Proceedings on non commutative geometry.

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Contents

Some words upfront.

This little book is meant as a generalization of classical topology and geometry into the realm of non-commutative topologies as well as geometries. These are objects of the operational kind where a space is glued together and the gluing procedure does not obey the commutativity and associativity properties of the set theoretical union. That is to say, there is substance to space like there are two sides to a wooden plate and space is not something which exists and void of properties at the same time. Likewise is this so for the distance function where measurement, or the act thereof, depends upon previous measurements leaving a trace in the non-abelian dust. This is a property which holds for sure in nature albeit we know the dust to be very subtle regarding our senses. We do not see neither feel it, but it is there and a necessary aspect for the creation of life. Until a few decades ago, physicists have righteously ignored these small aspects of geometrical sensitivity but it has become the time to investigate them properly. There exist a few distinct proposals from the mathematical side of how to deform the classical situation; for example Alain Connes focuses on function algebra's and Dirac operators. It is well known that the classical limit problem herein is well posed and answered for, albeit the general non-abelian situation has no obvious geometrical interpretation and it appears way too general in order for it to be useful. Another approach was taken by Majid, Vaes and others and hinges upon the concept of a Hecke algebra which is an object unifying a Lie-algebra and a Lie-group; an approach which seems certainly useful for highly symmetrical spaces such as Minkowski or (anti) de Sitter. It is nevertheless still grounded in the concept of "inertial coordinates" and generalizations towards curved geometry are highly suspicious and confused. The approach taken here resembles the one taken in my book on "geometrical quantum theory and applications" where classical physics is written as a peculiar case in the quantum language and quantum theory is rather seen as a bi-dual theirof. That is, there is a one to one mapping between a classical metric space and the quantization theirof. We shall go much further here and study connection theory from an abstract global point of view and develop quantum connections with differentiable sections. The book is short and intended for the beginning researcher; everything which follows is exactly defined and all relevant properties proven and commented upon.

Chapter 1

Classical logic and topology.

Spaces are usually defined as consisting out of elements and being composed by gluing of "standard" elements together. This requires cut and paste operations equivalent to taking the intersection $\cap$ and union $\cup$. In standard mathematics, we assume those operations to be perfect meaning there is no waste as well as no preference for how or the order in which they are performed. This last stance translates as $A \cap B = B \cap A$ as well as $A \cup B = B \cup A$ both properties being referred to as the commutativity of the respective operations. This is not necessarily so in nature, it does matter for example when I poor coffee in first in a bowl and then hot water later on. In this case the coffee dissolves and raises upwards causing for a homogeneous mixture. If I were to do it the other way around, the coffee would most likely keep on floating on the water. So this commutativity of the union is not obvious, it refers to the fact that items are hard objects and no particular law holds between them. They are independent as to speak; this stance of individualism is required in science, we would not learn anything from a holistic perspective. We have to subdivide and believe in holy freedom otherwise nothing can be said about the I and its relations to others. We moreover insist those operations to be associative meaning that $(A \cap B) \cap C = A \cap (B \cap C)$ and likewise so for the union. Now, we can talk! Denote with $A, B, C, \ldots$ so called sets; we have no idea yet what they are but we shall further specify some properties regarding the operations $\cap$ and $\cup$. The operations satisfy for sure $A \cap A = A \cup A = A$ and we demand the existence of a unique empty set $\emptyset$ such that

$$\begin{aligned} A \cap \emptyset &= \emptyset \\ A \cup \emptyset &= A \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

where this last rule is the same as the de-Morgan rule in Boolean logic. Set theory at this level is equivalent to the rules of classical logic where the A denote truisms and $\emptyset$ is given by "false". Then $A \cap A = A$ reads as A and A are both true is the same as A is true. A or A is true, denoted by $A \cup A$

is the same as A is true. A and false is always false whereas the vericacity of A or false just depends upon A. Finally A is true and B or C is true is the same as A and B is true or A and C is true. So set theory is classical logic, it is a definite speech about truisms of belonging to. We will later on think of devilish ways to escape this definite way of speaking about things which hinges upon many assumptions which could equally well be false. However, just as is the case for Greek and Roman architecture, the most simple rules can allow for very complicated ones to arise by means of building. The old Greek alway described elements or atomos as things which cannot be further subdivided; hence the following definitions. We say that A is a subset of B if and only if the intersection of A and B equals A which reads as $A \subseteq B \leftrightarrow A \cap B = A$. An atom $A \neq \emptyset$ is called a primitive set, that is, A has the property that if $B \subseteq A$ then $B = A$. The reader checks the obvious statement that $A \cap C \neq \emptyset$ is a subset of A; this follows from associativity and commutativity of the intersection because $A \cap (A \cap C) = (A \cap A) \cap C = A \cap C$ and therefore, by definition $A = A \cap C \subseteq C$ in case A is an atom or primitive set. Indeed, we can only speak of subparts when the operation of intersection is priceless. This suggests that primitive sets are as elements of a set and to emphasize that distinction we denote $A = \{\widehat{A}\}$ where $\widehat{A}$ is interpreted as an element and the brackets denote the bag. We use the symbolic notation $\widehat{A} \in B$ as an equivalent to the more primitive statement $A \cap B = A$.

The reader notices that we have defined elements from the operations $\cap, \cup$ whereas normally the opposite happens; you cannot crumble the bread further than up to its elementary fibers. This is a much more human way of dealing with language in the sense that the limitations attached to our operations define our notion of reality. The old approach starts from divine knowledge which nobody possesses; in order to make logic dynamical and attached to physical processes in space-time, mathematicians have invented the notion of a Heyting algebra instead of a Boolean one. We shall not go that far in this book but the interested reader should comprehend very well how this definition is tied to the one of classical relativistic causality. Our point of view also allows for quantal rules as long as the de-Morgan rule is suitably deformed; we shall discuss such logic in this book and make even further extensions towards non-associative and non-commutative cases. Extension of the material presented is left to the fantasy of the gifted reader. For example, an infinite straight line does not need to consist out of points, the latter being mere abstractions. Let us first investigate further implications of our rules before we move on to further limitation of the setting at hand. It is true that if $B \subseteq C$ then every element $\widehat{A}$ in B belongs to C. Indeed, $\widehat{A} \in B$ if and only if $A \cap B = A$ and therefore $A \cap C = (A \cap B) \cap C = A \cap (B \cap C) = A \cap B = A$ proving that $A \cap C = A$ and therefore $\widehat{A} \in C$. Differently, $\widehat{A} \in B$ if and only if $A \cap B = A$ which is equivalent to $(A \cap C) \cap B = A$ and therefore $A \cap C \neq \emptyset$ from which follows that $A \cap C = A$ because A is an atom. Hence, elements of subsets belong to the set itself. What about the intersection of two sets? First, we show that if $\widehat{A} \in B, C$ then $\widehat{A} \in B \cap C$: this holds because

$A \cap (B \cap C) = (A \cap B) \cap C = A \cap C = A$ and therefore $\widehat{A} \in B \cap C$. The other way around, we have that if $\widehat{A} \in B \cap C$ then $\widehat{A} \in B, C$ because the intersection is a subset of both. Hence, the elements in the intersection are precisely those which are in both of them. What about the union? We show that if $\widehat{A} \in B \cup C$ then either $\widehat{A} \in B$ or $\widehat{A} \in C$ because $A = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ implying that at least one of them is non empty and equal to A due to atomicity of the latter. Reversely, one has that if $\widehat{A} \in B$ then it is an element of $B \cup C$ because $A \cap (B \cup C) = A \cup (A \cap C)$ which equals $A \cup A$ or $A \cup \emptyset$ due to atomicity of A. In both cases we have that $A \cap (B \cup C) = A$ because $A \cup \emptyset = A = A \cup A$. Therefore, the elements in the union are in correspondence to the elements of one of the sets.

As suggested previously this does not imply that sets are fully specified by their elements nor that elements exist in the first place. For example, assume that $\mathcal{S}$ consists of $\emptyset, \{1\}, \{1, 2\}$, then $\{1\}$ is an atom, but $\{1, 2\}$ does not merely consist out of atoms. Standard set theory makes the assumption that

$$B = \{\widehat{A} | \widehat{A} \in B\}$$

meaning that a set equals a collection of its elements. In this case, we have just proved that $\cap$ and $\cup$ coincide with the usual operations of intersection and union. The reader might think this is all a bit abstract and utter "well, can I just not assume this without all these rules?". The simple answer is "no"; mathematicians are very scarce on their assumptions indeed! Why writing an extra sentence into the constitution when the latter is already a consequence of the former rules?! The next question one could pose then is "well on then, but how do you make up for all these theorems as well as the formal proofs?". The simple answer is that the results have to be in your mind prior to making up the concepts! A proof is no more as a logical confirmation of a kind of naturalistic observation in a way. Henceforth, it is merely an exercise to verify that the concepts lead to the appropriate results. This applies in the case of set theory due to the existence of the natural concept of an atom being equivalent to an element.

These are by far not the only rules of set theory which we shall slowly expand upon by means of more complicated objects and operations. Let us now deviate a bit and reflect further upon the commutation and associative properties of the intersection as well as union. We imagined that a set can be thought of as items in a bag; however, in reality our bag is a phantom bag given that the operations of emptying and resorting do not matter in taking the intersection or union. This would lead to complications involving the order of operations leading to a non-commutative logic which we shall study later on in this book. A true Frenchman would expect such rule to emerge in a way from the simple ones and indeed this is the case. Another field where such a thing happens is Riemannian geometry which is a generalization of flat Euclidean geometry.

We define the natural numbers n by means of the sum operation $n = 1 + 1 +$

$1 + 1 + \ldots + 1$ by means of the following prescription:

$$\begin{aligned} 0 &= \{\emptyset\} \\ n+1 &= \{n, \emptyset\}. \end{aligned}$$

Hence, $1 = \{\{\emptyset\}, \emptyset\}$, $2 = \{\{\{\emptyset\}, \emptyset\}, \emptyset\}$ etcetera; this is a partial dictionary made out the symbols $\emptyset, \{, \}$ which are part of any set theory. I have warned the reader that symbolic notation often is the most difficult part of set theory and the latter notation allows for a definition comprehensible by a computer albeit the latter uses binary representations. We define in the same way $n + m$ by means of the prescription $n + (m + 1) = \{(n + m), \emptyset\}$ where $n + 0 = 0 + n = n$. The reader shows that $n + m = m + n$ for every natural number m which is true by definition for $m = 0$. Indeed, suppose it is true for $m = k$, then we show it holds for $m = k + 1$. Indeed, $n + (k + 1) = \{n + k, \emptyset\} = \{k + n, \emptyset\} = \{k + (1 + (n - 1)), \emptyset\} = \{(k + 1) + (n - 1), \emptyset\} = (k + 1) + n$ where, in the first step, we have used the definition of the natural numbers, in the second the assumption that $k + n = n + k$ and finally, in the third step, the associativity of $+$. We pose that $\mathbb{N}$ is the set of all natural numbers, something which defines a set theory by means of taking all subsets of $\mathbb{N}$.

The operation $+$ maps two natural numbers onto a natural number; it is associative, commutative and has 0 as a neutral element implying that $0+n = n+0 = n$. For any n, it is possible to define an inverse $-n$ satisfying $n + (-n) = 0 = (-n) + n$ something we denote by $n - n = 0$; $n + (-m) = n - m$ is a natural number $n > m$ and minus a natural number if $n < m$. The set of natural numbers taken together with their inverse is called the entire numbers and is universally denoted by $\mathbb{Z}$. $\mathbb{Z}, +$ is called a commutative group given that the operation $+$ is interior, associative, has a neutral element and inverse.

As previously stated, one starts by making a distinction between elements of a set and sets themselves; we departed from the concept of an empty set $\emptyset$, the intersection and union and therefrom we deducted the first three axioms of set theory. The approach taken here is somewhat more general as we defined an element as a primitive set. Zermelo-Frankel set theory has plenty of more assumptions which have to do with infinity culminating into the axiom of choice. A fifth axiom deals with taking set theoretical differences

$$B \setminus C = \{\widehat{A} | \widehat{A} \in B \wedge \widehat{A} \notin C\}$$

and we shall always assume the difference set to exist. In the field of geometry, it is not only possible to take the union of two lines or the intersection theirof but we can also take the so called Cartesian product, defining a two dimensional sheet. More in particular, given two sets B, C, we define the Cartesian product $B \times C$ as the *set* of all tuples (x, y) such that $x \in B$ and $y \in C$ giving a six'th axiom in $\mathcal{S}$ and henceforth is this last one closed with respect to $\times$ from which holds

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

and

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

The existence of Cartesian products allows us to define relations where a *relationship* R between sets B and C constitutes a subset of $B \times C$. In case $B = C$ we can demand plenty of criteria. With the notation xRy we intend to say that x has a relation of type R to y if and only if $(x, y) \in R$; we call R reflexive if xRx for all $x \in B$, symmetric if xRy implies that yRx for all $x, y \in B$ and finally transitive if xRy and yRz imply that xRz. A reflexive, anti-symmetric, transitive relation is called to be a partial order and is noted by $\prec$ or $\leq$. A reflexive, symmetric and transitive relation is called an equivalence relation and is usually denoted by $\equiv$. One should think of an equivalence relation as a generalization of the equality sign given that it concerns objects with similar properties. One should prove that an equivalence relationship defined on a set A pulverizes it in equivalence classes $\overline{x}$ where

$$\overline{x} = \{y \in A | x \equiv y\}.$$

The reader verifies that $\overline{x} = \overline{y}$ if and only if $x \equiv y$ and therefore the intersection $\overline{x} \cap \overline{y} = \emptyset$ if they are not equivalent. A partial order is a generalization of a total order such as "Jon is larger as Elsa". A partial order allows for two objects to be not related at all.

We have defined the natural numbers by means of the operation $+$; $\mathbb{N}$ has a natural *total* order $\leq$ defined by $n \leq n$ and $n \leq n + 1$ and one takes the *transitive closure* therefrom which is defined by imposing transitivity on the existing relationship. This can be compared with lacing a chain. From the natural numbers we constructed the entire numbers $\mathbb{Z}$ and the definition of $\leq$ has a natural extension towards $\mathbb{Z}$. We construct now the rational numbers starting from $\mathbb{Z} \times \mathbb{N}_0$ and imposing the equivalence relationship $(m, n) \equiv (m', n')$ if and only if there exist a $k, l \in \mathbb{N}_0$ such that $km = lm'$, $kn = ln'$ where $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$. De *rational* numbers are henceforth defined as the equivalence classes defined by means of this *equivalence* relation.

The six axioms discussed are by far the most important ones of set theory and allow one to construct the rational numbers; the remaining two axioms concern infinity and are in general added to generalize aspects of the rational numbers to the real ones. We shall be very cautious here with the kind of infinity we shall allow for culminating into a thorough discussion of the axiom of choice. The seven'th axiom allows one to define subsets of sets: given a set D, the power set 2^D of all nontrivial subsets of D is a set and belongs to $\mathcal{S}$. This axiom leads to the construction of the ordinary numbers by Cantor. The definition the Cartesian product is extended to so called "index" sets something which requires a partial order $\prec$. An index set I is a set equipped with a partial order $\prec$ such that for any $x, y \in I$ there exists a $z \in I$ such that $x, y \prec z$. This condition is required and sufficient if we want to take unique limits such any reader should check. If this is not valid, then several sub limits could exist;

hence, we denote by

$$\times_{i \in I} A_i = \{(x_i)_{i \in I} | x_i \in A_i\}$$

where all I-tuples are partially ordered by $\prec$. Finally, we have the so called axiom of choice which can be formulated as follows: given sets A_i, $i \in I$, with I an index set, then the Cartesian product is nonempty. Another, but equivalent formulation is that there exists a function f from I to $\cup_{i \in I} A_i$ such that $f(i) \in A_i$. So, one can constitute a set by drawing an element from each set. This axiom has plenty of ramifications in some parts of mathematics, in particular functional analysis although some mathematicians have refuted it because some results appear too strong and give the transfinite an equal status to the finite situation. I have stated it already a few times: mathematics as such is not open to proof; it is a language and we have to make some grammatical choices. The reader has to reflect about these rules en be conscious of the fact that commutativity, associativity as well as the formation of a power set are the most simple of all symmetrical rules. An example which does not obey these rules has been constructed from this ideal situation; for example, we shall study later on non commutative or associative operations and construct those from the simple commutative situation. This leads to non commutative groups, quantum groups etcetera. This reminds us about the Egyptian architectural art followed by the Roman and French symmetrical ones: super simple, magnificent and logical.

One has to contemplate about topology as a refinement of set theory; it is to say, we limit ourselves to special sets being the so called open sets. In nature, an open set is an abstraction, an imaginary concept which has no real existence. An open surrounding has to be thought of as a voluminous object: for example, a straight line segment is the set of all real numbers between two extremal values denoted by $(a, b) = \{x | a < x < b\}$ with a natural length of $b - a$. A point is an example of a closed set and has vanishing volume or length. We now consider some properties regarding the set theoretical operations on the open segments (a, b): the union of two open segments is declared open by fiat whereas the intersection of two open segments is an open segment anew. Note that the union of open segments can be written as a disjoint union. Given a set D, we call a set $\tau(D)$ of subsets of D a topology if and only if

- $\emptyset, D \in \tau$,
- $A, B \in \tau$ implies that $A \cap B \in \tau$,
- $A_i \in \tau$ implies that $\cup_{i \in I} A_i \in \tau$ for every second countable index set I.

I stress again that this definition depends upon the commutativity as well as associativity of the intersection and union; it is possible to define a non-associative and non commutative topology by means of deformations. We shall study this from the viewpoint of logic further on and the reader may repeat these constructions almost ad-verbatim here. In this chapter, we start pedestrian by studying the classical case, where taking the union can be seen as putting landscape maps together; typically such charts overlap and all we demand is that the intersection of two charts is again a chart and that arbitrary many of them can be put together. There exist special subsets $E \subseteq D$ which are

- *closed* if and only if $E^c := D \setminus E \in \tau(D)$,
- *compact* if and only if for any coverage by means of open sets O_α of E there exists a finite sub-coverage $O_i; i = 1 \ldots n$ such that $E \subseteq \cup_{i=1}^{n} O_i$.

