Proceedings on non commutative geometry.

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Contents

Some words upfront.

This little book is meant as a generalization of classical topology and geometry into the realm of non commutative topologies as well as geometries. These are objects of the operational kind where a space is glued together and the gluing procedure does not obey the commutativity and associativity properties of the set theoretical union. That is to say, there is substance to space like there are two sides to a wooden plate and space is not something which exists and void of properties at the same time. Likewise is this so for the distance function where measurement, or the act thereof, depends upon previous measurements leaving a trace in the non-abelian dust. This is a property which holds for sure in nature albeit we know the dust to be very subtle regarding our senses. We do not see neither feel it, but it is there and a necessary aspect for the creation of life. Until a few decades ago, physicists have righteously ignored these small aspects of geometrical sensitivity but it has become the time to investigate them properly. There exist a few distinct proposals from the mathematical side of how to deform the classical situation; for example Alain Connes focuses on function algebra's and Dirac operators. It is well known that the classical limit problem herein is well posed and answered for albeit the general non-abelian situation has no obvious geometrical interpretation and it appears way too general in order for it to be useful. Another approach was taken by Majid, Vaes and others and hinges upon the concept of a Hecke algebra which is an object unifying a Lie-algebra and a Lie-group; an approach which seems certainly useful for highly symmetrical spaces such as Minkowski or (anti) de Sitter. It is nevertheless still grounded in the concept of "inertial coordinates" and generalizations towards curved geometry are highly suspicious and confused. The approach taken here resembles the one taken in my book on "geometrical quantum theory and applications" where classical physics is written as a peculiar case in the quantum language and quantum theory is rather seen as a bi-dual theirof. That is, there is a one to one mapping between a classical metric space and the quantization theirof. We shall go much further here and study connection theory from an abstract global point of view and develop quantum connections with differentiable sections. The book is short and intended for the beginning researcher; everything which follows is exactly defined and all relevant properties proven and commented upon.

Chapter 1

Classical logic and topology.

Spaces are usually defined as consisting out of elements and being composed by gluing of "standard" elements together. This requires cut and paste operations equivalent to taking the intersection $\cap$ and union $\cup$. In standard mathematics, we assume those operations to be *perfect* meaning there is no waste as well as no preference for how or the order in which they are performed. This last stance translates as $A \cap B = B \cap A$ as well as $A \cup B = B \cup A$ both properties being referred to as the commutativity of the respective operations. This is not necessarily so in nature, it does matter for example when I poor coffee in first in a bowl and then hot water later on. In this case the coffee dissolves and raises upwards causing for a homogeneous mixture. If I were to do it the other way around, the coffee would most likely keep on floating on the water. So this commutativity of the union is not obvious, it refers to the fact that items are hard objects and no particular law holds between them. They are independent as to speak; this stance of individualism is required in science, we would not learn anything from a holistic perspective. We have to subdivide and believe in holy freedom otherwise nothing can be said about the I and its relations to others. We moreover insist those operations to binary meaning that $(A \cap B) \cap C = A \cap (B \cap C)$ and likewise so for the union, a property which we call associativity of the respective operation. Now, we can talk! Denote with $A, B, C, \ldots$ so called sets; we have no idea yet what they are but we shall further specify some properties regarding the operations $\cap$ and $\cup$. The operations satisfy for sure $A \cap A = A \cup A = A$ and we demand the existence of a unique empty set $\emptyset$ such that

$$\begin{aligned} A \cap \emptyset &= \emptyset \\ A \cup \emptyset &= A \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

where this last rule is the same as the de-Morgan rule in Boolean logic. Set theory at this level is equivalent to the rules of classical logic where the A denote truisms and $\emptyset$ is given by "false". Then $A \cap A = A$ reads as A and

