The Inconsistency of Arithmetic

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Abstract. This paper presents an antinomy within the Peano arithmetic (PA). We derive two contradictory statements by showing that under two given assumptions a specific set is different in content on the one hand and identical on the other hand.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1, let \( \mathbb{N}_n \) denote the natural numbers starting from \( n > 1 \) and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Theorem. The Peano arithmetic (PA) is inconsistent.

Proof. We define the set

\[ S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \]

and we consider the following two cases.

\( (G) \): \( \forall x \in \mathbb{N}_4 \) \( \exists (pk, mk, qk) \in S_g \) \( x = m \).

\( \neg(G) \): \( \exists x \in \mathbb{N}_4 \) \( \forall (pk, mk, qk) \in S_g \) \( x \neq m \).

For each fixed \( k \geq 1 \), we set \( M(k) := \{ mk \mid (pk, mk, qk) \in S_g \} \). Then, by definition

(\( G \) ) \( \iff \) \( M(k) = k\mathbb{N}_4 \) for every \( k \geq 1 \)

(\( \neg(G) \) ) \( \iff \) \( M(k) \neq k\mathbb{N}_4 \) for every \( k \geq 1 \).

This implies that \( S_g \) does not have the same content in the cases \( (G) \) and \( \neg(G) \):

(\( I \) ) \( \exists \) sets \( S, S' \) such that \( S \neq S' \) and ( \( ((G) \Rightarrow S_g = S) \) and ( \( \neg(G) \Rightarrow S_g = S' \) ).

By using another criterion, we will now show the negation of (\( I \)), that is, that the set \( S_g \) has the same content in the cases \( (G) \) and \( \neg(G) \).

The whole range of \( \mathbb{N}_3 \) can be expressed by the triple components of \( S_g \), since every integer \( x \geq 3 \) can be written as some \( pk \) with \( k = 1 \) when \( x \) is prime, as some \( pk \) with \( k \neq 1 \) when \( x \) is composite and not a power of 2, or as \( (3+5)k / 2 \) when \( x \) is a power of 2, where \( p \in \mathbb{P}_3, k \in \mathbb{N} \).

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The case \(-\text{(G)}\) means that there is at least one \(n \geq 4\) such that for each fixed \(k \geq 1\) \(nk\) is different from all the \(mk\) generated in \(S_g\). Correspondingly, the case \(\text{(G)}\) means that there is no such \(n\). According to the above three types of expression by \(S_g\) triple components, for any \(n \geq 4\) given by \(-\text{(G)}\) we have the property

\[(C) \colon \forall k \in \mathbb{N} \exists (pk', mk', qk') \in S_g \quad nk = pk' \lor nk = mk' = 4k'.\]

So, every \(nk\) given by \(-\text{(G)}\) equals a component of some \(S_g\) triple that exists by definition.

Moreover, since all pairs \((p, q)\) of odd primes with \(p < q\) are used in \(S_g\) and so all arithmetic means \(m\) of two odd primes are generated, we have that an \(n \geq 4\) given by \(-\text{(G)}\) cannot be the arithmetic mean of a pair of odd primes not used in \(S_g\). This results in the property

\[(M) \colon \nexists p, q \in \mathbb{P}_3, p < q \quad n = (p + q) / 2.\]

Now, let \(S\) be a set such that \(\text{(G)} \Rightarrow S_g = S\) and let \(S'\) be a set such that \(-\text{(G)} \Rightarrow S_g = S'\).

Because the properties \((C)\) and \((M)\) hold for any \(n\) given by \(-\text{(G)}\), the content of \(S'\) can be written as the union of the following triples, which would otherwise be impossible.

\[(i) \quad S_g\) triples of the form \((pk' = nk, mk', qk')\) with \(k' = k\) in case \(n\) is prime, due to \((C)\)

\[(ii) \quad S_g\) triples of the form \((pk' = nk, mk', qk')\) with \(k' \neq k\) in case \(n\) is composite and not a power of 2, due to \((C)\)

\[(iii) \quad S_g\) triples of the form \((3k', 4k' = nk, 5k')\) in case \(n\) is a power of 2, due to \((C)\)

\[(iv) \quad \text{all remaining} \quad S_g\) triples of the form \((pk' = nk, mk', qk'), (pk', mk' = nk, qk')\) or \((pk', mk', qk' = nk)\)

\[\text{and of}\]

\[(v) \quad S_g\) triples of the form \((pk' \neq nk, mk' \neq nk, qk' \neq nk)\), i.e. those \(S_g\) triples where none of the \(nk\)’s equals a component.