Henceforth, the compact sets are those which can always be covered by means of a finite sub-cover such as for example a globe: irrespectful of how small you make the charts, the globe is covered by a finite number of them. Given a point $p \in D$, we say O is an open environment of p if and only if $p \in O$. Given a point p, a basis of open environments is given by a countable collection of open neighborhoods O_i of p, such that for any open V encompassing p it holds that there exists an index i such that $O_i \subseteq V$. One could moreover demand that $O_{i+1} \subseteq O_i$ by taking intersections but this is not mandatory however. Regarding the closed sets X, Y one has to verify the following truisms: (a) $\emptyset, D$ are closed (b) $X \cup Y$ is closed (c) $\cap_{i \in I} X_i$ is closed if and only if all X_i are as such. Sets such as $\emptyset, D$ which are open and closed at the same time are dubbed cloped. Given $B \subseteq D$, the intersection of all closed sets X encompassing B is closed and called the closure of B which we denote as $\overline{B}$. The closure of a set is therefore the smallest closed set encompassing the latter itself. In other words, one adds elements or points which are limits of elements in B. More concretely, we call x a limit point of a sequence $(x_i)_{i \in I}$ if and only if for every open neighborhood $\mathcal{O}$ of x it holds that there exists an index j such that $\forall j \prec i$ it holds that $x_i \in \mathcal{O}$. Now one shows that, using the properties of an index set, if y were another limit point then the open neighborhoods of x and y coincide. This motivates the following definition: a topology is Hausdorff if and only if all disjunct points x and y have open neighborhoods each with empty mutual intersection. It is to say that $x \in \mathcal{O}, y \in \mathcal{V}$ and $\mathcal{O} \cap \mathcal{V} = \emptyset$. For Hausdorff topologies it holds that the limit point of a sequence is unique. We now prove the following result for topologies with a countable basis: a set is closed if and only if it contains all its limit points. Indeed, suppose that B is closed, and $(x_i)_{i \in I}$ is a sequence in B with limit point $x \in D$, then it holds that $x \in B$ otherwise one can find an open neighborhood B^c of x which is disjoint with $(x_i)_{i \in I}$, something which contradicts the definition of a limit point. Reversely, suppose that any limit point of B belongs to B, then we show that B is closed; suppose it is not, then we find an $x \in \overline{B} \setminus B$ such that for any basis-open neighborhood $\mathcal{O}_n$ of x we find an element $x_n \in B \cap \mathcal{O}_n$ and as such it holds that x is a limit point of $(x_n)_{n \in \mathbb{N}} \in B$ and henceforth, by assumption, an element of B which leads to a logical contradiction. Later on, we give an example of a compact set in a non-Hausdorff topology with a sequence containing no subsequence with a limit point (in case you want to think about this; find an example in an infinite number of dimensions). We shall study further characteristics of compactness in the so called metric topologies, which are determined by a distance function

d.

So far, the treatment of topology appears to be very abstract and not very useful at all, one can think of any topology one wants to and indeed, all subsets of the real number system for example constitute a topology called the discrete topology. Indeed, in this case, all sets are cloped which suggests a huge triviality. The physical reality we live in appears by very close inspection much more peculiar given that we speak about distance functions and spheres such as for example the circle with radius of 10 kilometer around Brussels measured from the Grand Place in bird flight. On earth this procedure only goes wrong when one traverses half of the circumference; one step further in the same direction would replace that journey by a different one where one originally departs in the opposite direction. Therefore, at large distances, one can expect problems of this global nature and in quantum geometry, one suspects those issues can occur at small distances too. Typical scales here are much smaller as those of an atom. By definition, a distance function $d : X \times X \to \mathbb{R}^+$ defined on a set X satisfies

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ for each $x, y \in X$,
- $d(x, z) \leq d(x, y) + d(y, z)$ the so called triangle inequality.

A distance function defines a so-called Hausdorff topology with countable basis by means of the open balls

$$B(x, \epsilon) = \{z | d(x, z) < \epsilon\}$$

giving rise to a countable basis defined by $B(x, \frac{1}{n})$ where $n \in \mathbb{N}_0$. Two points x, y separated by means of a distance $d(x, y) > 2\epsilon$ can be surrounded by means of two disjoint balls $B(x, \epsilon), B(y, \epsilon)$ respectively. This representation of affairs is still a bit abstract given that one wants to measure angles as well contemplate a notion of orthogonality which is not so simple in this formalism. In other words, we require further specialization extending beyond the distance function only. Nevertheless, one can prove plenty of theorems in this primitive language relying solely upon those three axioms. A generalization consists in specifying that the distance function has a local origin; it is to say that the distance between two points can be chopped into arbitrarily small pieces. This leads to the notion of a path metric: d is a path metric if and only if the property holds that for any two points x, y there exists a z such that

$$d(x, z) = d(y, z) = \frac{d(x, y)}{2}.$$

In other words, every two points define at least one midpoint. We shall later on give a better representation of those facts.

We will study now an equivalence relationship between two topological spaces; in other words, when are two topological spaces the same? To determine that, we shall study topological mappings between two topological spaces X, Y. A mapping $f : X \to Y$ is defined by means of a subset F of the Cartesian product $X \times Y$; F obeys the law that for any $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in F$. y is then denoted as $f(x)$ and F is the graph of f. In human language, this signifies that each element chosen from X has precisely one image in Y. Concerning mappings, we formulate still the following extremal properties: (a) f is injective if and only if $f(x) = f(x')$ implies that $x = x'$ or each x has a different image (b) f is surjective if and only if for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$ or, in other words, every potential image is realized effectively. Finally, we say that f is a bijection if and only if it is injective as well as surjective; bijective mappings are equivalences between sets as we shall see now. Suppose $f : X \to Y$ and $g : Y \to Z$ then $g \circ f : X \to Z : x \to g(f(x))$ is the composition of these two mappings. Show that $g \circ f$ is injective if and only if g has this property on $f(X)$ and f obeys this law on X. Show that $g \circ f$ is surjective if and only if g is on $f(X)$; finally, show that $g \circ f$ is a bijection if and only if g is and f is injective. In case $f : X \to Y$ is a bijection, it becomes possible to define a unique inverse $f^{-1} : Y \to X$ by means of

$$f^{-1}(f(x)) = x$$

or $f^{-1} \circ f = \mathrm{id}_X$ where id_X constitutes the identity mapping on X. Derive here from that

$$f \circ f^{-1} = \mathrm{id}_Y$$

using the surjectivity of f. Finally, one shows that f^{-1} also is a bijection; we say henceforth that X and Y are equivalent if and only if there exists a bijection from X onto Y. Using the previous properties, one shows that this relation is reflexive, symmetric and transitive. Now, we are in position to define topological equivalences $f : X \to Y$; f is continuous if and only if the inverse of each open set O in Y, denoted by $f^{-1}(O)$, is open in X. For a continuous bijection, one has that f^{-1} is continuous if and only if $f(V)$ open is in Y for any open V. In case a function f satisfies this property, we call it an open mapping. An example of a continuous bijection for which the inverse is not continuous, is given by $f : (-1, 1) \to (-1, 0] \times \mathbb{Z}_2 : x \to (-|x|, \theta(x))$ where $|x| = -x$ if $x < 0$ and x if $x \geq 0$. $\theta(x) = 0$ for $x \leq 0$ and 1 otherwise; finally, $\mathbb{Z}_2 = \{0, 1\}$. The topology defined on $(-1, 0] \times \mathbb{Z}_2$ is the natural one of $(-1, 0]$ and is henceforth not Hausdorff on $\{0, 1\}$. One has that $f((-1, 0)) = (-1, 0) \times \{0\}$ is not open whereas $(-1, 0) \times \mathbb{Z}_2$ is. A topological equivalence is given by means of a bijection f which is continuous and open. Such mappings are called homeomorphisms and the reader verifies that this definition obeys all requirements of an equivalence relationship indeed.

We return to our study of metric topologies and in particular alternative characterizations of compactness. A sequence $(x_i)_{i\in I}$ is called Cauchy if and only if for each $\epsilon > 0$, there exists an i, such that for all $i \prec j, k$ one has that $d(x_j, x_k) < \epsilon$. In human language, this reads: if one proceeds sufficiently far in the sequence then the points reside arbitrarily close together. Such a property suggests the existence of a unique limit point x; a metrical space (X, d) for which any Cauchy sequence has a limit point is called complete. In case K is a compact set, then one shows that any sequence $(x_i)_{i\in I}$ has a subsequence with a limit point in K. The proof is simple, consider arbitrary finite (due to compactness) covers with balls of radius $\frac{1}{n}$; then one finds a sequence of balls $B(y_n, \frac{1}{n})$ such that finite intersections $\cap_{n=1}^{m} B(y_n, \frac{1}{n})$ contain an infinite number of $x_i \in K$. This defines a subsequence with as limit point

$$x = \cap_{n=1}^{\infty} B(y_n, \frac{1}{n})$$

in K. Reversely, suppose that any sequence in K has a Cauchy subsequence with a limit point in K, then K is compact. Choose a cover of K of open balls - without limitation of validity- $B(y_n, \epsilon_n)$ where $n \in \mathbb{N}$ and suppose that no finite sub-cover exists. Define then $B_m = \cup_{n=1}^{m} B(y_n, \epsilon_n)$, we henceforth arrive at the conclusion that for any m there exists an $m' > m$ such that $B_{m'} \cap \overline{B_m}^c \cap K \neq \emptyset$. In particular, we construct a sequence (x_m) with the property that for any m there is an $m' > m$ such that $x_{m'} \in \overline{B_m}^c$ for $k \geq m'$. This sequence cannot contain a Cauchy subsequence with some limit point x because $x \in B_m$ for m sufficiently large which is a contradiction. We just proved that a set is compact in a metric topology if and only if any sequence contains a Cauchy subsequence with limit point in K. Prove the following properties:

- define on $\mathbb{R}$ the function $d(x, y) = |y - x|$, show that this defines a metric (easy exercise),
- prove that in the metric topology on $\mathbb{R}$, the closed interval $[a, b]$ is compact (hint: use the decimal representation of real numbers) (difficult),
- suppose two topological sets X, Y, then the product topology $\tau(X \times Y)$ is the smallest topology containing $\tau(X) \times \tau(Y)$, where the last contains elements $U \times V$ with $U \in \tau(X)$ and $V \in \tau(V)$,
- show that the Cartesian product $K_1 \times K_2$ of two compact sets is compact in the product topology (average),
- a metrical space (X, d) is bounded if and only if there exists an $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in X$; show that a compact space is closed and bounded (easy).

Again, the reader might utter that this kind of considerations are far too general and that our world is much more detailed in the sense that light rays bend and twist around one and another and that this behavior is geometrical and continuous in nature. To describe these features in detail, one needs the notion of a local scalar product giving further rise to analytical geometry. Note the following: suppose that $\gamma : [a, b] \to X$ is a continuous curve joining x and y and define the length functional $L(\gamma)$ of γ where

$$L(\gamma) = \sup_{a=t_0<t_1<t_2<\ldots<t_{n+1}=b} \sum_{k=0}^{n} d(\gamma(t_k), \gamma(t_{k+1}))$$

and sup means taking the supremum of this sum over all finite partitions $a = t_0 < t_1 < t_2 < \ldots < t_{n+1} = b$ of the closed interval $[a, b]$. The supremum of a set of real numbers A is the smallest number larger or equal as any number $x \in A$. The supremum is also called the upper bound and the reader shows that by definition the supremum always exists and is unique by means of addition of the number $+\infty$. Likewise, one defines the infimum or under bound and one shows again it exists and is unique. Concerning the sum, one notices that breaking up an interval $[t_k, t_{k+1}]$ into two disjoint pieces by means of addition of an intermediate point $t_k < t_{k+\frac{1}{2}} < t_{k+1}$ the sum increases by means of the triangle inequality. Henceforth, splitting up an interval $[a, b]$ leads to a higher sum by means of the triangle inequality.

Now, we will formulate our main result; a complete metric space (X, d) defines a path metric d if and only if

$$d(x, y) = \min_{\gamma:[a,b]\to X, \gamma(a)=x, \gamma(b)=y} L(\gamma).$$

In other words, when the distance between two points equals the minimal length of a curve joining x to y we speak about a path metric space. The reader is advised to show this by means of using the midpoint property in order to construct such curve using that $L(\gamma) \geq d(x, y)$. Reversely, in case such a curve exists, one automatically finds a midpoint. A curve minimizing length is called a geodesic and in a path metric space, the length of a geodesic equals the distance between two points. Later on, we shall arrive at a more detailed characterization of geodesics when imposing more structure. Again, those primitive notions allow one to obtain a substantial amount of results some of which have been obtained by Mikhail Gromov and Peter Anderson. Studying those primitive metric spaces further on requires consultation of their work.

As one notices, our language is not rich enough to speak about notions such as perpendicularity, angles etcetera. One gradually learns that this book will become more and more specific, that the language gets more rich and complex allowing for stronger connections and results. Compactness or local compactness is an important notion because the (local) topology is finite in a way. Spaces which are not locally compact often do not allow for certain mathematical structures to exist because there is too much “room” or space such as is the case for integrals. We now arrive at very special building blocks: line segments, triangles and pyramids as well as higher dimensional generalizations thereof. We shall use those to describe certain topological spaces and characterize them: a central element herein is the concept of homology which leads to further categorical abstractions.

Whereas the previous topics were very abstract, we shall now continue to work with more tangible objects, things we know from everyday life. We shall use abstraction of these objects to deal with them in a more appropriate way. This has its advantages because it allows us to calculate with them; this actually is the main miracle of abstraction, that it allows us to do things. The topological spaces to be studied here are those which are modelled by means of the n-dimensional real space

$$\mathbb{R}^n = \times_{i=1}^n \mathbb{R} = \{(x_i)_{i=1}^n | x_i \in \mathbb{R}\}$$

which is the set of n-tuples of real numbers equipped with the product metrical topology of $\mathbb{R}$. One can extend the notion of a sum by means of the definition

$$(x_i) + (y_i) = (x_i + y_i)$$

and likewise can one define the scalar multiplication of a real number with an n-tuple vector by means of

$$r.(x_i) = (rx_i).$$

More in general, let R be a field and $G, +$ a commutative group, then we say that G is an R module in case there exists a scalar multiplication such that

$$1.g = g; \ (rs).g = r.(s.g); \ (r + s).g = r.g + s.g; \ r.(g_1 + g_2) = r.g_1 + r.g_2$$

for all $r, s \in R$ and $g, g_1, g_2 \in G$. In case $R = \mathbb{R}$ we call the module a real vector space. In $\mathbb{R}^n, +$, we have special vectors e_i, defined by the number 1 on the i'th digit and zero elsewhere; herefore, it holds that

$$\sum_{i=1}^{n} r_i.e_i = 0$$

if and only if it holds that all $r_i = 0$ and moreover all vectors can be written uniquely as

$$\sum_{i=1}^{n} r_i.e_i.$$

In case these properties hold for a set of vectors $\{v_i | i = 1 \ldots m\}$, then we call $\{v_i | i = 1 \ldots m\}$ a basis. One notices that we have used two integer numbers here, n for the e_i and m for all v_j; it is now a piece of cake to show that $n = m$. The reason is the following, because e_i is a basis, one can write the v_j uniquely as

$$v_j = \sum_{i=1}^{n} v_j^i e_i$$

and reversely

$$e_i = \sum_{j=1}^{m} e_i^j v_j.$$

Henceforth,

$$\sum_{i=1}^{n} v_j^i e_i^k = \delta_j^k;\; j, k : 1 \ldots m$$

and

$$\sum_{j=1}^{m} e_i^j v_j^l = \delta_i^l;\; i, l : 1 \ldots n$$

where $\delta_j^k = 1$ if and only if $j = k$ and zero otherwise. This system of equations is symmetrical in e and v and therefore $m = n$ given that both mappings are injective. Henceforth, n is a basis invariant and called the dimension of $\mathbb{R}^n, +$. Now, we have a sufficient grasp upon real vector spaces and we proceed by defining special building blocks mandatory for the construction of simplicial manifolds.

What follows is a generalization of simple cutting and pasting of higher dimensional triangles and pyramids. We may construct so called Euclidean bodies in this way and the old fashioned approach towards a classification of topological spaces upon a homeomorphism has been made as such. However, different lines of argumentation which are less constructivist can lead towards such classification too. Consider the space $\mathbb{R}^{n+1}$ and consider a basis $v_i; i = 0 \ldots n$, then the n simplex $(v_0 v_1 \ldots v_n)$ is defined by means of the closed space

$$(v_0 v_1 \ldots v_n) = \{\sum_{i=0}^{n} \lambda_i v_i | \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1\}.$$

This is all a bit abstract and in order to get a picture of how such space looks like, one imagines the $0, 1, 2, 3$ dimensional cases. A zero dimensional simplex (v_0) is simply a point, a one dimensional simplex is given by the line segment $(v_0 v_1)$ which may be embedded into the plane $\mathbb{R}^2$. A two dimensional simplex $(v_0 v_1 v_2)$ is given by a triangle which can be embedded into $\mathbb{R}^2$ whereas finally $(v_0 v_1 v_2 v_3)$ describes a pyramid in $\mathbb{R}^3$. In general, the simplex $(v_0 v_1 \ldots v_n)$ is a convex space meaning that the line segment between two points $x, y \in (v_0 v_1 \ldots v_n)$ completely belongs to $(v_0 v_1 \ldots v_n)$. The line segment between two points x, y is the set

$$\{\lambda x + (1 - \lambda) y | 0 \leq \lambda \leq 1\}.$$

Points of the simplex which do not belong to the interior of a line segment belonging entirely to the simplex are called extremal. Show by means of exercise that the only extremal points of $(v_0 v_1 \ldots v_n)$ are given by v_i. One calls the simplex the convex hull of the extremal points $\{v_i | i = 0 \ldots n\}$. We know now how a module is defined as well as a simplex which allows us for the definition of a linear operator. A mapping $A : V \to W$ between two R modules V, W is linear if and only if

$$A(rv_1 + sv_2) = rA(v_1) + sA(v_2)$$

for all $r, s \in R$ and $v_i \in V$. Show that A is injective if and only if $A(v) = 0$ implies that $v = 0$. Now, we shall work a bit more abstractly: we do not need at this point the property that $v_i \in \mathbb{R}^{n+1}$, something which was required for matters of representation. We shall temporarily proceed by insisting that the v_i, w_j are merely points which are not necessarily associated to vectors in some linear space. Note that a simplex $(v_0 v_1 \ldots v_n)$ naturally possesses an orientation defined by the order in which the vertices appear and that swapping two vertices reverses the orientation, meaning for example

$$(v_0, v_1) = -(v_1, v_0).$$

An n dimensional simplcial complex is defined as a collection of n distinct simplices such that any sub-simplex also belongs to it. We shall be interested in taking forml sums of simplices of the same dimension $k \leq n$; ab initio, you might want to impose certain constraints such as (a) no branching meaning that no more as two k dimensional simplices share the same $k-1$ dimensional sub-simplex. Also (b) you might want for every k simplex to appear exactly once into such a sum so that we can think of it as being single valued. Also, you might insist upon it being (c) oriented which in its most general sense would mean that the contribution of internal $k-1$ dimensional sub-simplices vanishes. This means that, upon taking a formal sum

$$\sum_i a_i(v_1^i, \ldots, v_k^i)$$

where all $(v_1^i, \ldots, v_k^i)$ are different, we have that in case

$$\sum_w \partial_w \partial_{w_1} \ldots \partial_{w_{k-1}} (v_1^l, \ldots, v_k^l) = \pm \sum_w \partial_w \partial_{w_1} \ldots \partial_{w_{k-1}} (v_1^s, \ldots, v_k^s) \neq 0$$

for at least two values $s \neq l$, then $\sum_w \partial_w \partial_{w_1} \ldots \partial_{w_{k-1}} \sum_i a_i(v_1^i, \ldots, v_k^i) = 0$ where ∂_v is the linear operator attached to any vertex v defined by $\partial_v(vv_0 \ldots v_i) = (v_0 \ldots v_i)$ in case none of the v_j equals v and zero otherwise, here it is assumed[1] that $() = 1$. This, taken together with condition (a) simply means that if precisely two k dimensional simplices share the same $k-1$ dimensional sub-simplex then the induced orientations differ. Let us for now keep things in the middle and see if those concerns really matter. We define the boundary operator $\partial_n : Z_n \to Z_{n-1}$ as the linear operator over $\mathbb{Z}$ mapping a simplex $(v_0 v_1 \ldots v_n)$ to

$$\partial_k(v_0 v_1 \ldots v_k) = \sum_{i=0}^{k} (-1)^i (v_0 \ldots v_{i-1} v_{i+1} \ldots v_k) = \sum_w \partial_w (v_0 v_1 \ldots v_k).$$

[1]Note that we deviate here slightly with the convention in the literature where $() = 0$. This will result in a zero'th homology froup with one generator less; it is worthwile keeping this in mind when discussing the definition of the Betti numbers.

