The Inconsistency of Arithmetic

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Abstract. This paper presents an antinomy within the Peano arithmetic (PA). We derive two contradictory statements by showing that under two given assumptions a specific set is different in content on the one hand and identical on the other hand.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1, let \( \mathbb{N}_n \) denote the natural numbers starting from \( n > 1 \) and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Theorem. The Peano arithmetic (PA) is inconsistent.

Proof. We define the set

\[ S_9 := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \]

and we consider the following two cases.

\[ (G): \forall x \in \mathbb{N}_4 \exists (pk, mk, qk) \in S_9 \quad x = m. \]

\[ \neg(G): \exists x \in \mathbb{N}_4 \forall (pk, mk, qk) \in S_9 \quad x \neq m. \]

For each fixed \( k \geq 1 \), we set \( M(k) := \{ mk \mid (pk, mk, qk) \in S_9 \} \). Then, by definition

\[ (G) \iff M(k) = k \mathbb{N}_4 \quad \text{for every } k \geq 1 \]

\[ \neg(G) \iff M(k) \neq k \mathbb{N}_4 \quad \text{for every } k \geq 1. \]

This implies that \( S_9 \) does not contain the same triples in the cases \( (G) \) and \( \neg(G) \):

\[ (I) \exists \text{ sets } S, S' \text{ such that } S \neq S' \text{ and } ( (G) \Rightarrow S_9 = S ) \text{ and } ( \neg(G) \Rightarrow S_9 = S' ). \]

Using another criterion, we will now show equality of the set \( S_9 \) in the cases \( (G) \) and \( \neg(G) \).

The case \( \neg(G) \) means that there is at least one \( n \geq 4 \) such that \( nk \) is different from all the \( mk \) for each \( k \geq 1 \), where all pairs \((p, q)\) of odd primes, that determine the numbers \( m \), are used in \( S_9 \).

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For each $k \geq 1$, such an $n_k$ can be written as some $p_k$ when $n$ is prime, as some $p_k'$ when $n$ is composite and not a power of 2, or as $4k'$ when $n$ is a power of 2; $p \in \mathbb{P}$; $k, k' \in \mathbb{N}$.

The expression $p_k'$ for $n_k$ with $k' = k$ or $k' \neq k$ is a first component of $S_9$ triples and the expression $4k'$ for $n_k$ is component of the triple $(3k', 4k', 5k')$. So, since $n_k$ equals some triple component $p_k'$ or $4k'$ that exists by definition of $S_9$ and since the $S_9$ triples are generated by the first and third components, the triples are the same in the case $n_k$ exists and in the case $n_k$ does not exist. I.e. the set $S_9$ stays as it is defined in case $\neg (G)$ holds and in case $(G)$ holds.

Analogously to the first part of the proof, we can formalize the above reasoning as follows.

We consider $n \in \mathbb{N}_4$ given by the case $\neg (G)$, that is, $n \neq m$ for all $(p_k, m_k, q_k) \in S_9$. Then, first of all, $n$ cannot be the arithmetic mean of a pair of odd primes not used in $S_9$. Thus, for $n$ only the following remains that we have shown above.

$$(C): \forall k \in \mathbb{N} \ \exists (p_k', m_k', q_k') \in S_9 \ \ n_k = p_k' \ \lor \ n_k = m_k'$$

Therefore, using that the difference between case $(G)$ and case $\neg (G)$ is the (non-) existence of $n_k$ for each $k \geq 1$, we can state:

\[
\forall \text{ sets } S, S' \\
\left( ((G) \Rightarrow S_9 = S) \text{ and } (\neg (G) \Rightarrow S_9 = S') \right) \\
\Rightarrow \left( (S_9 = S) \text{ and } ((C) \Rightarrow S_9 = S') \right), \text{ since } S_9 \text{ stays as it is defined if there is no } n_k \text{ and since in case of } \neg (G) S_9 \text{ stays as it is defined if all } n_k \text{ equal some } S_9 \text{ triple component.} \\
\Rightarrow \left( (S_9 = S) \text{ and } (S_9 = S') \right), \text{ since } (C) \text{ is true for any } n \text{ given by } \neg (G). \\
\Rightarrow S = S' \\
\iff (II) \not\exists \text{ sets } S, S' \text{ such that } S \neq S' \text{ and } \left( ((G) \Rightarrow S_9 = S) \text{ and } (\neg (G) \Rightarrow S_9 = S') \right). \quad \Box
\]

The proof uses a strengthened form of the strong Goldbach conjecture:

**Strengthened strong Goldbach conjecture (SSGB):** Every even integer greater than 6 can be expressed as the sum of two different primes.

SSGB is equivalent to saying that all integers $x \geq 4$ appear as $m$ in a component $m_k$ of $S_9$. Therefore, SSGB is equivalent to the case $(G)$ and the negation $\neg$SSGB is equivalent to the case $\neg (G)$. We have seen above that the $S_9$ triples are the same in these two cases. This means that both SSGB and $\neg$SSGB hold.