The triples in \((iv)\) comprise all \(S_g\) triples where \(nk\) occurs as a component redundantly to the occurrences in \((i) - (iii)\). So, we can split the triples in \((iv)\) as follows.

\[(iv, a) \quad S_g\) triples of the form \((pk', mk', qk' = nk)\) with \(k' = k\) in case \(n\) is prime

\[(iv, b) \quad S_g\) triples of the form \((pk', mk' = nk, qk')\) with \(k' = k\) in case \(n\) is prime

\[(iv, c) \quad S_g\) triples of the form \((pk', mk', qk' = nk)\) with \(k' \neq k\) in case \(n\) is composite and not a power of 2
(iv, d) $S_g$ triples of the form $(pk', mk' = nk, qk')$ with $k' \neq k$ in case $n$ is composite and not a power of 2

(iv, e) $S_g$ triples of the form $(pk', mk' = nk, qk')$ with $k' = k$ in case $n$ is composite

(iv, f) $S_g$ triples of the form $(pk' = nk, mk', qk')$ in case $n$ is a power of 2

(iv, g) $S_g$ triples of the form $(pk', mk', qk' = nk)$ in case $n$ is a power of 2

(iv, h) $S_g$ triples of the form $(pk', mk' = nk, qk')$ with $m \neq 4$ in case $n$ is a power of 2.

Apart from the triples of type (iv, b) and (iv, e) that cannot exist due to (M), $S'$ consists of all triples of type (i) to (v). This means that the content of $S'$ is composed of two complementary subsets of $S_g$, which we denote by $S_a$ and $S_{na}$. $S_a$ consists of all $S_g$ triples where one of the nk's equals one of the three components and $S_{na}$ consists of all those $S_g$ triples where none of the nk's equals a component.

Thus, we conclude that the set $S'$ has the same content as $S_g$, i.e. as if we do not assume the existence of $n$. So, the $n$ given by $\neg(G)$ actually does not exist and, for any $x \in \mathbb{N}_4$ in the place of $n$, the triples of type (iv, b) and (iv, e) actually exist.

On the other hand, by the non-existence of $n$, the set $S$ trivially equals $S_g$ since all triples of $S_g$ belong to $S$.

Therefore, we can state:

\[ \forall \text{ sets } S, S' \ ( (G) \Rightarrow S_g = S ) \text{ and } (\neg(G) \Rightarrow S_g = S') \Rightarrow (S = S_g) \text{ and } (S' = S_g) \]

\[ \Rightarrow S = S' \]

\[ \Leftarrow \]

(II) $\not\exists$ sets $S, S'$ such that $S \neq S'$ and $( (G) \Rightarrow S_g = S )$ and $(\neg(G) \Rightarrow S_g = S')$.

\[ \square \]

**Remark.** The proof uses a strengthened form of the strong Goldbach conjecture:

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

SSGB is equivalent to saying that all integers $x \geq 4$ appear as $m$ in a component $mk$ of $S_g$. Therefore, SSGB is equivalent to the case (G) and the negation $\neg$SSGB is equivalent to the case $\neg$(G). We have seen above that the set $S_g$ remains the same in these two cases. This immediately implies $\neg$SSGB $\Rightarrow$ SSGB, thereby proving SSGB (see details on
A proof of SSGB, that is derived from statement (II), means that (G) is true and \( \neg(G) \) is false. When using this result, (I) becomes trivially true and (II) becomes trivially false. This leads to the following logical conclusions:

\( * \) If (G) and \( \neg(G) \) are assumptions, we can derive the two contradictory statements (I) and (II) as shown in the above proof.

\( ** \) If (G) is proven, then (II) is no longer valid what is a contradiction since (II) is deduced in the proof above.

All in all, we have: 
\[
( \text{(II)} \land (\text{II} \Rightarrow \text{(G)} \Rightarrow \neg(\text{II}) ) ) \Rightarrow ( \text{(II)} \land \neg(\text{II}) ) .
\]

The two situations (\( * \)) and (\( ** \)) demonstrate that the inconsistency is independent of the state of knowledge.