One verifies that $\partial_{k-1}\partial_k S_k = 0$ given any sum of simplices. From the definition, it follows that the boundary of any linear combination of simplices is oriented in the previous sense. Also, by the same virtue, any closed sum S_k of simplices, meaning $\partial_k S_k = 0$, is oriented since $\sum_w \partial_w \partial_{w_1} \ldots \partial_{w_{k-1}} S_k = (-1)^{k-1} \partial_{w_1} \ldots \partial_{w_{k-1}} \left(\sum_w \partial_w S_k \right) = 0$ given that $\partial_v \partial_w = -\partial_w \partial_v$. Now, one may wonder whether any closed S_n can be written as a linear combination of closed simplicial complexes satisfying (a) and (b). We shall first prove that this is the case for $n = 1$; take any one dimensional complex $S_1 = \frac{1}{2} \sum_{ij} a^{ij} (v_i v_j)$ where $a_{ij} = -a_{ji}$ where a factor $\frac{1}{2}$ has been included because each simplex is summed over twice. Assume now that the simplex is closed meaning that $\sum_i a_{ij} = 0$ and choose the smallest positive a_{ij}. Then at the vertex j, one certainly has some k such that $a_{jk} \geq a_{ij}$, proceed towards k and substract a_{ij} from a_{jk}. One repeats this procedure a sufficient number of times until the curve comes back to itself defining a_{ij} times a canonical loop obeying a and b. Now, the remainder contains at least one edge less and is also closed; hence, upon repetition of this procedure we arrive at our result. Now, a one dimensional simplicial omplex is rather trivial as each such structure can be consistently oriented. This is no longer true in two dimensions and we shall generalize here the construction of a Mobius strip in order to provide for a counterexample. I will not provide all details but the reader will see how it works. Take an oriented square with four corner boundary points in order (according to the orientation of the boundary) given by $1, 2, 3, 4$ and identify the line segments 12 with 34 and 23 with 41, then the reader notices that, given a representation in terms of a simplicial complex, all the interior lines are cancelled when taking the boundary of this simplicial complex, but the "boundary lines" between 12 and 23 are doubled. So, this sum is not closed; to compensate for this, take a second identical construction but now with opposite orientation on the "boundary lines" and glue them together. Then, the boundary of this doubled complex vanishes but there is no way to undo the bifurcation at the lines 12 and 23 which are now adjacent to four half planes instead of two. It appears that we have to live with such "anomalies" as there is no way to exclude them, therefore we consider all formal linear combinations of k dimensional simplices. To repeat, we consider a formal linear combination T_k of k simplices closed if and only if $\partial_k T_k = 0$ and exact if and only if $T_k = \partial_{k+1} S_{k+1}$ for some S_{k+1}. It is clear that exact simplicial complexes are closed using the crucial property of a boundary operator and we define accordingly the $\mathbb{Z}$ modules $C_k(S_n)$ of all closed k sums and $E_k(S_n)$ of all exact k sums, where $E_n(S_n) = \{0\}$ and $C_0(S_n)$ equals $\mathbb{Z}^{V-1}$ with V the number of points or vertices in S_n. Clearly it holds that $E_k(S_n) \subseteq C_k(S_n)$ and we define the homology classes $H_k(S_n)$ as the quotient module

$$H_k(S_n) = \frac{C_k(S_n)}{E_k(S_n)}$$

being the $\mathbb{Z}$ module of $E_k(S_n)$ equivalence classes in $C_k(S_n)$. We say that two closed sums T_k, Y_k are equivalent if and only if $T_k - Y_k \in E_k(S_n)$. So far for the general theory of simplical complexes, we now arrive to the very important sub theory of topological spaces A homeomorphic to a simplicial complex S_n; the important step herein consists in proving that $H_k(A)$ is well defined because homeomorphic simplicial sums define the same homology module. The reader may try to show this fact by him or herself as a kind of difficult exercise but it it clear that the statement is rather obvious. Indeed, the boundary operator is defined independently of the simplicial decomposition. The dimension of $H_k(S_n)$ plus one, in case $k = 0$, is called the k-th Betti number b_k of the simplicial complex S_n. The reader now makes the following exercises: take a two dimensional spherical surface and show that $b_2 = 1, b_1 = 0, b_0 = 1$. The two torus T_2 is defined by taking an oriented square and glue opposite sides to one and another; show that $b_2 = 1, b_1 = 2, b_0 = 1$. In general, one defines the Euler number of a two dimensional simplicial complex S_2 as

$$\chi(S_2) = D - L + V$$

where D is the number of triangles and L the number of line segments. One can show that the Euler number is a topological invariant; calculate that the Euler number of a two sphere is given by $2 = b_2 - b_0 + b_1 = 1 - 0 + 1$ and that of a torus by $0 = 1 - 2 + 1$. In general, one shows that

$$\chi(S_n) := \sum_{i=0}^{n} (-1)^i V_{n-i} = \sum_{i=0}^{n} (-1)^i b_{n-i}$$

where V_i equals the number of i dimensional sub-simplices. To start with the calculation of the dimension of a homology class, note that an element of $H_k(S_n)$ corresponds to a closed k dimensional connected surface which cannot be contracted to a point. Concerning the calculation of b_1 on the two sphere, it is clear that any closed curve can be reduced to a point whereas on the two torus two fundamental circles do existwhich are not the boundary of a two dimensional simplicial complex. Consider two closed surfaces A_2 and B_2 and remove a two disk from both of them; now, paste each of the remainders along the circular boundaries resulting in a new closed surface denoted by $A_2 \diamond B_2$. Show that the operation $\diamond$ is associative as well as commutative with as identity element the two dimensional surface S^2. Calculate that the Euler number of the n-fold crossproduct of T_2 equals $2 - 2n$; more in particular, it holds that

$$\chi(A_2 \diamond B_2) = \chi(A_2) + \chi(B_2) - 2.$$

Later on, we shall study the notion of a manifold and one of the most important results is that any closed, compact, connected and oriented two dimensional topological space is homeomorphic to S^2 or an n-fold product $T_2 \diamond T_2 \diamond \ldots \diamond T_2$. This formula can be generalized towards any dimension, where the connected sum is then defined by means of cutting out the interior of a ball and identifying the boundaries; the reader verifies that in general

$$\chi(A_n \diamond B_n) = \chi(A_n) + \chi(B_n) - 2$$

for n even and

$$\chi(A_n \diamond B_n) = \chi(A_n) + \chi(B_n)$$

for n odd. This implies that closed, compact, connected as well as orientable two dimensional manifolds are completely characterized topologically by means of the Euler number. For closed manifolds, one shows that $b_{n-i} = b_i$ something which is called Betti duality, a result which may be proved by definition of a duality operator $\star$ on the simplicial complexes such that $S_n^\star$ is homeomorphic to S_n and $H_k(S_n)$ is mapped bijectively to $H_{n-k}(S_n^\star)$. One can imagine $\star$ as a natural generalization of the following operation on a one dimensional simplicial complex S_1: it maps every line segment r to a point $r^\star$ and each point p to a line segment $p^\star$ such that $\star$ interchanges the operation $\subseteq$ meaning $r^\star \subseteq p^\star$ if and only if $p \subseteq r$. $S_1^\star$ is a closed simplicial complex if and only if S_1 is in case no branching occurs; the Euler number changes in case S_1 shaws branching as the reader verifies. Henceforth, the manifold condition is mandatory and Betti duality does not hold for general closed simplicial complexes. The reader should prove that two circles having a common point show bad behavior under the duality transformation. The notion of a variety is henceforth really special and our result, that closed two dimensional and oriented varieties are classified by the Euler number only does not hold in higher dimensions. Here ends our discussion of simplicial homology which can be summarized by a chain of operations $\partial_k : Z_k(S_n) \to Z_{k-1}(S_n)$ met $\partial_0 : Z_0(S_n) \to \mathbb{Z}$ and[2] $\partial_{k+1}\partial_k = 0$. Such a structure is called a chain and those objects enjoy plenty of beautiful characteristics which are much more primitive as the topological point of departure. An initial point for higher mathematics therefore!

It is clear, from the simplicial point of view, that topological spaces of dimension n cannot be classified by means of the Betti numbers. The reader is invited to show this by means of braiding three closed surfaces in different ways. Later on, we shall study the Euler number from the viewpoint of vectorfields, akin Morse theory, as well as closed differential forms determined by the homology classes.

Exercise: the Poincaré conjecture.
The conjecture of Poincaré is that every 3 dimensional compact, closed topological space $\mathcal{M}$ which is path connected and has trivial first homotopy class, is homeomorphic to the 3 dimensional sphere. Note that I am speaking of homotopy instead of homology which is another and much crazier way of constructing topological invariants; the reader is encouraged to wade through the literature on homotopy which is full of rich results (for example, the homotopy groups are also labelled by a discrete index, referring to dimension, but they may well be nontrivial beyond the dimension of the embedding space). The first homotopy group consists out of all equivalence classes of continuous closed curves with a base point which cannot be continuously deformed into one and another while keeping the base point fixed. Obviously, for the torus, the homology group is $\mathbb{Z} \oplus \mathbb{Z}$ which basically means that you consider linear combinations over $\mathbb{Z}$ of two independent generators, whereas the first homotopy group is the same given that one must show that the two generator commute. However, this is no longer

[2] In the literature, this is zero given that there one takes another definition of ∂_0.

true upon considering the torus and cutting out a small disc. The boundary of the latter is obviously the boundary of the complement of the disc (and is therefore trivially zero in the homology group) but it cannot be deformed into any existing element in the homotopy theory. Nevertheless one may trivially prove that if the first homotopy group vanishes, then the first homology croup must vanish too; the reverse is not true however. Note that this theorem does not hold in higher dimensions as for example $S^2 \times S^2$ provides for a counterexample. The conjecture has been proved by Perelman some years ago and the proof heavily relies upon techniques from differential geometry, something which we shall study further on in this book. Taking the classification of closed, compact and connected two dimensional topological spaces we have just given, I once constructed the following simple argument. As of today, I do not know where my error resides and I encourage the reader to think about it.

- Show that $\mathcal{M}$ allows for a path metric d.
- Consider an arbitrary point p and show that for sufficiently small r, the surface $L_r := \{x|d(p,x) = r\}$ is homeomorphic to the 2 dimensional sphere S^2.
- Show that there exists a critical point r_0 such that L_{r_0} is no longer a sphere.
- In case L_{r_0} is a point, the theorem is proved; otherwise we have a compact 2 dimensional topological space obtained from the sphere by means of identification of k dimensional subspaces where k can range from 0 to 2.
- Show that the subsequent connected components of the topological space for $r > r_0$ are again two dimensional connected, closed spaces[3] which can only close up to a point in a three dimensional closed space in case they are homeomorphic to the sphere S^2.
- Subsequently, to close the topological space, all components different from some S^2 and possibly the S^2 themselves must be pasted together leading to a nontrivial first homotopy class which is forbidden.
- Consequently $\mathcal{M}$ is a 3 dimensional sphere.

Simplicial gravitation.
Simplicial metric spaces are very simple and entirely characterized by means of distances $d(v_0v_1)$ defined on the line segments (v_0v_1). One defines the following operators: $x_w(v_0 \ldots v_i) = (wv_0 \ldots v_i)$ and $\partial_w(wv_0 \ldots v_i) = (v_0 \ldots v_i)$ in case

[3]This seems to be the crucial step! It is certainly true for the theorem in two instead of three dimensions where a circle possibly biffurcates into two circles which, in case they rejoin, gives rise to a nontrivial homotopy. The reader may convince himself of that by studying the example of a torus versus a long “saussage”. In both cases, we have that for generic points x the circles of radius r around x identify at some points but split later again into two distinct circles which in case of the torus rejoin and in case of the saussage individually collapse to a point.

none of the v_j equals w. The remaining cases where this last condition is violated lead to the null simplex with as boundary conditions $\partial_w(w) = \mathbf{1}$, $x_w\mathbf{1} = (w)$ where $\mathbf{1} = ()$ is the empty simplex. From this, it follows that $(x_w)^2 = 0$ as well as $(\partial_w)^2 = 0$. One verifies that the operator $\partial = \sum_{w \in S} \partial_w$ is the usual boundary operator what shows that ∂_w constitutes the appropriate derivative operator defined by means of the boundary operator ∂. The empty simplex constitutes the neutral element regarding the cross product $*$ defined by means of

$$(v_0 \ldots v_i) * (w_0 \ldots w_j) = (v_0 \ldots v_i w_0 \ldots w_j).$$

One simply verifies that $x_w x_v = -x_v x_w$ and likewise for the operators ∂_v, ∂_w. Henceforth, the creation operators associated to a vertex generate a Grassmann algebra; moreover, it holds on the vector space of simplices that

$$\partial_v x_w + x_w \partial_v = \delta(v, w)$$

such that the ∂_v represent Grassmann annihilation operators. Bosonic line segment operators are consequently defined by means of

$$\partial_{(vw)} = \partial_w \partial_v$$

and such operators satisfy

$$\partial_{(vw)}(yz) = \delta(v, y)\delta(w, z) - \delta(v, z)\delta(w, y)$$

giving rise to an oriented derivative. The simplex algebra is henceforth defined by means of polynomials spanned by monomials which are formal products of simplices $(v_0 \ldots v_j)$ for all $j : 0 \ldots n$. Mind that this formal product does not equal the crossproduct implying that $\mathbf{1}$ does not constitute the neutral element. Given that on general spaces bi relations carry an evaluation by means of the metric d it is natural to limit the function algebra to two simplices $(v_0 v_1)$ given that other simplices do not procure for independent variables. The bosonic character of $\mathbf{1}$ implies that the ∂_v, x_w constitute Fermionic Leibniz operators on the function algebra. Indeed, one has that

$$\partial_v((w)Q) = \partial_v((x_w\mathbf{1})Q) = \partial_v x_w(\mathbf{1}Q) - \partial_v(\mathbf{1}x_w Q) =$$

$$(k+1)\delta(v, w)\mathbf{1}Q - x_w(\mathbf{1}\partial_v Q) - \partial_v(\mathbf{1}x_w Q)$$

which reduces to

$$(k+1)\delta(v, w)\mathbf{1}Q - (x_w)\partial_v Q - \mathbf{1}x_w\partial_v Q - \mathbf{1}\partial_v x_w Q = \delta(v, w)\mathbf{1}Q - (x_w)\partial_v Q$$

where k denotes the degree of the monomial Q given by the number of factors. This follows immediately from the Leibniz rule given that the operator

$$x_w \partial_v + \partial_v x_w = \delta(v, w)$$

is bosonic. Henceforth, the even simplex variables behave bosonically whereas the odd ones fermionic. Indeed,

$$\partial_v((wz)Q) = \partial_v((x_w(z))Q) = \partial_v(x_w((z)Q) + ((z)x_w Q)) = -x_w\partial_v((z)Q) - (z)(\partial_v x_w Q)$$

which reduces to

$$= x_w((z)\partial_v Q) - (z)(\partial_v x_w Q) = (wz)\partial_v Q.$$

Given that the usual derivatives of a function are defined by means of the infinitesimal intervals $(x - |\epsilon|, x + |\epsilon|)$ where $f(v + \epsilon, v - \epsilon)$ gets identified with the coordinate function $f(x)$. This is logical given that the $v \pm \epsilon$ are fermionic and independent such that the intervals $(v - \epsilon, v + \epsilon) \sim x$ are bosonic. Note that products of the form $(v - \epsilon)(v + \epsilon)$ can be further derived such that

$$\partial_x f(x) = \mathbf{L}\left[\partial_{(v-\epsilon, v+\epsilon)} f(v - \epsilon, v + \epsilon)\right]$$

where $\mathbf{L}$ merely retains the monomials depending exclusively of the line segments. This phenomenon clearly occurs in $(vw)^2$ whose (vw) derivative equals

$$2(vw) - 2(v)(w).$$

To obtain the standard commutation-relations on the function algebra generated by (vw) we define

$$\widehat{x}_{(vw)} Q := x_{(vw)} x_{\mathbf{1}} Q$$

where Q is a polynomial defined on the edges (r, s) and $x_{(vw)}$ is a bosonic Leibniz operator defined by

$$x_{(vw)}(v_0 \ldots v_j) = (vwv_0 \ldots v_j).$$

By definition, one has that

$$x_{(vw)}(rs) = 0$$

if and only if r or s equals v, w and moreover

$$(x_{(vw)} + x_{(rs)})((vw) + (rs)) = 2(vwrs)$$

which vanishes unless (r, s) is the opposite side of a pyramid which we shall forbid from now on. In particular, this does not apply to geodesics

$$\gamma(v_0 v_i) := (v_0 v_1) + (v_1 v_2) + \ldots (v_{i-1} v_i)$$

which satisfy

$$x_{\gamma(v_0 v_i)} := \sum_{j=1}^{i} x_{(v_{j-1} v_j)}$$

and therefore

$$x_{\gamma(v_0 v_i)} \gamma(v_0, v_i) = 0.$$

Next, we define the derivatives

$$\partial_{\gamma(v_0, v_i)} := \sum_{j=1}^{i} \partial_{(v_{j-1} v_j)}$$

and consider the operator

$$\widehat{\partial}_{\gamma(v_0,v_i)} = \mathbf{L} \circ \partial_{\gamma(v_0,v_i)}$$

and one calculates that

$$\widehat{\partial}_{\gamma(v_0,v_i)} \widehat{x}_{\gamma(v_0,v_i)} - \widehat{x}_{\gamma(v_0,v_i)} \widehat{\partial}_{\gamma(v_0,v_i)} = 1$$

on the function algebra generated by the monomials Q of the form $(\gamma(v_0, v_i))^k$ where $k > 0$. We have now a tool to do physics; in particular,generated by the monomials Q of the form $(\gamma(v_0, v_i))^k$ where $k > 0$. We have now a tool to do physics; in particular,

$$\mathbf{E}P(\gamma(v_0, v_i)) = P(\sum_{j=1}^{i} d(v_{j-1} v_j))$$

is the evaluation function. The reader is invited to expand this theory further as well as to implement the Fourier transformation from chapter fourteen on conic tangent spaces. Hint: integrate in "hyperbolic" or "spherical" coordinates by replacing the $n-1$ sphere with the level surface $H^{n-1}(\epsilon, v_0) = \{x | d(v_0, x) = \epsilon\}$ for ϵ sufficiently small such that $H^{n-1}(\epsilon, v_0)$ belongs to the star neighborhood of v_0. See chapter thirteen for more information.

