Direct Sum Decomposition of a Linear Vector Space

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Abstract

The direct sum decomposition of a vector space has been explored to bring out a conflicting feature in the theory. We decompose a vector space using two subspaces. Keeping one subspace fixed we endeavor to replace the other by one which is not equal to the replaced subspace. Proceeding from such an effort we bring out the conflict. From certain considerations it is not possible to work out the replacement with an unequal subspace. From alternative considerations an unequal replacement is possible.

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Introduction

The concept of linear vector spaces[1] is fundamental to the edifice of physics and mathematics. Nevertheless this theory is not free from conflicts. We decompose a vector space using two subspaces[2]. Keeping one subspace fixed we endeavor to replace the other by one which is not equal to the replaced subspace. Proceeding from such an effort we bring out the conflict. From certain considerations it is not possible to work out the replacement with an unequal subspace. From alternative considerations an unequal replacement is possible.

Basic Considerations and Calculations

We consider the direct sum[3] decomposition of a vector space V in subspaces $A$and B .

$$V = A \oplus B \quad (1)$$

The dimensions of the three spaces have been stated below.

$$Dim(V) = n, Dim(A) = k \Rightarrow Dim(B) = n - k$$

If possible let there be a subspace B' ,distinct from B such that

$$V = A \oplus B' \quad (2)$$

Since $Dim(V) = n, Dim(A) = k$,we have , $Dim(B') = n - k$

We denote by m, the dimension of the intersection, $B \cap B'$

Number of linearly vectors in the subset $A \cup B \cup B'$isequal to the number of linearly independent vectors in $A + those\ in\ (\ B - B \cap B') + those\ in\ (B' - B \cap B') + those\ in\ B \cap B'$

$$k + (n - k - m) + (n - k - m) + m \leq n \quad (3)$$

In relation to relation (3) we have to take note of the fact that $A \cap B = A \cap B' = A \cap B \cap B' = \{0\}$

Equation (3) implies

$$m \geq n - k$$

It is not possible to have m (=the dimension of $B \cap B'$) greater than $n - k$, the dimension of Bor of B'

Therefore,

$$m = n - k \;\; (4)$$

$$\Rightarrow B = B' \; (5)$$

A Pair of Theorems

We consider a basis of $V = \{e_1, e_2, e_3, \ldots\ldots e_k, e_{k+1}, e_{k+2} \ldots. e_n\}$

We further assume $\{e_1, e_2, e_3, \ldots\ldots e_k\}$is a basis of A and that $\{e_{k+1}, e_{k+2} \ldots. e_n\} \equiv \{e_{k+j}\}$forms a basis of B

We have relation (1): $V = A \oplus B$

Now we consider a set of n vectors $\{e_1, e_2, e_3, \ldots\ldots e_k, e'_{k+1}, e'_{k+2} \ldots. e_n{}'\}$ where $e'_{k+j} = e_{k+j} + \alpha_j$ and $\alpha_j \in A; j = 1,2, \ldots n - k;\ \alpha_j \neq 0[e_{k+j} \in B$,defined earlier in this section.

Theorem 1:We prove that

$e'_{k+j} \in \overline{A \cup B} \qquad or, e'_{k+j} = e_{k+j} + \alpha_j \in \overline{A \cup B}$

Assume that $e'_{k+j} \in A$

Now,

$$e_{k+j} = e'_{k+j} - \alpha_j$$

On the left side $e_{k+j} \in B$

On the right side both $e'_{k+j}$and $\alpha_j$belong to $A \Rightarrow e'_{k+j} - \alpha_j \in A; j = 1,2, \dots n-k$. This not possible taking note of the fact that $A \cap B = \{0\}$ and that all the vectors involved are non zero vectors.

Therefore $e'_{k+j} \notin A$

Next let $e'_{k+j} \in B$

Now,

$$e'_{k+j} - e_{k+j} = \alpha_j$$

On the left side $e'_{k+j} - e_{k+j}$belongs to B since each $e'_{k+j}$and e_{k+j}belong to B.On the right side of the above $\alpha_j \in A$.This is not possible taking note of the fact that $A \cap B = \{0\}$ and that all the vectors involved are non zero vectors.

Therefore $e'_{k+j} \notin B$

Therefore as claimed we have,

$$e'_{k+j} = e_{k+j} + \alpha_j \in \overline{A \cup B}$$

Theorem 2:The set $\{e_1, e_2, e_3, \dots \dots e_k, e'_{k+1}, e'_{k+2} \dots . e_n{}'\}$ form as basis with respect to the space V.

We consider the equation

$$\sum_{i=1}^{k} C_i e_i + \sum_{j=1}^{n-k} C_{k+j} e'_{k+j} = 0 \quad (6)$$

Now,

$$e'_{k+j} = e_{k+j} + \alpha_j$$

Therefore,

$$\sum_{i=1}^{k} C_i e_i + \sum_{j=1}^{n-k} C_{k+j} \left(e_{k+j} + \alpha_j\right) = 0$$

$$\sum_{i=1}^{k} C_i e_i + \sum_{j=1}^{n-k} C_{k+j} e_{k+j} + \sum_{j=1}^{n-k} C_{k+j} \alpha_j = 0$$

$$\alpha_j = \sum_{l=1}^{k} D_{jl} e_l = \sum_{i=1}^{k} D_{ji} e_i$$

Therefore,

$$\sum_{i=1}^{k} C_i e_i + \sum_{j=1}^{n-k} C_{k+j} e_{k+j} + \sum_{j=1}^{n-k} \sum_{i=1}^{k} C_{k+j} D_{ji} e_i = 0$$

$$\sum_{i=1}^{k} \left(C_i + \sum_{j=1}^{n-k} C_{k+j} D_{ji} \right) e_i + \sum_{j=1}^{n-k} C_{k+j} e_{k+j} = 0$$

Since

$$\{e_1, e_2, e_3, \ldots\ldots e_k, e_{k+1}, e_{k+2} \ldots. e_n\}$$

forms a linearly independent set in that they are the basic vectors for V,we have,

$$C_{k+j} = 0; j = 1,2,3 \ldots n - k \ (7.1)$$

$$C_i + \sum_{j=1}^{n-k} C_{k+j} D_{ji} = 0; i = 1,2,3 .. k; i = 1,2,3 \ldots k \ \ (7.2)$$

Since from (7.1)

$$C_{k+j} = 0; j = 1,2,3 \ldots n - k$$

we have

$$C_i = 0; i = 1,2,3 \ldots k$$

Therefore,

$$\{e_1, e_2, e_3, \ldots\ldots e_k, e'_{k+1}, e'_{k+2} \ldots. e_n{}'\}$$

comprise a linearly independent set of vectors . Since there are 'n' such vectors, n being the dimension of V, they span V.

Therefore the above set is a basis for V.

The Conflict

Let us consider B' spanned by

$$\{e'_{k+1}, e'_{k+2} \ldots. e_n{}'\}$$

Since

$$\{e_1, e_2, e_3, \ldots\ldots e_k, e'{}_{k+1}, e'{}_{k+2} \ldots. e'{}_n\}$$

spans V, we have the direct sum decomposition,

$$V = A \oplus B'$$

We also do have from(5)

$$B' = B$$

Now from theorem I, we have, $e'_{k+j} \in \overline{A \cup B} \Rightarrow e'_{k+j} \in \overline{A \cup B'} \Rightarrow e'_{k+j} \notin B'$

But e'_{k+j} is a basic vector of B'

Thus we have arrived at a contradiction.

Conclusion

A conflict in the theory of vector spaces can have serious consequences in the areas of mathematics and physics opening up gateways to fundamental research

References

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