{3}}} \tag{3}$$

Entry 3.

$$\frac{\pi}{\sqrt{12+6\sqrt{3}}}= \\ =\sum_{n=0}^{\infty}(-2)^{-n}\left(\frac{F(n+1,1;2n+(7/4);1/2)}{8n+3}+\frac{F(n+1,1;2n+(9/4);1/2)}{8n+5}\right) \tag{4}$$

$$\frac{\pi}{\sqrt{12+6\sqrt{3}}}=\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{2n}(-1)^k\left(\frac{1}{4k+3}+\frac{1}{4k+5}\right)\sum_{m=0}^{[k/2]}(-1)^m\binom{n}{k-m}\binom{k-m}{m} \tag{5}$$

Entry 4. If $f(n,k)=\sum_{m=0}^{[k/2]}(-1)^m\binom{n}{k-m}\binom{k-m}{m}$, $n\geq 0, 0\leq k\leq 2n$, then

$f(n,k),\quad 0\leq n\leq 5,\quad 0\leq k\leq 2n$

1										
1	1	−1								
1	2	−1	−2	−1						
1	3	0	−5	0	3	−1				
1	4	2	−8	−5	8	2	−4	1		
1	5	5	−10	−15	11	15	−10	−5	5	−1

Final Remarks:

- $\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=3.141592...$
- $F(a,b;c;x)={}_2F_1(a,b;c;x)$ is the hypergeometric function.

References

1. Boros, G. and Moll, V.H.: Irresistible Integrals, Cambridge University Press, 2004.