This is all we have to say basically at the classical level regarding topology for general enough spaces; further specialization is obviously always possible and can lead to very rich results such as is the case for the de Rahm theorem connecting the exterior derivative to the boundary operator by means of the Hodge theorem. Another step away then consists in abstraction of this duality from the point of category theory and in particular long left and right exact sequences attached to the exterior derivative and boundary operator respectively. In my opinion, this topic is too specialist to be treated here and philosophically, the very gist of classicality has been treated and resides in the axioms of Boolean logic or the algebraic structure generated by $\wedge, \cap$ and $\times$. This author has recently suggested an interesting extension of this formalism by extending those operations to semi-group ones where the semi reflects the fact that the inverse is not necessarily unique. That is, given a set A, an anti-set obeys

$$A \times A^{\times} = \{1\}$$

where the last one is a set with one element 1 and henceforth serves as the identity element for $\times$. To represent an anti-set in the set-like fashion; denote that if $A = \{x | x \in A\}$ and $A^{\times} = \{\omega_A^{\star}\}$ where $\omega_A : A \to \{1\}$ is the constant mapping onto 1 and $\star$ is the associated duality relation, then

$$A \times \{\omega_A^{\star}\} = \omega_A(A) = \{1\}.$$

So, taking inverses regarding the Cartesian product naturally leads to a notion of duality which may be interpreted as an anti-event. Likewise, we can demand

$$A \cup A^{\cup} = \{\emptyset\}$$

as well as

$$A \cap A^{\cap} = \Omega$$

where Ω is supposed to be a maximal set. $A^{\cup}$ can be seen to exist out of the negative events $-x$ where $x \in A$. The latter induces the following natural rules

$$-(A \cup B) = (-A) \cup (-B), \; -(A \cap B) = (-A) \cap (-B)$$

and henceforth it is a Boolean algebra isomorphism. In logic, the notion of a negative primitive sentence has no obvious philosophical meaning whereas this is the case for the negative of an event as one absorbing the other. The Boolean $\neg$ operation satisfies

$$A \cap \neg A = 0, \; A \cup \neg A = 1, \; \neg(A \cup B) = \neg A \cap \neg B$$

and has as set theoretical counterpart the complementation operation. However, Boolean logic does possess another operation called xor instead of or which does allow for these things to happen; A xor B is true if and only if exactly one of them is true and the other is false. In set theory, the equivalent is given by the disjoint union

$$A \sqcup B = (A \cup B) \setminus (A \cap B)$$

and in such a case $A^{\sqcup} = A$. However, such a thing is rather mundaine and $-x$ should really be seen as eating x meaning

$$\{x, -x\} = \{\emptyset\}$$

which is a serious departure from the negative of integer numbers given that the equivalence $\{5, -5\} = \{\emptyset\}$ does not exist. This brings along some subtleties with the complementation operation A^c given that $(-\Omega) \cup \Omega = \{\emptyset\}$ and henceforth $\{\emptyset\}^c = \{\emptyset\}$ in the enlarged setting of negative events breaking hereby the equivalence with the logical operator $\neg$. This is logical from a philosophical point of view given that $-$ is associated to death and therefore pressuposes creation whereas $\neg$ pertains to an eternal truism. This gives problematic aspects regarding the intersection operation

$$A \cap \{\emptyset\} = B$$

where, in last instance, B is any subset of A. This is obviously not desirable and is resolved by insisting that

$$(-A) \cap B = -(A \cap B)$$

where A, B are ordinary sets. This implies that the notion of element becomes superfluous given that

$$\{x\} \cap \{-x\} = \{-x\}$$

which is a situation intermediate between the classical and quantum; there is no contradiction with our previous treatment of set theory given that $\{x\}$ is no longer a primitive set but $\{-x\}$ is (so x is no longer an element). So, even if we start out with a standard set theory, adding all anti sets to it leads to a new theory where the elements of the old sets are no longer elements in the full set theory but their anti elements are. It is natural to posit that the boundary operator ∂ commutes with $-$ meaning $\partial(-A) = -(\partial A)$ and obviously it holds as well that

$$\partial A^c = -\partial A$$

assuming Ω has no boundary and the minus sign in this case refers to the opposite orientation (and has nothing to do with negative events). We leave such exotisms for future exploration.

Chapter 2

Quantum logic and topology.

We now treat quantal set theory from an axiomatic point of view and connect this with the subject of quantal logic developed by Von Neumann a century ago. The central idea in quantum theory is that a proposition is associated to a linear space and the mathematics of linear spaces is provided for by the (set theoretical) intersection $\cap$, the (direct) sum $+$ $(\oplus)$ (replacing the union) and the tensor product $\otimes$ as a substitute for the Cartesian product. Given a complex vector space $\mathcal{H}$ as well as scalar product $\langle v|w\rangle$ defined upon it for $v, w \in \mathcal{H}$. The scalar product between v and w is supposed to be equal to the product of the oriented length of the projection of w upon v times the length of v. This quantity satisfies, by means of simple experience, the following properties:

$$\begin{aligned} \langle v|w\rangle &= \langle w|v\rangle \\ \langle v|aw + bu\rangle &= a\langle v|w\rangle + b\langle v|u\rangle \\ \langle v|v\rangle &\geq 0 \,\text{where equality holds if and only if}\, v = 0. \end{aligned}$$

Hilbert spaces carry some natural topologies; to define those, we show that the scalar product defines in a canonical fashion a metric d. We first prove that the quantity $||v||$ defined by

$$||v|| = \sqrt{\langle v|v\rangle}$$

and called a norm has identical properties to those of the modulus of a complex number. An important step herein is the so called Cauchy-Schwartz identity

$$|\langle v|w\rangle| \leq ||v||||w||$$

meaning that the projection of w on v multiplied with the length of v is less or equal to the product of the lengths of v and w, a result one expects to hold trivially. The formal proof goes as follows:

$$0 \leq ||v + \lambda w||^2 = ||v||^2 + |\lambda|^2 \, ||w||^2 + 2\text{Re}\left(\overline{\lambda}\langle w|v\rangle\right)$$

where $\mathrm{Re}(a+ib)=a$ is the real part of the complex number $z=a+bi$. One verifies that the real part of the complex number z may be written as $\frac{1}{2}(z+\overline{z})$ whereas the imaginary part equals $-i\frac{1}{2}(z-\overline{z})$. The modulus of a complex number is defined by means of

$$|z|=\sqrt{z\overline{z}}=\sqrt{a^2+b^2}$$

and satisfies

$$|z+z'|^2=|z|^2+|z'|^2+(z\overline{z'}+\overline{z}z')$$

whereas the last term equals, up to a factor of two,

$$aa'+bb'$$

and the absolute value is bounded from above by $|a|\,|a'|+|b|\,|b'|$. The square of this last expression is given by

$$a^2a'^2+b^2b'^2+2\,|a|\,|a'|\,|b|\,|b'|\leq\left(a^2+b^2\right)\left(a'^2+b'^2\right)=|z|^2\,|z'|^2$$

and consequently one has that

$$|z+z'|^2\leq(|z|+|z'|)^2$$

and hitherto

$$|z+z'|\leq|z|+|z'|$$

a formula known as the triangle inequality. Consequently, we may define a metric on the complex plane by means of

$$d(z,z')=|z-z'|\,.$$

Returning to the proof of the triangle inequality, one notices that we may pick λ such that

$$\mathrm{Re}\left(\overline{\lambda}\langle w|v\rangle\right)=-\,|\lambda|\,|\langle v|w\rangle|$$

whereas, in general, the left hand side is always larger than the right hand side. Therefore, we have that

$$0\leq||v||^2+|\lambda|^2\,||w||^2-2\,|\lambda|\,|\langle v|w\rangle|$$

which is a quadratic polynomial inequality in the positive variable $|\lambda|$. The existence of at most one positive root demands that

$$0\leq 4\,|\langle v|w\rangle|^2-4||v||^2||w||^2$$

which proves the result and equality only holds if and only if $w=-\lambda v$. Consequently,

$$||v+w||^2\leq||v||^2+||w||^2+2\,|\langle v|w\rangle|\leq||v||^2+||w||^2+2||v||\,||w||=(||v||+||w||)^2$$

which proves the triangle inequality for the norm. Consequently, each Hilbert space $\mathcal{H}$ defines a canonical metric topology by means of

$$d(v, w) = ||v - w||$$

and we demand that $\mathcal{H}$ is complete in this topology. This condition is extremely important for the theory of linear operators but let us start by making some preliminary observations. Two non-zero vectors v, w are perpendicular to one and another if and only if $\langle v|w\rangle = 0$ and we say v is normed if and only if $||v|| = 1$. Given two Hilbert spaces $\mathcal{H}_i$, the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ constitutes again a Hilbert space spanned by pure vectors $v_1 \otimes v_2$ where $v_i \in \mathcal{H}_i$. Regarding sums $\sum_{i=1}^{n} z_i v^i \otimes w^i$, the following equivalences are in place

$$\begin{aligned} z(v \otimes w) &\equiv (zv) \otimes w \equiv v \otimes (zw) \\ v \otimes w_1 + v \otimes w_2 &\equiv v \otimes (w_1 + w_2). \end{aligned}$$

We define $\mathcal{H}$ as the linear space of such equivalence classes and make a completion in the metric topology defined by means of the scalar product

$$\langle v_1 \otimes w_1 | v_2 \otimes w_2\rangle := \langle v_1|v_2\rangle\langle w_1|w_2\rangle.$$

In a similar vein, the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ is defined by means of the equivalences

$$\begin{aligned} z(v \oplus w) &\equiv (zv) \oplus (zw) \\ v_1 \oplus w_1 + v_2 \oplus w_2 &\equiv (v_1 + v_2) \oplus (w_1 + w_2) \end{aligned}$$

with as scalar product

$$\langle v_1 \oplus w_1 | v_2 \oplus w_2\rangle := \langle v_1|v_2\rangle + \langle w_1|w_2\rangle.$$

One verifies that a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$ is provided by means of $v_i \otimes v_j$ where the v_i constitute a basis of $\mathcal{H}_1$ and w_j of $\mathcal{H}_2$. A basis for $\mathcal{H}_1 \oplus \mathcal{H}_2$ is provided by $v_i \oplus 0, 0 \oplus w_j$. Linear subspaces of Hilbert spaces are characterized by means of Hermitian projection operators. What follows also holds in an infinite number of dimensions, but for simplicity of the presentation, we shall confine ourselves to finite dimensions. Given a linear operator A, we define its adjoint $A^\dagger$ by means of

$$\langle v|Aw\rangle = \langle A^\dagger v|w\rangle$$

for all $v, w \in \mathcal{H}$. An operator is called self adjoint in case $A = A^\dagger$ and normal in case $AA^\dagger = A^\dagger A$. The reader is advised to make the following exercises.

- Let P, Q be two Hermitian projection operators meaning that $P^2 = P$, $Q^2 = Q$, $P^\dagger = P, Q^\dagger = Q$. Show that $P + Q$ constitutes a Hermitian projection operator if and only if $PQ = QP = 0$. Show that the same holds for PQ if and only if $PQ = QP$.
- Two Hermitian projection operators P, Q are orthogonal if and only if $PQ = 0$; we define the partial order $\leq$ by means of $P \leq Q$ if and only if $QP = PQ = P$. Prove explicitly that $\leq$ defines a partial order on the set of Hermitian projection operators. In particular, it holds that $P \leq Q$ and $Q \leq P$ implies that $P = Q$. Also, $P \leq Q$ and $Q \leq R$ leads to $P \leq R$.

- We call the set of Hermitian projection operators on a vector space, equipped with $\leq$, a raster. Show that for any P, Q there exists a minimal projection operator $P \vee Q$ such that $P, Q \leq P \vee Q$ and any R such that $P, Q \leq R$ satisfies $P \vee Q \leq R$. On the other hand, one may construct a maximal projection operator $P \wedge Q \leq P, Q$. Show that $\vee, \wedge$ do not in general obey the rule of de Morgan

$$P \wedge (R \vee Q) \neq (P \wedge R) \vee (P \wedge Q).$$

- In terms of subspaces, $P \vee Q$ is the projection operator on $V + W$, whereas $P \wedge Q$ on $V \cap W$ where $V\,(W)$ is the image of $P\,(Q)$.

- Show that the raster possesses a unique minimum as well as maximum provided by 0 and 1 respectively.

- Show that there exist minimal nonzero Hermitian projection operators, called atoms. Every Hermitian projection operator may be written as a sum of orthogonal atoms.

Quantum logic.
Given that in the previous exercise $\vee$ and $\wedge$ may be conceived as "or" and "and" respectively, it becomes possible to understand quantal logic by means of using Hermitian projection operators as propositions. Reflect on this and retrieve classical pointer propositions by considering a complete set of orthogonal projection operators. In such case, $P \vee Q = P + Q$ and $P \wedge Q = PQ$.

Quantum set theory.
$\mathcal{S}$ consists out of sets which are given by objects P, Q and we have again $\wedge, \vee$ where $P \wedge P = P = P \vee P$ with minimal and maximal elements $0, 1$ replacing the empty set and the entire universe. The distinction with classical set theory is to be found in the de-Morgan rule; in a way, it would be nice if we could find a logical rule in terms of $\wedge$, $\vee$ which would deliver us with the Hilbert space setting where P, Q may be seen as Hermitian projection operators. This has been the topic of research of the Geneva school for plenty of years and was rather extensively documented for by Piron.

In particular, the set of propositions must give rise to a so called orthomodular lattice defined by

- sets P, Q,

- a minimal 0 and maximal element 1,

- commutative and associative operations $\vee, \wedge$ satisfying $P \vee P = P \wedge P = P$ as well as $P \wedge 0 = 0$, $P \wedge 1 = P = P \vee 0$, $P \vee 1 = 1$.

- a partial order $\leq$ defined by $P \leq Q$ if and only if $P \wedge Q = P = Q \wedge P$ with 0 as unique minimal element and 1 as maximal one where $P \vee Q$ is the supremum of P, Q and $P \wedge Q$ the infimum,

- A linear structure: if one takes Q, P then $PQ \neq QP$ and therefore it does not hold that $(PQ)^2 = PQ$ assuming the product satisfies the standard requirements of commutativity and associativity. To compensate for this, one should consider something like

  $$P \wedge Q = \frac{1}{2}(PQ + QP)$$

  which still does not lead to $(P \wedge Q)^2 = P \wedge Q$. However, there is a unique natural fix given by

  $$P \wedge Q = \lim_{n \to \infty} \left(\frac{1}{2}(PQ + QP) \right)^n$$

  which we will take as a standard formula. We moreover demand that $P + Q \in \mathcal{S}$ if and only if $PQ = 0$ in which case it coincides with $P \vee Q$,

- atomisticity, meaning every set P can be written as $P = \vee_\alpha Q_\alpha$ with the $Q_\alpha \neq 0$ orthogonal primitive propositions $Q_\alpha Q_\beta = 0$ for $\alpha \neq \beta$,

- a reality notion given by an involution $\dagger$ such that $P^\dagger = P$.

The reader notices that the first four axioms could give rise to classical set theory and that the linearity really decides upon the case being "quantal".

Non-commutative Quantum logic.
We generalize the operations $\wedge$ and $\vee$ to a context in which they are no longer commutative; this procedure holds as well for the classical Boolean logic or the quantual logic explained above where the de Morgan rule gets a minor blow. It is natural to interpret $\wedge$ as well as $\vee$ as mappings $\wedge, \vee : P \times P \to P : (x, y) \to x \wedge y$, $(x, y) \to x \vee y$ where P denotes the lattice of propositions defined by means of a linear Euclidean space in the quantal case. Define the mapping $S : P \times P \to P \times P : (x, y) \to (y, x)$ and consider $\wedge^{'}_{(V,W)} := W \circ \wedge \circ S \circ V$ as well as $\vee^{'}_{(V,W)} = W \circ \vee \circ S \circ V$ where $V : P \times P \to P \times P$ is required to be invertible as well as is the case for $W : P \to P$. Requiring $\wedge^{'}_{(V,W)}$ to satisfy $\left(\wedge^{'}_{(V,W)} \right)^{'}_{(V,W)} = \wedge$ it is sufficient and mandatory that $W^2 = 1$ as well as $S \circ V \circ S \circ V = 1$. This demand is of a special algebraic nature which we dub by the name of an involution; so we are going to study involutive deviations from quantal logic. An involution gives rise to a notion of duality; in particular self-duality is defined by the condition that

$$\wedge^{'}_{(V,W)} = \wedge, \vee^{'}_{(V,W)} = \vee.$$

It is natural to propose first S symmetrical logics; these are given by

$$\wedge^{'(V,W)} \circ S = \wedge^{'(V,W)}, \vee^{'(V,W)} \circ S = \vee^{'(V,W)}.$$

This can only happen by choosing V such that

$$V \circ S = S \circ V$$

reducing a previous condition to

$$V^2 = 1$$

whereas it still holds that

$$\wedge'^{(V,W)} = W \circ \wedge \circ S \circ V.$$

In case $\wedge$, $\vee$ coincide with the standard Boolean or Quantal operations denoted by $\wedge_d$, $\vee_d$ where $d = c, q$ one has that

$$\wedge_d \circ S = \wedge_d, \; \vee_d \circ S = \vee_d.$$

In such a case,

$$\wedge := \wedge_d'^{(V,W)} = W \circ \wedge_d \circ V$$

a small simplification of the previous formula and $\vee$ is defined in a similar way. Now, to remain entirely clear, it is so that the d index should be the same in $\wedge$, $\vee$ but (V, W) becomes (R, T) for $\vee$ whereas the former pertains to $\wedge$. We now isolate the "de Morgan expression" $a \wedge (b \vee c)$:

$$\wedge \circ (1 \times \vee)(a, b, c) = W \wedge_q V(1 \times T \vee_q R)(a, b, c).$$

