Ordinary Differential Equation
Alternate Solution (Co-efficient not Limited to Polynomial)

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Abstract

The laws of Physics and some other related courses are generally written as differential equations. Therefore, all of science and engineering use differential equations to some extent. A good knowledge of differential equations will be an integral part of your study in science and/or engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned.

Definition:A differential equation is an equation involving an unknown function and its derivative(s). A differential equation is an **ordinary differential equation** if the unknown function depend on only one of the independent variable.

Definition:A function is said to be a solution of a differential equation on some interval if it possesses requisite number of derivative and satisfy the equation upon substitution.

Exactness

In this paper, I will try to explore the use of integrating factor to solve higher order differential equation. Given

$$A(x)\frac{dy}{dx} + B(x)y = C(x) \tag{1}$$

Where $A(x),\ B(x),$ and $C(x)$ are specified real valued functions defined on an open interval I of $\mathbb{R}$. Where $A(x),\ B(x)$ and $C(x)$ are continuous on I.

Case I: If $A(x)\frac{dy}{dx} + B(x)y = [A(x)y]' = C(x)$, then

$$A'(x) - B(x) = 0 \tag{2}$$

Proof:

$[A(x)y]' = C(x)$
$[A(x)y]' = A(x)\frac{dy}{dx} + A'(x)y = C(x)$
Compare with equation 1
$A'(x) = B(x)$
$A'(x) - B(x) = 0$

Case II: If $A'(x) - B(x) \neq 0$, and there exists $U(x)$ in I such that, $U(x) = \frac{D}{A(x)} \exp^{\int \frac{B(x)}{A(x)} dx}$, where D is a constant. Then

$$[U(x)A(x)y]' = U(x)C(x)$$

Proof:

$A(x)\frac{dy}{dx} + B(x)y = C(x)$
Multiply through by U(x),

$$U(x)A(x)\frac{dy}{dx} + U(x)B(x)y = U(x)C(x) \tag{3}$$

By case I above, $[U(x)A(x)y]' = U(x)C(x)$
$U(x)A(x)\frac{dy}{dx} + [U(x)A'(x) + U'(x)A(x)]y = U(x)C(x)$
Compare with equation 3
$U(x)A'(x) + U'(x)A(x) = U(x)B(x)$
$U(x)A'(x) + U'(x)A(x) - U(x)B(x) = 0$
$U'(x)A(x) + [A'(x) - B(x)]U(x) = 0$
Divide through by $A(x)U(x)$
$\frac{U'(x)}{U(x)} + \frac{A'(x)-B(x)}{A(x)} = 0$
Solving the first order differential equation and simplifying further, we are left with

$$U(x) = \frac{D}{A(x)} \exp^{\int \frac{B(x)}{A(x)} dx}$$

Similarly, given

$$A(x)\frac{d^2y}{dx^2}+B(x)\frac{dy}{dx}+C(x)y=D(x) \tag{4}$$

Where $A(x),\ B(x),\ C(x)$ and $D(x)$ are specified real valued functions defined on an open interval I of $\mathbb{R}$. Where $A(x),\ B(x),\ C(x)$ and $D(x)$ are continuous on I.

Case III: If $A(x)\frac{d^2y}{dx^2}+B(x)\frac{dy}{dx}+C(x)y=[A(x)y']'+[T(x)y]'=D(x)$, then

$$A''(x)-B'(x)+C(x)=0 \tag{5}$$

Where $T(x)=B(x)-A'(x),\quad T'(x)=C(x)$

Proof:

$[A(x)y']'+[T(x)y]'=D(x)$
$[A(x)y']'+[T(x)y]'=A(x)\frac{d^2y}{dx^2}+[A'(x)+T(x)]\frac{dy}{dx}+T'(x)y=D(x)$
Compare with equation (4)
$B(x)=A'(x)+T(x),C(x)=T(x)$
$B'(x)=A''(x)+T'(x),C'(x)=T'(x)$
$A''(x)-B'(x)+C(x)=0$

Case IV: If $A''(x)-B'(x)+C(x)\neq 0$, and there exists $U(x)$ in I such that, $U(x)=\int\frac{P}{A^2(x)}\exp^{\int\frac{B(x)}{A(x)}dx}dx+Q,$ where P, Q are constants. Then

$$[U(x)A(x)y']'+[T(x)y]'=U(x)C(x)$$

Proof:

$A(x)\frac{d^2y}{dx^2}+B(x)\frac{dy}{dx}+C(x)y=D(x)$
Multiply through by U(x),

$$U(x)A(x)\frac{d^2y}{dx^2}+U(x)B(x)\frac{dy}{dx}+U(x)C(x)y=U(x)D(x) \tag{6}$$

By case III above,
$[U(x)A(x)y']'+[T(x)y]'=U(x)D(x)$
$U(x)A(x)\frac{d^2y}{dx^2}+[U(x)A'(x)+U'(x)A(x)+T(x)]\frac{dy}{dx}+T'(x)y=U(x)D(x)$
Compare with equation 6
$U(x)A'(x)+U'(x)A(x)+T(x)=U(x)B(x),\qquad T'(x)=U(x)C(x)$
Differentiate the first part and then substitute the second part
$U(x)A''(x)+2U'(x)A'(x)+U''(x)A(x)+T'(x)=U(x)B'(x)+U'(x)B(x)$
$U(x)A''(x)+2U'(x)A'(x)+U''(x)A(x)+U(x)C(x)=U(x)B'(x)+U'(x)B(x)$
Since $A''(x)-B'(x)+C(x)=0$ by the existence of $U(x)$, then
$2U'(x)A'(x)+U''(x)A(x)=U'(x)B(x)$
$U''(x)A(x)+[2A'(x)-B(x)]U'(x)=0$
Solve the second order differential equation, simplify further to get
$U=\int\frac{P}{A^2(x)}\exp^{\int\frac{B(x)}{A(x)}dx}dx+Q$

Remark: The above method above can be extended to n-th order differential equation.

Examples:

1. Solve $y''+\frac{1}{x}y'-\frac{1}{x^2}y=0$ completely
   Since $A''(x)-B'(x)+C(x)=0$, then
   $[y']'+[\frac{1}{x}y]'=0$
   $y'+\frac{1}{x}y=A$
   $xy'+y=Ax$
   $[xy]'=Ax$
   $xy=Bx^2+C,\quad$ where $B=\frac{A}{2}$
   $y=Bx+\frac{C}{x}$
2. Solve $y''+(\cot x)y'-(\csc^2 x)y=0$ completely
   Since $A''(x)-B'(x)+C(x)=0$, then

$$
\begin{aligned}
&[y']' + [(\cot x)y]' = 0 \\
&y' + (\cot x)y = A \\
&[y \sin x]' = A \sin x \\
&y \sin x = B \cos x + C, \quad \text{where } B = -A \\
&y = B \cot x + C \csc x
\end{aligned}
$$

Note: If $A''(x) - B'(x) + C(x) \neq 0$, the solution is more rigorous than existing method.

Reference

Advanced Engineering Mathematics by H K Dass