It is subsequently natural to call T - $(\wedge_q, V)$ compatible if and only if $\wedge_q V(1 \times T) = T' \wedge_q V$ for some $T' : P \to P$. Likewise, it is natural to call V - $\vee_q$ compatible if and only if $V(1 \times \vee_q) = (1 \times \vee_q)V'$ for some $V' : P^3 \to P^3$. Under these assumptions, the previous expression reduces to

$$WT'(\wedge_q(1 \times \vee_q))V'(1 \times R)$$

which was the desirable separation. It is furthermore natural to suggest further restrictions

$$WT' = 1, \; V'(1 \times R) = 1_3.$$

Truth evaluators ω

The material presented below constitutes an extension of the notes I have received once from Rafael Dolnick Sorkin; in classical Boolean logic one disposes of truth evaluator ω of logical sentences which constitutes a homomorphism from the set of propositions $P, \vee_c, \wedge_c$ to $\mathbb{Z}_2, +, .$ where 0 is interpreted as false and 1 as true and $\vee_c$ is the so called exclusive *or* in the sense that $a \vee_c b$ is true if and only if exactly one of them is true. It is to say that

$$\omega(a \vee_c b) = \omega(a) + \omega(b), \; \omega(a \wedge_c b) = \omega(a)\omega(b).$$

In quantum logic, there is no such thing as a truth evaluator; one can only say wether a particular assertion is true or false with a certain probability. A quantum reality is then a particular choice of mapping from P to $\mathbb{Z}_2$ but it makes no sense any longer to speak about a homomorphism because the de-Morgan rule fails in general: the lattice is not distributive. As such, it may very well be that you have a quantal reality ω for which $\omega(a) = \omega(b) = 1$, but $\omega(a \wedge_q b) = 0$. To get an idea of what more general realities are about, let us describe a classical system in a quantum mechanical fashion. An example is give by means of the weather, "the sun shines", modelled by $|l\rangle$, or "it is dark" given by $|d\rangle$. Quantum mechanically, one disposes of a complex two dimensional Euclidean space spanned by the extremal vectors $|l\rangle, |d\rangle$. Consider now a general state

$$|\psi\rangle = \alpha|l\rangle + \beta|d\rangle$$

and study the class of truth functionals ω which merely depend upon

$$\frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2}, \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2}$$

something which reduces to a parameter $0 \leq \lambda \leq 1$ due to

$$\frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2} + \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2} = 1.$$

When all truth evaluators merely depend upon this parameter only, the complex plane may be reduced to the line segment connecting both extremal vectors $|l\rangle, |d\rangle$ to one and another. An example of such a gneralized reality is provided by

$$\omega^l_\epsilon : [0, 1] \to \mathbb{Z}_3$$

given by means of the prescription

$$\omega^l_\epsilon(\sqrt{\lambda}|l\rangle + \sqrt{(1-\lambda)}|d\rangle) = \chi(\lambda + \epsilon - 1) + 2\chi(\lambda - \epsilon)\chi(1 - \epsilon - \lambda).$$

ω^l and is henceforth connected to the question whether the light shines and ϵ is the tolerance of the observer. This truth evaluator says "yes", given by means of 1, in case $1 - \epsilon \leq \lambda \leq 1$, under determined or "vague" 2 when $\epsilon \leq \lambda \leq 1 - \epsilon$ and no, given by 0, when $0 \leq \lambda \leq \epsilon$. We have that χ is the so called characteristic function defined on the real numbers by means of $\chi(x) = 1$ in case $x \geq 0$ and zero otherwise. The issue is that we departed from a quantum mechanical description of the weather and by reduction of the allowed questions arrived to a classical system where, moreover, ω^l_ϵ is nonlinear.

Most physicists would suggest at this moment that we did not make a sufficient distinction between classical and quantum logic as yet because $\wedge_q, \vee_q$ are commutative, assiociative but $\wedge_q$ is not distributive with regard to $\vee_q$ which is the case for $\wedge_c$, $\vee_c$. In our most general setting, one has that $\wedge$ and $\vee$ are neither commutative, nor associative

$$\vee(1\times\vee)(a,b,c) = T\vee_d R(1\times T\vee_d R)(a,b,c) \neq T\vee_d R(T\vee_d R\times 1)(a,b,c) = \vee(\vee\times 1)(a,b,c)$$

and likewise so for $\wedge$. The main distinction between classical and quantum logic resides in the fact that the set of propositions constitutes a distributive lattice in the former case whereas it does not in the latter; this results in the statement that the classical rule

$$\mu(a|b)\mu(b) = \mu(b|a)\mu(a)$$

is no longer true in the quantal case. Here, μ is the probability measure that a is true; in other words, the truth determinations of a and b depend upon the order in which they occur. This has so far not been accounted given that a homomorphism $\vee_{c,q}, \wedge_{c,q}$ does not make any distinction in the order of the factors. Therefore, classically, for our homomorphism $\omega_c(a \wedge_c b)$ is determined by the unordered tuple $\{\omega_c(a), \omega_c(b)\}$. Quantum mechanically, it is as such that the reality $\omega_q(a \wedge_q b)$ is not provided by the ordered couple $(\omega_q(a), \omega_q(b))$ as elements of $\mathbb{Z}_2$ but also depends upon a, b themselves. It is not so that

$$\mu_{|v\rangle}(a|b) = \frac{\mu_{|v\rangle}(a \wedge_q b)}{\mu_{|v\rangle}(b)}$$

due to commutativity of $\wedge_q$ as well as $a \wedge_q b = 0$ for distinct one dimensional Hermitian projection operators a, b on a Hilbert space $\mathcal{H}$. The exact formula is given by

$$\mu_{|v\rangle}(a|b) = \frac{\mathrm{Tr}(|v\rangle\langle v|bab)}{\mathrm{Tr}(|v\rangle\langle v|b)}$$

and the reader notices that the non-commutativity of a and b is of vital importance. Henceforth, the ontological mapping defined in quantum theory is given by $\kappa : P \to \mathbf{L}(\mathcal{H})$ where P is the set of prepositions with a yes or no answer onto the lattice of Hermitian projection operators defined on the Hilbert space of states of the system. The classical Lagrange formula

$$\mu(a|b)\mu(b) = \mu(b|a)\mu(a)$$

where μ is determined by the state of the system is abandoned upon provided that $\wedge_q$ a la Von Neumann offers no alternative. The natural question henceforth is whether we may find a natural $\wedge$ as well as a consistent set of realities

$$\omega_q^\rho : P \to \mathbb{Z}_2 \times [0, 1]$$

attached to density matrices ρ defined on $\mathcal{H}$, such that

$$\omega_q^\rho(a) = (1, \lambda)$$

and

$$\omega_q'^\rho(a) := (0, 1 - \lambda)$$

is defined as the complementary observation. It is clear that ω_q is not always given by a homomorphism; prior to proceeding, it is important to understand $\vee_q$. It is clearly so that in quantum theory, we have an extended ontology; we do not only pose the question "what is the probability that $a \wedge_c b$ holds given that a as well as b are true" such as the case in classical logic, but we insist on the formulation "what is the chance that $a \wedge_q b$ holds given that a after b has been experimentally established". The right answer is easy if $a \wedge b$ is represented by the Hermitian operator bab which is logical given that the order of measurement matters. In general, one shows that

$$a \wedge_q b = \lim_{n \to \infty} \left(\frac{1}{2}(ab + ba) \right)^n$$

and in the framework of our deformation theory $\wedge$ is given by means of

$$V(a, b) = (1, bab)$$

at least this is so for atomistic elements a, b. For atomistic elements, $\frac{bab}{\mathrm{Tr(ab)}}$ is again a rank one Hermitian projection operator; however for projection operators of general rank, this is no longer the case. Here, we have to extend our definition of V as going from $P \times P \to C \times C$ where C are the so called positive operators on Hilbert space. An operator A is positive if and only if A is self adjoint and

$$\langle v|A|v\rangle > 0$$

for all $v \neq 0$. As an exercise, the reader understands that the definition of $\leq$ extends to the Hermitian operators by means of $A \leq B$ if and only if

$$\langle v|(B - A)|v\rangle > 0.$$

Show that in such a case, the definitions of $\wedge_q$ and $\vee_q$ can be extended as the largest Hermitian operator smaller or equal than A, B and the smallest Hermitian operator greater or equal to A, B respectively. The proof of this statement hinges on the so-called spectral decomposition theorem for Hermitian operators, something which we shall study in the next section. Briefly, it says that any Hermitian operator A can be written as

$$A = \sum_i \lambda_i P_i$$

where the λ_i are the real eigenvalues and the P_i Hermitian projection operators such that $P_i P_j = \delta_{ij} P_i$. Therefore, take A, B and order all eigenvalues

$$\lambda_0 < \lambda_1 \ldots < \lambda_k$$

with $k \leq 2n$ where n is the dimension of Hilbert space. Note that some of the λ_i may belong to A as well as B; in that case, we consider the projection operators $R_i := P_i \vee_q Q_i$ where the Q_i refer to B otherwise R_i equals P_i or Q_i. Start now with λ_0, the smallest eigenvalue, and consider the operator $C_0 = \lambda_0 R_0$; clearly $C_0 \leq A, B$. Proceed now towards the minimal λ_j such that $S_j := \vee_{i=2}^{j} R_i$ obeys $[S_j, R_0] = 0$ and consider the projection operator

$$T_1 := S_j(1 - R_0)$$

then the reader verifies that this is an Hermitian projection operator and that $T_1 R_0 = 0$. In case no such j exists, then define $A \wedge_q B = \lambda_0 R_0 + \lambda_1 (1 - R_0)$, otherwise proceed with $C_1 := \lambda_0 R_0 + \lambda_1 T_1$. The reader now understands that he has to look at λ_{j+1} and construct the smallest $S_k := \vee_{i=j+1}^{k} R_i$ such that

$$[S_k, R_0 + T_1] = 0.$$

In case no such k exists $A \wedge_q B = \lambda_0 R_0 + \lambda_1 T_1 + \lambda_{j+1}(1 - R_0 - T_1)$ otherwise we consider

$$C_2 = \lambda_0 R_0 + \lambda_1 T_1 + \lambda_{j+1} T_2$$

where $T_2 = S_k(1 - R_0 - T_1)$ and the procedure continues. It is obvious that the final result is the optimal Hermitian operator which is smaller or equal to both A, B. The construction of $\vee_q$ is similar, but then one starts at the largest eigenvalue of both operators. W is henceforth determined on the rank 1 matrices by means of the identity. Therefore, for rank one projectors a, b it holds that

$$a \wedge b = T \circ \wedge_q \circ R(a, b) = bab.$$

Subsequently, one has that

$$\omega_q^{\rho}(a) = (1, \mathrm{Tr}(\rho a))$$

or

$$\omega_q^{\rho}(a) = (0, 1 - \mathrm{Tr}(\rho a))$$

for a of rank one. Clearly, by definition

$$\omega_{q,1}^{\rho}(a|b) := \frac{\pi_2(\omega_q^{\rho}(a \wedge b))}{\pi_2(\omega_q^{\rho}(b))}$$

equals the probability that a is measured after b. Here π_j equals the projection on the j'th factor. Elaborate further on this theory and determine a suitable $\vee$ operation. Hint: the latter is cannot be given by $a \vee b = a + b$ in the deformation framework provided that $\vee_q$ does not allow one to determine the projection of a on b as is given by $\mathrm{Tr}(ab)$. This is something which is mandatory to extract the sum operation. To define $\vee$ it is advised to use the classical rule

$$\neg(a \vee_c b) = (\neg a) \wedge_c (\neg b)$$

and using $\neg\neg = 1$, it holds that

$$a \vee b = \neg((\neg a) \wedge (\neg b)).$$

In quantum theory, $\neg(a)$ is provided by $1 - a$ and henceforth, we arrive at

$$a \vee b = 1 - (1 - a) \wedge (1 - b)$$

which leads to a violation of the de Morgan rule given that

$$a \wedge (b \vee c) = a.(1 - (1 - b).(1 - c)) = -ab - ac + abc$$

whereas

$$(a \wedge b) \vee (a \wedge c) = 1 - (1 - ab).(1 - ac) = -ab - ac + abac.$$

General exercise.
Determine matrix representations of deformed logic's in terms of commutative albeit possible non-associative ones. It is to say that

$$\wedge = (\tilde{\wedge}_{ijk})_{i,j,k:1\ldots n}$$

where

$$\tilde{\wedge}_{ijk}(a_j, b_k) = \tilde{\wedge}_{ijk}(b_k, a_j)$$

constitute S symmetrical logics on the product space $\times_n P$ where P provides for elementary propositions. Classify first the S symmetric deformations of Boolean logic on general proposition sets.

Chapter 3

Classical metric spaces and connection theory thereupon.

In this section, we expand pon well known notions in differential geometry to the extend that one can define Riemann and Torsion functions on a general path metric space. The construction goes by means of a generalized conncection which gives rise to the definition of a non commutative and non associative sum even ultra-locally in case the limit exists. In that vein may these classical constructions be perceived as quantuantum mechanical given that precise values of the curvature and torsion tensors may not be within reach. In the next section, we try to define non-comutative geometry from the operational point of view wich is a considerably less flexible language.

Let X be any topological space (we do not insist upon it being metrical yet) and consider an equivalence relation $R \subset X \times X$ which is topologically open. R defines vectors, that is $(x, y) \in R$ is a vector connecting x with y; the correspondance to the usual vectors on a manifold being that (x, y) has to be thought of as the vector at x such that thenimage of the exponential map equals y, so they defined in a way relative to a meric and not a coordinate system. As said in the introduction, the notion of transport can easily be generalized and is defined by means of the following

$$\nabla_X : \{(x, y, z) : y, z \in R(x, \cdot)\} \to X \times X : (x, y, z) \to \nabla_{(x,y)}(x, z) = (y, w)$$

is called the transported relation regarding (x, z) over (x, y) from x to y and as such it indicates a preffered path or geodesic at least locally. ∇_X should obey the following further properties: (a) for any x, there exists an open O around it, such that $\{x\} \times O \subset R$ and such that for any $y, z \in O$ holds that $\nabla_{(x,y)}(x, z) \in R$, allowing one to define the composition of two transporters (b) ∇_X is continuous in the product topology (c) $\nabla_X(x, x, z) = (x, z)$, $\nabla_X(x, y, x) =$

(y, y) indicating that transport over the zero vector is the identity map and the zero vector gets transported into the zero vector. Before we proceed, it is useful to defientwo projections $\pi_1 : R \to X : (x, y) \to x$ and $\pi_2 : R \to X : (x, y) \to y$. We shall impose a furter condition on R which is that for any x and sufficiently small neighborhood O around it, that for any y, z, p, q it holds that $(\pi_2(\nabla_{(x,y)}(x, z)), \pi_2(\nabla_{(x,p)}(x, q))) \in R$ meaning that for sufficiently small vectors sufficiently small vectors around a point, the resulting endpoints of the parallel transport again constitute a vector. Another, useful operation is the reversion P which maps (x, y) into (y, x), something which has to do with the linear structure of vectors. To localize, the reversion, we define $\tilde{P}(x, y)$ as $\nabla_{P(x,y)}(P(x, y)) \in R(x)$, so again, taking the minus sign is a geometrical operation. On R, it is now possible to define two kinds of (non-commutative) sums; the first one is mere composition, that is

$$(x, y) \circ (y, z) = (x, z)$$

being non local operation and the second one

$$(x, y) \oplus (x, z) = \nabla_{(x,y)}(x, z) \circ (x, y)$$

being a local operation. The reader notices that the reversion also defines a minus operation

$$(x, y) \ominus (x, z) = (x, y) \oplus \tilde{P}(x, z).$$

So, the reader understands that the local notion of a sum is a geometrical one and not one which merely originates from the manifold structure. Now, we can easily define the torsion functor

$$T : X \times R(x) \times R(x) \to R(x) : (x, y, z) \to ((x, y) \oplus (x, z)) \ominus ((x, z) \oplus (x, y))$$

and we shall prove that in a way this coincides with the usual definition in case y, z converge to x at the same rate. The Riemann function may be defined in a sufficiently small neighborhood of x as

$$R(x, p, q, r) = ((x, p) \oplus ((x, q) \oplus (x, r))) \ominus ((x, q) \oplus ((x, p) \oplus (x, r))).$$

The reader notices here that we did not include the commutator in this definition as we have no natural substitute for a vectorfield, neither commutator and all draggings are supposed to define commuting vectorfields anyway. We shall investigate these two definitions in further detail in the next section. There is no meaningful topological way to define this, you need a metric for that. Finally, we may consider functions between two metrical spaces $(X, d_X), (Y, d_Y)$ with vector structures R, T and transporters ∇_X, ∇_Y defined upon it: we then say that $F : X \to Y$ is differentiable in a surrounding of $x \in X$ in case for any open $\mathcal{V} \subset T(F(x))$ there exists an open neighborhood $\mathcal{O} \subset R(x)$ such that the canonical bi-continuous mapping $DF(w, v) : (w, v) \in \mathcal{O}^2 \to \mathcal{V}^2$, $v, w \in \mathcal{O}$ defined by $(F(v), F(w)) = DF(v, w)$ satisfing

$$DF(((x, y) \oplus (x, w)) = DF(\nabla_{(x,y)}(x, w)) \circ DF(x, y)$$

also obeys

$$\frac{d_2(DF\left((x,y)\oplus(x,w)\right)\ominus\left(DF(x,y)\oplus DF(x,w)\right))}{\epsilon}\to 0$$

in case $d_1(x,y)=\epsilon a,\ d_1(x,w)=\epsilon b$, where $a,b>0$ constants, which is the linearity condition. To define the torsion and Riemann "tensor", we need additional information. A connection is called weakly metric compatible if and only if

$$d(\nabla_{(xy)}(xz))=d((xz))$$

which is, by itself insufficient to select for an "integrable" class of connections; for example, consider $\mathbb{R}^2$ with the standard Euclidean metric and define the connection $\nabla_{(x,y)}(x,z)=(y,y+R(z-x))$ where R is the rotation over the minimum of the angle θ between the vector $y-x$ and $z-x$ and $\pi-\theta$ in opposite orientation to the one defined by $z-x$ and $y-x$. Then the reader convinces himself that the angle is not preserved and that the torsion function vanishes identically. So, we must insist upon a stronger metric compatibility which says that the angles are preserved. For doing this, we need a path metric defined by the property that for any $x,y\in X$ it holds that there exists a $z\in X$ such that

$$d(x,z)=d(y,z)=\frac{d(x,y)}{2}.$$

The latter is equivalent to stating that there exists a curve, called a geodesic, $\gamma:[0,1]\to X$ which minimizes the length functional L for paths with endpoints x,y and, moreover, $L(\gamma)=d(x,y)$. The latter is defined by

$$L(\gamma)=\sup_{0=t_0<t_1\ldots<t_n=1,n>0}\sum_{j=0}^{n-1}d(\gamma(t_j),\gamma(t_{j+1}))$$

and γ can be parametrized in arc-length parametrization by means of the Radon Nikodym derivative. Furthermore, this only makes sense if the geodesic connecting two points x,y close enough to one and another exists and is unique so that we can associate vectors to geodesics. Consider a point $x\in X$ and take a sequence of points y_n,z_n placed on two half geodesics emanating from x converging in the limit for n to infinity towards x. In case the limit

$$\lim_{n\to\infty}\frac{d(x,y_n)^2+d(x,z_n)^2-d(y_n,z_n)^2}{2d(x,y_n)d(x,z_n)}$$

exists, we define the angle $\theta_x(y,z)$ between both geodesics by equating the latter expression to $\cos(\theta_x(y,z))$. So, we must also require that ∇_X preserves angles; in short, $\theta_x(y,z)=\theta_p(\pi_2(\nabla_{(x,p)}(x,y)),\pi_2(\nabla_{(x,p)}(x,z)))$ for x,p,y,z sufficiently close to one and another. Obviously, this is still not enough given that one may consider the connection $\nabla_{(x,y)}(x,z)=(y,y-(z-x))$ and notice that $(x,y)\oplus(x,y)=(x,x)=0$. The reader sees immediately that angles as well as distances are preserved and that the torsion vanishes since $(x, y) \oplus (x, z) = (x, y - (z - x))$ and $(x, z) \oplus (x, y) = (x, z - (y - x))$ so that

$$((x, y) \oplus (x, z)) \ominus ((x, y) \oplus (x, z)) = (x, y - (z - x)) \oplus \tilde{P}(x, z - (y - x)) =$$

$$(x, y - (z - x)) \oplus (x, y - (z - x)) = (x, x) = 0$$

since $\tilde{P}(x, z-(y-x)) = \nabla_{(z-(y-x),x)}(z-(y-x), x) = (x, x-(z-y)) = (x, y-(z-x))$. So, therefore we need to impose the strongest form, which amounts to an integrability condition which is that the d geodesics are auto-parallel curves meaning that for any geodesic γ from x to y in arclength parametrization, it holds that

$$\nabla_{(\gamma(t),\gamma(s))}(\gamma(t), \gamma(s)) = (\gamma(s), \gamma(2s - t))$$

for $s > t$ sufficiently small. In that case, we find back the ordinary Levi-Civita connection with vanishing torsion in case for metrics on a manifold. To allow for torsion, one may impose that for any vector x, y sufficiently small, there exists a unique curve γ from x to y in arclength parametrization such that for $t < s$ sufficiently small, the above condition holds. We shall henceforth insist upon the last integrability condition. To give a nontrivial example of our construction, take two manifolds glued together at a point p, with identified induced metrics on both meaning there exist two orthonormal basis at p which are identified by means of a linear mapping $T : T\mathcal{M}_p \to T\mathcal{N}_p : v \to T(v)$ and T^{-1} of course for the opposite directions. Then, for general vectors $a \in \mathcal{M}_p$ corresponding to a unique vector (p, x) and $b \in T\mathcal{N}_p$ corresponding to a unique vector (p, y), one can define $a \oplus b \equiv a \oplus_{\mathcal{M}} T^{-1}(b)$ in $\mathcal{M}$ resultng in a vector (p, z) and vice versa for $b \oplus a \equiv b \oplus_{\mathcal{N}} T(a)$. So, usually, the torsion function does not vanish, but it does so for infinitesimal vectors $a = \epsilon a'$, $b = \epsilon b'$ keeping a' and b' fixed. In the limit for ϵ to zero (as we shall show in full detail below) will $a \oplus T^{-1}(b)$ reduce to $\epsilon(a' + T^{-1}b') + O(\epsilon^2)$ so that in first order of ϵ, we have that

$$(a \oplus T^{-1}(b)) \ominus (b \oplus T(a)) = \epsilon(a' + T^{-1}(b') - T^{-1}(b' + T(a'))) + O(\epsilon^2) = O(\epsilon^2)$$

and we will show below that even the second order term in ϵ vanishes in case the torsion tensors are anti-podal. Notice that differentiability is a priori a metric dependent concept but as the reader may verify, this is not the case for smooth metrics and general metric compatible connections defined by scalar products on a manifold. Here, the metric locally trivializes and the connection gives subleading corrections so that the sum reduces to the ordinary one. Let us work this out in full detail here so that the reader understands that the usual manifold definitions follow from ours. Given a metric tensor, $g_{\mu\nu}$ the reader verifies that the general connection is given by

$$\widehat{\Gamma}^{\delta}{}_{\mu\nu} = \Gamma^{\delta}{}_{\mu\nu} - \frac{1}{2}\left(T_{\mu\nu}{}^{\delta} + T_{\nu\mu}{}^{\delta} - T^{\delta}{}_{\mu\nu}\right)$$

where

$$T^{\delta}{}_{\mu\nu}$$

is the Torsion tensor which is anti-symmetric in $\mu\nu$ and in the previous expression, lowering and raising of indices has been done by means of the metric tensor. Now, take two vectors V, W at x, take $\epsilon > 0$ and consider the exponential map defined by ϵV, equivalent to (x, y) and ϵW, equivlent to (x, z) respectively. Up to second order in ϵ those are given by

$$y = x + \epsilon V - \frac{\epsilon^2}{2}\widehat{\Gamma}(V, V)$$

and likewise for W. Parallel transport of ϵW along ϵV gives

$$W(y) = \epsilon W - \epsilon^2\widehat{\Gamma}(V, W)$$

and likewise for V, W interchanged. Hence,

$$\nabla_{(x,y)}(x, z) = \left(y, x + \epsilon V - \frac{\epsilon^2}{2}\widehat{\Gamma}(V, V) + \epsilon W - \epsilon^2\widehat{\Gamma}(V, W) - \frac{\epsilon^2}{2}\widehat{\Gamma}(W, W)\right)$$

and likewise for V, W interchanged. The reader notices that $\widehat{\Gamma}(V, V)$ can be retrieved from the geodesic equation and therefore $\widehat{\Gamma}(V, W)$ from the transport equation, both in order ϵ^2. We shall make this now precise. One sees now that

$$(x, y) \oplus (x, z) = \left(x, x + \epsilon(V + W) - \frac{\epsilon^2}{2}\left(\widehat{\Gamma}(V + W, V + W) + T(V, W)\right)\right)$$

implying that

$$\pi_2\left((x, z) \oplus (x, y)\right) = x + \epsilon\left(W + V - \frac{\epsilon}{2}T(W, V)\right)$$
$$-\frac{\epsilon^2}{2}\widehat{\Gamma}\left(W + V + \frac{\epsilon}{2}T(W, V), W + V + \frac{\epsilon}{2}T(W, V)\right).$$

Hence,

$$((x, y) \oplus (x, z)) \ominus ((x, z) \oplus (x, y)) = (x, x + \epsilon^2 T(W, V) + O(\epsilon^3))$$

so, as promised, the torsion tensor emerges in leading order ϵ^2. To make this precise in our setting, consider the generalized geodesics γ_y, γ_z in arclength parametrization representing with $\gamma_y(0) = x$, $\gamma_y(1) = y$ and likewise for γ_z. Furthermore, choose any reference direction γ_q then we have that with

$$\widehat{T} := T(s) := \pi_2(T(x, \gamma_y(s), \gamma_z(s)))$$

that

$$\theta(\widehat{T}, \gamma_q), \;\; \lim_{s \to 0} \frac{d(x, T(s))}{s^2}$$

are well defined and fully capture the Torsion tensor without coordinates. In order to find the Riemann tensor, we need to be a bit more careful and expand terms up to the third power of ϵ; more in particular,

$$(x, y) := \left(x, x + \epsilon V - \frac{\epsilon^2}{2}\widehat{\Gamma}(V, V) - \frac{\epsilon^3}{6}\left((V\widehat{\Gamma})(V, V) - \widehat{\Gamma}(\widehat{\Gamma}(V, V), V) - \widehat{\Gamma}(V, \widehat{\Gamma}(V, V))\right)\right)$$

and

$$W(y) = W - \epsilon\widehat{\Gamma}(V,W) - \frac{\epsilon^2}{2}\Big((V(\widehat{\Gamma}))(V,W) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),W) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,W))\Big)$$

so that

$$\pi_2\left((x,y)\oplus(x,z)\right) = x+\epsilon V-\frac{\epsilon^2}{2}\widehat{\Gamma}(V,V)-\frac{\epsilon^3}{6}\Big((V\widehat{\Gamma})(V,V) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),V) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,V))\Big) +$$
$$\epsilon\left(W - \epsilon\widehat{\Gamma}(V,W) - \frac{\epsilon^2}{2}\Big((V\widehat{\Gamma})(V,W) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),W) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,W))\Big)\right)$$
$$-\frac{\epsilon^2}{2}\left(\widehat{\Gamma}(W,W) - \epsilon\left(\widehat{\Gamma}(\widehat{\Gamma}(V,W),W) + \widehat{\Gamma}(W,\widehat{\Gamma}(V,W)) - (V\widehat{\Gamma})(W,W)\right)\right)$$
$$-\frac{\epsilon^3}{6}\Big((W\widehat{\Gamma})(W,W) - \widehat{\Gamma}(\widehat{\Gamma}(W,W),W) - \widehat{\Gamma}(W,\widehat{\Gamma}(W,W))\Big).$$

We seek now for the associated geodesic of time ϵ which maps to this endpoint; that is we have to solve for

$$Z(V,W,\epsilon) = V + W - \frac{\epsilon}{2}T(V,W) + \frac{\epsilon^2}{6}K(V,W)$$

such that

$$x + \epsilon Z - \frac{\epsilon^2}{2}\widehat{\Gamma}(Z,Z) - \frac{\epsilon^3}{6}\Big((Z\widehat{\Gamma})(Z,Z) - \widehat{\Gamma}(\widehat{\Gamma}(Z,Z),Z) - \widehat{\Gamma}(Z,\widehat{\Gamma}(Z,Z))\Big)$$

equals the previous expression up to third order in ϵ. This leads to

$$(W\widehat{\Gamma})(W,W) - \widehat{\Gamma}(\widehat{\Gamma}(W,W),W) - \widehat{\Gamma}(W,\widehat{\Gamma}(W,W) + (V\widehat{\Gamma})(V,V) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),V) -$$
$$\widehat{\Gamma}(V,\widehat{\Gamma}(V,V)) + 3(V\widehat{\Gamma})(W,W) - 3\left(\widehat{\Gamma}(\widehat{\Gamma}(V,W),W) + \widehat{\Gamma}(W,\widehat{\Gamma}(V,W))\right) +$$
$$3\Big((V\widehat{\Gamma})(V,W) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),W) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,W))\Big)$$

must be equal to

$$-K(V,W) - \frac{3}{2}\left(\widehat{\Gamma}(V+W,T(V,W)) + \widehat{\Gamma}(T(V,W),V+W)\right) + ((V+W)\widehat{\Gamma})(V+W,V+W) -$$
$$\widehat{\Gamma}(\widehat{\Gamma}(V+W,V+W),V+W) - \widehat{\Gamma}(V+W,\widehat{\Gamma}(V+W,V+W))$$

which leads to

$$K(V,W) = (W\widehat{\Gamma})(V,V) + (W\widehat{\Gamma})(V,W) + (W\widehat{\Gamma})(W,V) + (V\widehat{\Gamma})(W,V) - 2(V\widehat{\Gamma})(V,W) - 2(V\widehat{\Gamma})(W,W)$$
$$-\frac{5}{2}\widehat{\Gamma}(\widehat{\Gamma}(V,W),V) + 2\widehat{\Gamma}(\widehat{\Gamma}(V,V),W) + \frac{1}{2}\widehat{\Gamma}(\widehat{\Gamma}(W,V),V) +$$
$$\frac{1}{2}\widehat{\Gamma}(\widehat{\Gamma}(W,V),W) + \frac{1}{2}\widehat{\Gamma}(\widehat{\Gamma}(V,W),W) - \widehat{\Gamma}(\widehat{\Gamma}(W,W),V) + \frac{1}{2}\widehat{\Gamma}(V,\widehat{\Gamma}(V,W)) +$$

$$\frac{1}{2}\widehat{\Gamma}(W,\widehat{\Gamma}(V,W)) + \frac{1}{2}\widehat{\Gamma}(V,\widehat{\Gamma}(W,V)) +$$
$$\frac{1}{2}\widehat{\Gamma}(W,\widehat{\Gamma}(W,V)) - \widehat{\Gamma}(V,\widehat{\Gamma}(W,W)) - \widehat{\Gamma}(W,\widehat{\Gamma}(V,V)).$$

The kinetic term can be rewritten as

$$2\left((W\widehat{\Gamma})(V,W) - (V\widehat{\Gamma})(W,W)\right) + \left((W\widehat{\Gamma})(V,V) - (V\widehat{\Gamma})(W,V)\right) + 2\left((V\widehat{\Gamma})(W,V) - (V\widehat{\Gamma})(V,W)\right)$$
$$+\left((W\widehat{\Gamma})(W,V) - (W\widehat{\Gamma})(V,W)\right)$$

which suggests for two distinct Riemann tensors and two derivatives of torsion tensors. Further computation yields that

$$K(V,W) = 2\widehat{R}(W,V)W + \widehat{R}(W,V)V + 2\widehat{\nabla}_V T(W,V) + \widehat{\nabla}_W T(W,V) + \frac{1}{2}T(V,T(V,W)) + \frac{1}{2}T(W,T(W,V)).$$

The reader must note here that we used the following defition of the Riemann tensor

$$\widehat{R}(X,Y)Z = \widehat{\nabla}_X\widehat{\nabla}_Y Z - \widehat{\nabla}_Y\widehat{\nabla}_X Z - \widehat{\nabla}_{[X,Y]}Z;$$

Note also that $K(V,\lambda V) = 0$ and the reader immediately calculates that

$$Z(S,Z(V,W,\epsilon),\epsilon) = S + V + W - \frac{\epsilon}{2}\left(T(V,W) + T(S,V) + T(S,W)\right) +$$
$$\frac{\epsilon^2}{6}(K(S,V+W) + K(V,W) + 3T(S,T(V,W)))$$

and therefore

$$D(S,V,W,\epsilon) := Z(Z(S,Z(V,W,\epsilon),\epsilon), -Z(V,Z(S,W,\epsilon),\epsilon)) = -\epsilon T(S,V) +$$
$$\frac{\epsilon^2}{6}\left(K(V,W) + K(S,V+W) - K(V,S+W) - K(S,W)\right) +$$
$$\frac{\epsilon^2}{6}\left(3T(S,T(V,W)) - 3T(V,T(S,W)) + 3T(S+V+W,T(S,V))\right)$$

and the expression of order $\frac{\epsilon^2}{6}$ reduces to

$$2\left(\widehat{R}(S,V)W + \widehat{R}(W,S)V + \widehat{R}(V,W)S\right) + 6\widehat{R}(V,S)W + 3\widehat{R}(V,S)V + 3\widehat{R}(V,S)S + 3\widehat{\nabla}_S T(V,S) + 3\widehat{\nabla}_V T(V,S)$$
$$+\widehat{\nabla}_V T(W,S) + 2\widehat{\nabla}_W T(V,S) - \widehat{\nabla}_S T(W,V) + 3T(S,T(V,W)) - 3T(V,T(S,W)) + 3T(S+V+W,T(S,V))$$

In the absence of torsion, our vectorfield reduces to

$$\frac{\epsilon^2}{2}(2\widehat{R}(V,S)W + \widehat{R}(V,S)V + \widehat{R}(V,S)S).$$

In general, the reader may enjoy observing that $D(S,V,W,\epsilon) = -D(V,S,W,\epsilon)$; in order to eliminate the quadratic terms in the above expression, it is useful to consider

$$E(S,V,W,\epsilon) := D(S,V,W,\epsilon) - D(S,V,-W,\epsilon) =$$

$$\epsilon^2\left(\frac{2}{3}\left(\widehat{R}(S,V)W+\widehat{R}(W,S)V+\widehat{R}(V,W)S\right)-2\widehat{R}(S,V)W+\frac{1}{3}\widehat{\nabla}_V T(W,S)\right)$$

$$+\epsilon^2\left(-\frac{2}{3}\widehat{\nabla}_W T(S,V)+\frac{1}{3}\widehat{\nabla}_S T(V,W)+T(S,T(V,W))+T(V,T(W,S))+T(W,T(S,V))\right)$$

so that we now have a tensor! The reader immediately notices that in the absence of torsion this expresion reduces to

$$-2\epsilon^2\widehat{R}(S,V)W$$

by means of the first Bianchi identity, so we would have isolated the Riemann curvature. In general, the first Bianchi identity reads

$$\widehat{R}(S,V)W+\widehat{R}(W,S)V+\widehat{R}(V,W)S=$$

$$T(T(S,V),W)+T(T(W,S),V)+T(T(V,W),S)+\widehat{\nabla}_S T(V,W)+\widehat{\nabla}_W T(S,V)+\widehat{\nabla}_V T(W,S)$$

so that the above expression reduces to

$$\epsilon^2\left(-2\widehat{R}(S,V)W+\widehat{\nabla}_S T(V,W)+\widehat{\nabla}_V T(W,S)+\frac{1}{3}\left(T(S,T(V,W))+T(V,T(W,S))+T(W,T(S,V))\right)\right).$$

In order to get rid of the torsion terms, the reader may verify that

$$\frac{1}{3}\left(E(S,V,W,\epsilon)+E(W,S,V,\epsilon)+E(V,W,S,\epsilon)\right)=$$

$$\epsilon^2\left(T(S,T(V,W))+T(V,T(W,S))+T(W,T(S,V))\right)$$

using the first Bianchi identity again. So, therefore

$$\frac{8}{9}E(S,V,W,\epsilon)-\frac{1}{9}E(V,W,S,\epsilon)-\frac{1}{9}E(W,S,V,\epsilon)=\epsilon^2\left(-2\widehat{R}(S,V)W+\widehat{\nabla}_S T(V,W)+\widehat{\nabla}_V T(W,S)\right)$$

There is no way to further reduce this and eliminate the remaining derivatives of the Torsion tensor and the reader is invited to play a bit around and consider different sum operations in order to extract those. Finally, we return to the case without torsion, which is considerably easier and we now turn the prescription into our novel language; the reader may verify that to third order in ϵ our definition of $E(S,V,W,\epsilon)$ coincides with

$$E(x,p,q,r):=\left[((x,p)\oplus((x,q)\oplus(x,r)))\ominus((x,q)\oplus((x,p)\oplus(x,r)))\right]\ominus$$

$$\left[\left((x,p)\oplus((x,q)\oplus\tilde{P}(x,r))\right)\ominus\left((x,q)\oplus((x,p)\oplus\tilde{P}(x,r))\right)\right]$$

and we have applied the same limiting procedure as we did tor the torsion tensor previously. The reader may repeat that exercise and define $E(x,p,q,r)(s)$ with $s\in\mathbb{R}_+$ and show that

$$d(E(x,p,q,r)(s))\sim 2s^3||\widehat{R}(S,V)W||.$$

Considering the angle with a reference direction, the entire Riemann tensor may be retrieved in a coordinate independent way. Note also that we have a very nice "arithmetic" interpretation of torsion and curvature; that is, they express the failure of $\oplus$ to be commutative and perhaps associative to some extend. In the next section, we shall abandon the case with torsion and give an entirely different prescription for the Riemann tensor. This treatment shall be more basic and rough, which may not be a bad thing given the connections constructed so far are extremely subtle. We now finish this section by some comments upon differentiability and how the usual bundle apparatus of differential geometry may be generalized to our setting.

Given that we dispose of a local notion of a (non-commutative) sum whos infinitesimal version may very well become commutative and associative as explained previously and moreover, we have a natural notion of scalar multiplication by means of our generalized exponential map which associates to a vector (x, y) a unique geodesic γ in arclength parametrization such that $\gamma(0) = x$ and $\gamma(s) = y$, then we define for any sufficiently small positive real number λ,

$$\lambda(x, y) = (x, \gamma(\lambda s))$$

and in case λ is negative we suggest

$$\lambda(x, y) := (-\lambda)\tilde{P}(x, y)$$

and the reader immediately verifies that these definitions induce the usual ones on the tangent bundle of a manifold. The reader should understand therefore, that it is natural to speak of directions at x defined by means of the geodesics (with respect tot the connection, so they don't need to be the geodesics of the metric) and that also in our general context of a non-commutative and nonlinear sum meaning that

$$\lambda((x, y) \oplus (x, z)) \neq (\lambda(x, y)) \oplus (\lambda(x, z))$$

the very concept of a linearly independent and generating set of directions at x is still a well defined concept albeit I believe this does not imply that each vector can be written in a unique way by means of $\oplus$ and scalar multiplication. So, the concept of a basis is somewhat less restrictive but it is still well defined as a minimal set of independent and generating directions. The dimension is then an ordinary integer defined by the number of basis directions; these observations allow one to transport the entire cnstruction of tangent and cotangent spaces to our setting. But beware, we work very differently here as in the case of the ordinary theory; here it are the connections which determine the tangent bundle as well as its dimension, a much more intrinsic approach as the usual one where the backbone differential structure defines the connections. So, a linear functional, or covector, is defined by means of a continuous functional ω_X on the displacements (x, y) satisfying

$$\frac{1}{\epsilon}\left(\omega_X((x, z) \oplus (x, y)) - \omega_X((x, y)) - \omega((x, z))\right) = 0$$

and

$$\frac{1}{\epsilon}\left(\omega_X(\lambda(x,y)) - \lambda\omega_X((x,y))\right) = 0 \in \mathbb{R}$$

in the limit for $d(x,z) = \epsilon a,\ d(x,y) = \epsilon b$ for $a, b > 0$ constant and $\epsilon \to 0$. Note that we cannot request $\omega_X((y,z) \oplus (x,y)) = \omega_X((x,y)) + \omega((y,z))$ for finite displacements given that the sum operation allows for ambiguities non-locally. Furthermore, if ω_X were a field, then we could define it to constant meaning that

$$\omega_X(\nabla_{(x,y)}(x,z)) = \omega_X((x,z)).$$

Just as in ordinary functional analysis, we can define the weaker notions of continuity and differentiability of functions regarding convergence properties with respect to linear functions which all define semi-norms when suitably rescaled in the infinitesimal limit given by

$$||(x,y)|| := |\omega_X((x,y))|.$$

All proceeds now in a fairly trivial way: given our geodesics (with or without torsion), we have, as mentioned before directions which are endowed with a natural notion of length and angles between them. You can consider generalizations of tensors in those directions which upon suitable rescaling in the infinitesimal limit might become ordinary linear objects. We leave such developments to the reader.

3.1 Riemannian geometry.

In this section, we shall take a very different point of view as in the previous one; the latter was delicate and subtle and very much in line with the standard manifold treatment. Note that we have sidestepped the issue of existence of connections something which seems not totally obvious to prove and might be too delicate for practical purposes. For example, regarding hyperbolic spaces with conical singularities, it is rather obvious that no connection exists at the singular points. To give away the detail, take a couple of equilateral flat triangles (all angles having 60 degrees) and glue them together along their edges such that one has the situation where an interior vertex meets $n > 6$ triangles; in either the internal angle meaure exceeds 360 degrees. Take now any half line starting from the vertex, then it will have an angle of π with all other half lines in a range of $(n-6)2\pi$. Obviously, it is impossible for any mapping to preserve angles when it returns to a normal region where the measure of the circle equals 2π. The situation is the reverse for conical spherical spaces where no mapping towards such points exist. Nevertheless, our coarse grained notion of curvature is still able to capture the curvature around such vertex whereas local curvature fails. I invite the reader to think about this; after all, the integrability condition was together with preservation of distances by far the most important criterion. But it is not sufficient either, so maybe we should be clever enough to find a weaker condition as the preservation of angles which amounts in the manifold case to precisely that. For example, a weaker criterion would be that the angles with the direction of propagation need to be preserved as well as the angles amongst themselves as long as both angles with respect to the direction of propagation are less than π. This definition would certainly fit all path metrical spaces and coincide with the usual lore of differential geometry. This does not change anything to what we have said in the previous section, but merely generalizes the setting to which it be applied. Nevertheless, the downside of the connection theory is that in general it is impossible to give a concrete prescription something which made the Christoffel connection so powerful. There are people who think you should give an easy prescription to calculate curvature even without constructing geodesics which might be a very daunting if not impossible task for a general path metric. Now, I am someone who is very found of geodesics, which are barely manageable in a general curved Riemannian space but I also sympathize with such an idea. The least you should know, I believe are distances and the work done in this section does precisely that. The price to pay is that we cannot speak any longer of vectors, but we have to directly calculate the scalar invariants.

With this in mind, we work now on general path metric spaces (X, d). We have the following definitions:

- Alexandrov curvature: in flat Euclidean geometry, the midpoint r of a line segment $[ab]$ satisfies
$$\vec{xr} = \frac{1}{2}(\vec{xa} + \vec{xb})$$
for any x. Hence, one arrives at
$$d(x,r)^2 = \frac{1}{4}(d(x,a)^2 + d(x,b)^2 + 2d(x,a)d(x,b)\cos(\theta_x(a,b))).$$
We define the nonlocal Alexandrov curvature as
$$R(x,y,z) = \frac{-2d(x,y)^2 - 2d(x,z)^2 + d(y,z)^2 + 4d(x,r)^2}{d(x,y)^2 d(x,z)^2 \sin^2(\theta_x(y,z))}.$$
Taking again geodesic segments between (x,y) and (x,z) parametrized by ϵ and corresponding to the vectors V, W respectively then, as before
$$y = x + \epsilon V - \frac{\epsilon^2}{2}\Gamma(V,V) - \frac{\epsilon^3}{6}\left((V\Gamma)(V,V) - \Gamma(\Gamma(V,V),V) - \Gamma(V,\Gamma(V,V))\right)$$
and
$$d(x,y)^2 = \epsilon^2 h(V,V)$$
by the very property of the exponential map. To find the midpoint between y and z we solve for
$$x + \epsilon V - \frac{\epsilon^2}{2}\Gamma(V,V) - \frac{\epsilon^3}{6}\left((V\Gamma)(V,V) - \Gamma(\Gamma(V,V),V) - \Gamma(V,\Gamma(V,V))\right)$$

$$+\epsilon Z - \frac{\epsilon^2}{2}\Gamma_y(Z,Z) - \frac{\epsilon^3}{6}\left((Z\Gamma_y)(Z,Z) - \Gamma_y(\Gamma_y(Z,Z),Z) + \Gamma_y(Z,\Gamma_y(Z,Z))\right) =$$

$$x + \epsilon W - \frac{\epsilon^2}{2}\Gamma(W,W) - \frac{\epsilon^3}{6}\left((W\Gamma)(W,W) - \Gamma(\Gamma(W,W),W) - \Gamma(W,\Gamma(W,W))\right)$$

leading to

$$Z := W - V + \epsilon\left(\Gamma(V,V) - \Gamma(W,V)\right) +$$

$$\epsilon^2\left(\frac{1}{2}(V\Gamma)(V,V) - \frac{2}{3}(V\Gamma)(W,V) + \frac{1}{3}(V\Gamma)(W,W) + \frac{1}{6}(W\Gamma)(V,V) - \frac{1}{3}(W\Gamma)(W,V)\right)$$

$$+\epsilon^2\left(\frac{2}{3}\Gamma(W,\Gamma(V,V)) - \frac{1}{3}\Gamma(W,\Gamma(W,V)) - \Gamma(V,\Gamma(V,V)) + \frac{1}{3}\Gamma(V,\Gamma(W,W)) + \frac{1}{3}\Gamma(V,\Gamma(V,W))\right).$$

This implies that the midpoint has coordinates, up to third order in ϵ given by

$$r = x + \epsilon\left(\frac{V+W}{2}\right) - \frac{\epsilon^2}{2}\Gamma\left(\frac{V+W}{2}, \frac{V+W}{2}\right) - \frac{\epsilon^3}{6}\left(\frac{V+W}{2}\Gamma\right)\left(\frac{V+W}{2}, \frac{V+W}{2}\right)$$

$$-\frac{\epsilon^3}{6}\left(\frac{1}{2}R(V,W)V + \frac{1}{2}R(W,V)W\right)$$

$$-\frac{\epsilon^3}{6}\left(-2\Gamma\left(\Gamma\left(\frac{V+W}{2}, \frac{V+W}{2}\right), \frac{V+W}{2}\right)\right)$$

This shows that

$$d(x,r)^2 = \frac{\epsilon^2}{4}\left(h(V,V) + h(W,W) + 2h(V,W)\right) - \frac{\epsilon^4}{6}h(R(V,W)V,W) + O(\epsilon)^6$$

and because

$$d(y,z)^2 = \epsilon^2\left(h(V,V) + h(W,W) - 2h(V,W)\right) + \frac{\epsilon^4}{3}h(R(V,W)V,W)$$

the Alexandrov curvature equals

$$-\frac{h(R(V,W)V,W)\epsilon^4 + \ldots}{3\epsilon^4(h(V,V)h(W,W) - h(V,W)^2) + \ldots}$$

which in the limit for ϵ to zero provides for $\frac{1}{3}$ times the sectional curvature. The reader might have guessed this result apart from the front factor based upon the symmetries of the Alexandrov curvature and the Riemann tensor.

- We now arrive to the notion of Riemann curvature; here, we shall ahve to take midpoints of midpoints. To understand why this is the case, consider the following expression

$$h\left(R\left(\frac{V+X}{2}, \frac{W+Y}{2}\right)\frac{V+X}{2}, \frac{W+Y}{2}\right) =$$

$$-\frac{1}{16}\left(h(R(V,W)V,W)+h(R(V,Y)V,Y)+h(R(X,W)X,W)+h(R(X,Y)X,Y)\right)+$$
$$\frac{1}{4}\left(h\left(R\left(\frac{V+X}{2},W\right)\frac{V+X}{2},W\right)+h\left(R\left(V,\frac{W+Y}{2}\right)V,\frac{W+Y}{2}\right)\right)$$
$$+\frac{1}{4}\left(h\left(R\left(X,\frac{W+Y}{2}\right)X,\frac{W+Y}{2}\right)+h\left(R\left(\frac{V+X}{2},Y\right)\frac{V+X}{2},Y\right)\right)$$
$$+\frac{1}{8}h(R(V,Y)X,W)+\frac{1}{8}h(R(X,Y)V,W)$$

Now, to undo the symmetrization in the curvature terms

$$\frac{1}{8}h(R(V,Y)X,W)+\frac{1}{8}h(R(X,Y)V,W)$$

note that by means of the Bianchi identity, this can be rewritten as

$$-\frac{1}{4}h(R(Y,X)V,W)+\frac{1}{8}h(R(V,X)Y,W)$$

so that we have broken the coefficient symmetry. Considering therefore the expression

$$h\left(R\left(\frac{V+X}{2},\frac{W+Y}{2}\right)\frac{V+X}{2},\frac{W+Y}{2}\right)-h\left(R\left(\frac{V+Y}{2},\frac{W+X}{2}\right)\frac{V+Y}{2},\frac{W+X}{2}\right)=$$
$$-\frac{1}{16}\left(h(R(V,Y)V,Y)+h(R(X,W)X,W)-h(R(V,X)V,X)-h(R(Y,W)Y,W)\right)+$$
$$\frac{1}{4}\left(h\left(R\left(\frac{V+X}{2},W\right)\frac{V+X}{2},W\right)+h\left(R\left(V,\frac{W+Y}{2}\right)V,\frac{W+Y}{2}\right)\right)$$
$$-\frac{1}{4}\left(h\left(R\left(\frac{V+Y}{2},W\right)\frac{V+Y}{2},W\right)+h\left(R\left(V,\frac{W+X}{2}\right)V,\frac{W+X}{2}\right)\right)+$$
$$\frac{1}{4}\left(h\left(R\left(X,\frac{W+Y}{2}\right)X,\frac{W+Y}{2}\right)+h\left(R\left(\frac{V+X}{2},Y\right)\frac{V+X}{2},Y\right)\right)$$
$$-\frac{1}{4}\left(h\left(R\left(Y,\frac{W+X}{2}\right)Y,\frac{W+X}{2}\right)+h\left(R\left(\frac{V+Y}{2},X\right)\frac{V+Y}{2},X\right)\right)$$
$$+\frac{3}{8}h(R(X,Y)V,W)$$

which is the result we needed. Denoting by $\widehat{(y,z)}$ the midpoint between y,z, we arrive at the following prescription for the curvature

$$S(x,y,z,p,q)=-8\left(S(x,\widehat{(y,p)},\widehat{(z,q)})-S(x,\widehat{(p,z)},\widehat{(y,q)})\right)$$
$$-\frac{1}{2}\left(S(x,p,z)+S(x,y,q)-S(x,p,y)-S(x,z,q)\right)$$

$$+2\left(S(x,\widehat{(p,y)},q)+S(x,\widehat{(q,z)},p)-S(x,\widehat{(p,z)},q)-S(x,\widehat{(y,q)},p)\right)$$

$$+2\left(S(x,\widehat{(z,q)},y)+S(x,\widehat{(p,y)},z)-S(x,\widehat{(q,y)},z)-S(x,\widehat{(p,z)},y)\right)$$

where

$$S(x,y,z)=-2d(x,y)^2-2d(x,z)^2+d(y,z)^2+4d(x,\widehat{(y,z)})^2.$$

The reader verifies that all symmetries of the Riemann tensor hold, meaning

$$S(x,y,z,p,q)=-S(x,z,y,p,q)=-S(x,y,z,q,p)=S(x,p,q,y,z)$$

and

$$S(x,y,z,p,q)+S(x,p,y,z,q)+S(x,z,p,y,q)=0.$$

This concludes our definition of the Riemann tensor.

- We shall now first define a notion of measure attached to any metric very much like the canonical volume element attached to a Riemannian metric tensor; there are several ways to proceed here. Define for any subset $S\subset X$, the outer measure of scale $\delta>0$ and dimension d as

  $$\mu^d_\delta(S)=\inf\{\sum_i r_i^d|B(x_i,r_i)\text{ is a countable cover of open balls of radius } r_i<\delta \text{ around } x_i \text{ of } S\}.$$

  Obviously, the $\mu^d_\delta(S)$ increase as δ decreases so we define

  $$\mu^d(S)=\lim_{\delta\to 0}\mu^d_\delta(S).$$

  The reader verifies that this defines a measure on the Borel sets of X and morover $\mu^d(S)$ is a decreasing function of d which is infinity for $d=0$, in case X does not consist out of a finite number of points, and 0 for $d=\infty$. Upon defining α as

  $$\alpha=\inf\{d|\mu^d(X)=0\}=\sup\{d|\mu^d(X)=\infty\}$$

  an equality which holds as the reader should prove and it is $\mu^\alpha(S)$ which is of interest. α is called the Hausdorff dimension of X. I invite the reader to "localize" this concept such that one can speak of the local dimension of a space at a point and not just a global one.

- We define now a one parameter family of "scalar products" by means of

  $$g^\epsilon(x,a,b)=\frac{d(x,a)d(x,b)\cos(\theta_x(a,b))}{\epsilon^2}.$$

  The reader notices the scaling here as we shall be interested in taking the limit for ϵ to zero in a well defined way. Note that we could replace the metric compatibility of our connections in the previous section by the single demand that $g^{\epsilon}(x,a,b)$ is preseved under transport meaning that

$$g^{\epsilon}(y,\pi_2(\nabla_{(x,y)}(x,a)),\pi_2(\nabla_{(x,y)}(x,a))) = g^{\epsilon}(x,a,b).$$

We want now, in full analogy with the standard treatment in differential geometry define contractions of the Riemann "tensor" in order to construct the Ricci and Einstein tensor. Note that we do not necessarily dispose of a connection here and therefore we have no addition of vectors, seen as defining a direction. Therefore, we cannot rely upon the notion of a dual tensor associated to our functionals defind in the previous section. Nevertheless, we want to construct a notion of inverse which coincides in the latter cases with the more advanced linear concept. To set the ground for this discussion, note that there exists a natural generalization of the Dirac delta function regarding the Hausdorff measure. That is, there exists a symmetric $\delta(a,b)$ such that for all continuous functions f on X, it holds that

$$\int_X d\mu^{\alpha}(a)\delta(a,b)f(a) = f(b).$$

Defining now the nonlinear dual $\widehat{a}$ of a as

$$\widehat{b}(a) = \delta(a,b)$$

we define inverses $g^{\epsilon}(x,\widehat{a},\widehat{b})$ as

$$\frac{\int_{B(x,\epsilon)} d\mu^{\alpha}(b)g^{\epsilon}(x,\widehat{a},\widehat{b})g^{\epsilon}(x,b,c)}{\mu^{\alpha}(B(x,\epsilon))} = \delta(a,c).$$

The existence of a uniqueness of the inverse follows from the fact that the former defines a Toeplitz operator with trivial kernel. It is to say, $g^{\epsilon}(x,\widehat{a},\widehat{b})$ is the standard Green's function of the metric regarding the Hausdorff measure. This holds of course only if the measure is well behaved and we leave such details to the reader.

Prior to defining contractions with the metric tensor, remark that

$$\int_{B(x,\epsilon)}\int_{B(x,\epsilon)} d\mu^{\alpha}(b)d\mu^{\alpha}(a)g^{\epsilon}(x,\widehat{a},\widehat{b})g^{\epsilon}(x,b,a)$$

is ill defined and requires "a point splitting" procedure to obtain a well defined answer. Concretely, we consider

$$\alpha\frac{\int_{B(x,\epsilon)}\int_{B(x,\epsilon)} d\mu^{\alpha}(b)d\mu^{\alpha}(a)\int_{B(a,\delta)} d\mu^{\alpha}(c)g^{\epsilon}(x,\widehat{c},\widehat{b})g^{\epsilon}(x,b,a)}{\mu^{\alpha}(B(x,\epsilon))^2}$$

an expression which is independent of $\delta > 0$. Note that the dimension α has been added here to restore for the correct trace.

- The reader may now define the rescaled Riemann curvature tensors $S(x, y, z, p, q, \epsilon) := \frac{S(x,y,z,p,q)}{\epsilon^4}$ and consider contractions with $g^{\epsilon}(\widehat{y}, \widehat{q})$ to define the Ricci tensor $S(x, z, p, \epsilon)$ and from thereon the Ricci scalar. We leave this as an exercise to the reader.

Chapter 4

Quantum metrics.

As before, we consider a quantum space $(\mathcal{S}, \wedge, \vee, +, .)$ as a space of Hermitian projection operators on Hilbert space $\mathcal{H}$ carrying, besides ordinary algebra, notions of quantal logic. Then, a metric geometry is characterized by a bi-function

$$d : \mathcal{S} \times \mathcal{S} \to \mathcal{H}^+$$

satisfying

$$d(P,P) = 0,\ d(P,Q) = d(Q,P) > 0,\ \tilde{d}(P,Q) + \tilde{d}(Q,R) \geq \tilde{d}(P,R)$$

where $\tilde{d}$ is a classical metric associated to d in a way to be explained below. Here $\mathcal{H}^+$ is the cone of all positive operators. One notices furthermore that the so-called inner algebra automorphisms $\alpha(A) = gAg^\dagger$ with $gg^\dagger = g^\dagger g = 1$ which correspond to a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$, in case $\mathcal{H} = L^2(\mathcal{M}, \mu)$ for some compact manifold $\mathcal{M}$ and measure μ such that the measure is preserved, meaning $\phi_\star \mu = \mu$, act on functions f by means of $g(f) = f \circ \phi$. An easy calculation reveals that

$$\int_{\mathcal{M}} d\mu(x)\overline{f(\phi(x))}g(\phi(x)) = \int_{\mathcal{M}} d(\phi_\star^{-1}\mu)(y)\overline{f(y)}g(y) = \int_{\mathcal{M}} d\mu(y)\overline{f(y)}g(y)$$

where in the second step we have used the usual diffeomorphism invariance of the integral and in the third step the invariance of the measure, which shows that g is unitary. Of course not every unitary transformation corresponds to a diffeomorphism. If, moreover, the distance function is preserved, then we recover the Killing fields.

The reader must have noticed that in the above, we spoke about a classical distance attached to a quantum one. Indeed, this is a crucial aspect of our vision, which is that the quantum world needs a classical backbone, something which has been disussed at length by the founding fathers of quantum mechanics such as Niels Bohr and Werner Heisenberg. We shall proceed therefore in the opposite way and start from a classical geometry and show how, within an appropriate language, it can be made into a quantum one. To start out simple, consider flat $n+1$ dimensional Euclidean space compactified on a $n+1$-dimensional torus with length between $-L$ and L in every orthonormal direction. Points are determined by distributional states $\delta(x-z) := |x\rangle$ where the last symbol is the Dirac notation of a vector and the unit, or identity operator, is given by

$$1 = \int dx |x\rangle\langle x|.$$

Consider a translation $T(h)$ defined by $(T(h)f)(x) = f(x+h)$ where the sum operation is taken modulo $2L$, then $T(h)$ is a unitary operator and $T(h)^{\dagger} = T(-h)$. Therefore, $T(h) + T(-h)$ is self adjoint and one sees that

$$0 \leq (T(h) \pm T(-h))^2 = T(h)^2 + T(-h)^2 \pm 2$$

and therefore

$$-2 \leq T(h)^2 + T(-h)^2 \leq 2.$$

Also, we have that $T_r T_s = T_{r+s}$ and therefore

$$\int_{T^{[0,L]}_{n+1}} dr \int_{T^{[0,L]}_{n+1}} ds\, r.s\, T_{-r} T_s = \int_{T^{[0,L]}_{n+1}} dr \int_{T^{[-L,L]}_{n+1}} ds\, (r+s).r\, T_s$$

$$= \int_{T^{[0,L]}_{n+1}} dr \int_{T^{[-L,L]}_{n+1}} ds\, (r^2 + r.s)\, T_s =$$

$$\left(\int_{T^{[0,L]}_{n+1}} dr r^2 \right) \left(\int_{T^{[-L,L]}_{n+1}} ds\, T_s \right) + \frac{L^{n+2}}{2} \int_{T^{[-L,L]}_{n+1}} (1,1,1,\ldots,1).s T_s$$

which is a positive operator. Moreover,

$$E = \int_{T^{[-L,L]}_{n+1}} ds\, T_s$$

obeys $E T_t = T_t E$ as well as $E^2 = 2^{n+1}(2L)^{n+1} E$ given that we have periodic boundary conditions and therefore E must equal the (distributional if $L = \infty$) matrix

$$2^{n+1} |1\rangle\langle 1|$$

given that $\langle 1|1\rangle = (2L)^{n+1}$ in the functional representation. Hence, our formula reads

$$2^{n+1} \left(\int_{T^{[0,L]}_{n+1}} dr r^2 \right) |1\rangle\langle 1| + \frac{L^{n+2}}{2} \int_{T^{[-L,L]}_{n+1}} (1,1,1,\ldots,1).s T_s$$

which is the expression of our concern. Therefore, a good definition of an operator valued distance d is determined by the "positive scalar product" operator

$$\langle A|B\rangle_{\text{op}} = \int_{T^{[0,L]}_{n+1}} dr \int_{T^{[0,L]}_{n+1}} ds\, r.s\, A^{\dagger} T_{-r} T_s B.$$

Notice that the quantity

$$C(|x\rangle\langle x|, |y\rangle\langle y|) = \int_{T^{[-L,L]}_{n+1}} dh h^2 |x\rangle\langle x|T(h)|y\rangle\langle y|$$

equals

$$C(|x\rangle\langle x|, |y\rangle\langle y|) = \tilde{d}(x,y)^2 \, |x\rangle\langle y|$$

where $\tilde{d}$ is the distance on the $n+1$ torus. By definition, the distance formula equals

$$d(|x\rangle\langle x|, |y\rangle\langle y|)^2 = \int_{T^{[0,L]}_{n+1}} dr \int_{T^{[0,L]}_{n+1}} ds \, r.s \, (|y\rangle\langle y| - |x\rangle\langle x|) \, T_{r-s} \, (|y\rangle\langle y| - |x\rangle\langle x|) =$$

$$\frac{(n+1)L^{n+3}}{3} (|x\rangle\langle x| + |y\rangle\langle y| - |x\rangle\langle y| - |y\rangle\langle x|) + \frac{1}{2} \int_{T^{[0,L]}_{n+1}} [(x-y)].s \, (|x\rangle\langle y| + |y\rangle\langle x|) \, .$$

Unfortunately, this does not provide for the correct distance function; a better definition would be

$$d(|x\rangle\langle x|, |y\rangle\langle y|)^2 = -\int_{T^{[-L,L]}_{n+1}} dr \, r^2 \, (|y\rangle\langle y| - |x\rangle\langle x|) \, T_r \, (|y\rangle\langle y| - |x\rangle\langle x|)$$

which is a Hermitian quantity due to $T_r^\dagger = T_{-r}$ and change of integration variable $r \to -r$ but not a positive definite one since

$$d(|x\rangle\langle x|, |y\rangle\langle y|)^2 = \tilde{d}(x,y)^2 \, (|x\rangle\langle y| + |y\rangle\langle x|)$$

and the right hand side is an operator with mixed positive negative eigenvalues. A natural root, up to a factor ∞, would be provided by

$$d(|x\rangle\langle x|, |y\rangle\langle y|) = \frac{1}{2}\tilde{d}(x,y) \, ((|x\rangle + |y\rangle)(\langle x| + \langle y|) - (|x\rangle - |y\rangle)(\langle x| - \langle y|)) \, .$$

Insisting upon retaining positivity of d as well as keeping the quantity relational forces one to consider only the second part of this expression, in either

$$\widehat{d}(|x\rangle\langle x|, |y\rangle\langle y|) = \tilde{d}(x,y)(|x\rangle - |y\rangle)(\langle x| - \langle y|)$$

and it remains to verify the triangle inequality

$$B(x,y) := \widetilde{d}(x,y) \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, we obtain a quantum triangle inequality

$$B(x,y) + B(y,z) - B(x,z) \sim \begin{pmatrix} \widetilde{d}(x,y) - \widetilde{d}(x,z) & -\widetilde{d}(x,y) & \widetilde{d}(x,z) \\ -\widetilde{d}(x,y) & \widetilde{d}(x,y) + \widetilde{d}(y,z) & \widetilde{d}(y,z) \\ \widetilde{d}(x,z) & \widetilde{d}(y,z) & \widetilde{d}(y,z) - \widetilde{d}(x,z) \end{pmatrix}$$

which is not necessarily a negative definite matrix due to the determinant test. Hence,

$$\widetilde{d}(x,y) + \widetilde{d}(y,z) \geq \widetilde{d}(x,z)$$

is not conclusive for $\widehat{d}$ which justifies our axiom that the triangle inequality merely needs to be obeyed classically. However, this problem may be circumvented by noticing that the expression

$$d(|x\rangle, |y\rangle)^2 := -(\langle x| - \langle y|)\frac{1}{2}\int_{T_n^{[-L,L]}} ds\, ||s||^2\, T_s(|x\rangle - |y\rangle) = \widetilde{d}(x,y)^2$$

which is clearly a satisfying formula. Evidently,

$$d(|x\rangle, |y\rangle)$$

restricted to those "atomic" states satisfies the full triangle inequality given that $\widetilde{d}$ does. This suggests different results for scalar valued quantal distances on states than it does for density matrices.

4.1 Differential geometry.

In the previous section we has cast flat, compactified, Euclidean geometry into a new functional analytic jacket.. Quantum geometry obviously necessitates spaeaking in a weaker (distributional) sense about points given that they are "atomistic" in a much weaker sense than it is for classical vectors projection operators in the Hilbert algebra of fuctions on a metric space. The reader must have noticed by now that

$$\widetilde{d}(x,y) = d(|x\rangle, |y\rangle) = \frac{1}{4}(\langle x| - \langle y|)\int_{\times_n [-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dh\, T_h\, \widehat{d}(|x\rangle\langle x|, |y\rangle\langle y|)\int_{\times_n [-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dh\, T_h\,(|x\rangle - |y\rangle)$$

where the integration in the last formula serves to eliminate the factor of ∞ in the distributional scalar products. Therefore, it is appealing to consider

$$\triangle(x,y;z) := d(|x\rangle\langle x|, |y\rangle\langle y|) + d(|y\rangle\langle y|, |z\rangle\langle z|) - d(|x\rangle\langle x|, |z\rangle\langle z|)$$

but it cannot be related to the previous formula due to the alternating character of $|x\rangle - |y\rangle + |z\rangle$ providing for $|x\rangle + |z\rangle$ in the sandwish for $\widehat{d}(|x\rangle\langle x|, |y\rangle\langle y|)$. This can be remedied by considering the vector $|x\rangle + e^{i\frac{\pi}{3}}|y\rangle + e^{i\frac{2\pi}{3}}|z\rangle$ providing for an expression of the kind

$$\widetilde{d}(x,y) + \widetilde{d}(y,z) - \widetilde{d}(x,z)$$

such that one has that

$$(\langle x| + e^{-i\frac{\pi}{3}}\langle y| + e^{-i\frac{2\pi}{3}}\langle z|)\int_{\times_n [-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dh\, T_h\, \triangle(x,y;z)\int_{\times_n [-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dh\, T_h\,(|x\rangle + e^{i\frac{\pi}{3}}|y\rangle + e^{i\frac{2\pi}{3}}|y\rangle) \geq 0$$

for ϵ suffiiently small due to the classical triangle inequality. So, this is our classical-quantum correspondance: for a general state over all points the triangle inequality is not obvious.

The reader must correctly understand that underlying the quantum geometry is a fixed classical one just as is the case in this author's work on quantum gravity. We now generalize this work to a curved classical background by means of the exponential map which is after all immediately determined by the geodesic equation and vierbein and generalizes the idea of a translation group towards abelian group bundles and non-abelian semigroups. That is, locally, we may write

$$T_{[T_x(v)]}(w) = T_x((w \oplus v)_x)$$

where $w \oplus v$ is uniquely given if we demand that geodesics do not leave a certain open region $\mathcal{O}$ around x and $T_x(v) = \exp_x(v)$. On the other hand $T_x(v)$ may be thought of as representing a translation on the tangent space at x in which case the usual law

$$T_x(w)T_x(v) = T_x(v+w)$$

holds. We shall be interested in the first representation which is isomorphic to the second in flat Euclidean space with respect to a global inertial frame so that there, the x dependency can be dropped in T_x as well as $\oplus_x = +$. Specifically, the global action T is

$$(T(v)f)(x) := f(T_x(v(x)))$$

where $v(x)$ is a vectorfield on $\mathcal{M}$. The element $v(x)$, seen as an ultralocal vector, may also serve as $T_{v(x)}$ on the flat geometry modelled at x. It is the exponential map which connects both representations as we shall see soon. One also has

$$[T(w)(T(v)f)](x) := [T(v \oplus w)f](x) = f(T_x(v \oplus w)_x) = f(T_{T_x(w(x))}(v(T_x(w(x))))).$$

Therefore, the right framework for curved geometry is the one of the induced non-abelian sum on the vectorfields something which has been extensively used in the previous chapter where we studied general connections (in this case, the notion of $\oplus$ was different as herein since we did not dispose at that point of a tangent space[1]). This calls for an extension of our previous setting; one could work with the Hilbert-algebra $\mathcal{H}$ of functions on $\mathcal{M}$ where $\mathcal{M}$ is compact, equipped with the real *Leibniz* topological dual $\mathcal{H}^{\star,L}$ on $\mathcal{M}$ defined by the continuous, real linear functionals D satisfying[2]

$$D(fg) = D(f)g + fD(g).$$

The Leibniz rule is there to ensure the locality aspect and enables one to define $D(x)$ which is what we need; notice that the previous definition of $\mathcal{H}^{\star,L}$ does

[1]The reader may enjoy generalizing the content below in that more general framework of chapter three.

[2]Hence $\mathcal{H}^{\star,L}$ equals $C^\infty(\mathcal{M})$ equipped with something as the Schwartz topology to ensure continuity of D.

not depend upon the choice of H whereas quantum mechanically it might (when taking the closure in a suitable scalar product). Given that $\mathcal{H}^{\star,L}$ is infinite dimensional, we cannot integrate over it. Note that we have something as a pull back defined by

$$f_\star D$$

where $[(f_\star D)(g)](x) = [D(g \circ f^{-1})](f(x))$ for $f \in \mathrm{Diff}(\mathcal{M})$ which is an "automorphism" or derivation of $\mathcal{H}^{\star,L}$. Formulated more algebraically, every automorphism χ of $\mathcal{H}^{\star,L}$ induces a whole class of mappings $\chi^n \circ \chi_\star D : \mathcal{H}^{\star,L} \to \mathcal{H}^{\star,L}$ by means of

$$[\chi^n \circ \chi_\star D](f) = \chi^n[D(\chi^{-1}(f))].$$

Indeed, one checks that

$$[\chi_\star D](fg) = \chi[D(\chi^{-1}[fg])] = \chi^n[D(\chi^{-1}(f))]\chi^{n-1}(g) + \chi^{n-1} f \chi^n[D(\chi^{-1}[g])]$$

which shows its sanity and such concept has been previously developed by Michor. As in the previous chapter, is better to fix a point x and drag $D(x)$ along the geodesics emanating from it. This prepares the setting for a generalization of the geometry defined in the previous section. The crucial part is to use the standard spectral theorem on $\mathcal{H}$ to know that every element can be written as a sum of complex multiples of Hermitian idempotents which in their turn can be written as an integral of distributional atomistic characters (a Hilbert algebra is a commutative $C^\star$ algebra as well as a Hilbert space, where the $C^\star$ algebra is represented on itself). Therefore, the position "basis" of atoms always is a basis of orthogonal elements in the general distributional sense. A classical metric is defined in the following way: pick a point x and a scalar product $h_x(v(x), w(x))$ on $\mathcal{H}^{\star,L}(x)$ which we assume locally to be isomorphic, as a vector space, to $\mathbb{R}^n$. The pull back of h_x is defined as

$$(\chi_\star h)_{\chi^{-1}(x)}((\chi_\star v)_{\chi^{-1}(x)}, (\chi_\star w)_{\chi^{-1}(x)}) := h_x(v(x), w(x)).$$

If one were to define the h field outside of x by means of

$$[(T_x^{-1}(v))_\star h]_{T_x(v)} = h_x$$

or

$$h_{T_x(v)} = (T_x)_\star(v) h_x.$$

To rectify this, note that T defines the full connection and therefore the parallel transporter which we denote with $\widehat{T}$. With conventions pointed out as in the proceeding chapter and all explicit calculations therein, we arrive that

$$(\epsilon v) \oplus (\epsilon w) = \epsilon(v + w) + O(\epsilon^3).$$

As is well known from differential geometry, this issue *does* depend upon the choice of h_x if the latter is nondegenerate and symmetric and of fixed signature. Indeed, take a matrix field $O(x)$, then the connection associated to $O(x)g(x)O^T(x)$ is given by

$$O(x)\gamma(x)O^T(x) \otimes O^T(x) + \frac{1}{2}(O^T)^{-1} g^{-1} O^{-1}(\text{first derivatives of O}).$$

There are in general $\frac{n^2(n+1)}{2}$ equations and n^2 variables so that inconsistencies arise. This issue is pretty easily solved by demanding that

$$\lim_{\epsilon \to 0} \frac{\widehat{T}_{\epsilon v} h - h}{\epsilon} = 0$$

for the appropriate metric h and any field $v \in \mathcal{H}^{\star, L}$. Consistency then implies that

$$\lim_{\epsilon \to 0} \frac{\widehat{T}_{\epsilon v} \widehat{T}_{\epsilon w} h - \widehat{T}_{\epsilon(v+w)} h}{\epsilon^2} = 0$$

for any fields v, w and the two conditions on T which define one parameter subgroups and restrict the coincidental behaviour of $\oplus$, together with the fact that $\widehat{T}$ must define an infinitesimal isometry of the metric field, fix the classical geometry entirely.

Taking this excessive infomation regarding the translations into account it becomes utterly clear how to define the quantum distance at hand on a general metrically complete manifold of any topology using the existence of a distance minimizing geodesic (Hopf-Rinow theorem). However, we still rely upon the geometrical concept of a classical geodeic while doing quantum geometry and I do not comprehend how one could uplift this limitation. A straightforwar suggestion would be to proceed with non-abelian semigroups defined on vector (fields) using the connections defined in the previous chapter and take that as the backbone of a non commutative geometry; however, the functional language employed here does not really fundamentally contribute in that regard. Perhaps, there is some slight way out here if one were to generalize this connection therory even further algebraically.