A are both true is the same as A is true. A or A is true, denoted by $A \cup A$ is the same as A is true. A and false is always false whereas the vericacity of A or false just depends upon A. Finally A is true and B or C is true is the same as A and B is true or A and C is true. So set theory is classical logic, it is a definite speech about truisms of belonging to. We will later on think of devilish ways to escape this definite way of speaking about things which hinges upon many assumptions which could equally well be false. However, just as is the case for Greek and Roman architecture, the most simple rules can allow for very complicated ones to arise by means of building. The old Greek alway described elements or atomos as things which cannot be further subdivided; hence the following definitions. We say that A is a subset of B if and only if the intersection of A and B equals A which reads as $A \subseteq B \leftrightarrow A \cap B = A$. An atom $A \neq \emptyset$ is called a primitive set, that is, A has the property that if $B \subseteq A$ then $B = A$. The reader checks the obvious statement that $A \cap C \neq \emptyset$ is a subset of A; this follows from associativity and commutativity of the intersection because $A \cap (A \cap C) = (A \cap A) \cap C = A \cap C$ and therefore, by definition $A = A \cap C \subseteq C$ in case A is an atom or primitive set. Indeed, we can only speak of subparts when the operation of intersection is priceless. This suggests that primitive sets are as elements of a set and to emphasize that distinction we denote $A = \{\widehat{A}\}$ where $\widehat{A}$ is interpreted as an element and the brackets denote the bag. We use the symbolic notation $\widehat{A} \in B$ as an equivalent to the more primitive statement $A \cap B = A$.

The reader notices that we have *defined* elements from the operations $\cap, \cup$ whereas normally the opposite happens; you cannot crumble the bread further than up to its elementary fibers. This is a much more human way of dealing with language in the sense that the limitations attached to our operations define our notion of reality. The old approach starts from divine knowledge which nobody possesses; in order to make logic dynamical and attached to physical processes in space-time, mathematicians have invented the notion of a Heyting algebra instead of a Boolean one. We shall not go that far in this book but the interested reader should comprehend very well how this definition is tied to the one of classical relativistic causality. Our point of view also allows for quantal rules as long as the de-Morgan rule is suitably deformed; we shall discuss such logic in this book and make even further extensions towards non-associative and non-commutative cases. Extension of the material presented is left to the fantasy of the gifted reader. For example, an infinite straight line does not need to consist out of points, the latter being mere abstractions. Let us first investigate further implications of our rules before we move on to further limitation of the setting at hand. It is true that if $B \subseteq C$ then every element $\widehat{A}$ in B belongs to C. Indeed, $\widehat{A} \in B$ if and only if $A \cap B = A$ and therefore $A \cap C = (A \cap B) \cap C = A \cap (B \cap C) = A \cap B = A$ proving that $A \cap C = A$ and therefore $\widehat{A} \in C$. Differently, $\widehat{A} \in B$ if and only if $A \cap B = A$ which is equivalent to $(A \cap C) \cap B = A$ and therefore $A \cap C \neq \emptyset$ from which follows that $A \cap C = A$ because A is an atom. Hence, elements of subsets belong to the set itself. What about the intersection of two sets? First, we show that if $\widehat{A} \in B, C$ then $\widehat{A} \in B \cap C$: this holds because $A \cap (B \cap C) = (A \cap B) \cap C = A \cap C = A$ and therefore $\widehat{A} \in B \cap C$. The other way around, we have that if $\widehat{A} \in B \cap C$ then $\widehat{A} \in B, C$ because the intersection is a subset of both. Hence, the elements in the intersection are precisely correspond to those which are in both of them. What about the union? We show that if $\widehat{A} \in B \cup C$ then either $\widehat{A} \in B$ or $\widehat{A} \in C$ because $A = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ implying that at least one of them is non empty and equal to A due to atomicity of the latter. Reversely, one has that if $\widehat{A} \in B$ then it is an element of $B \cup C$ because $A \cap (B \cup C) = A \cup (A \cap C)$ which equals $A \cup A$ or $A \cup \emptyset$ due to atomicity of A. In both cases we have that $A \cap (B \cup C) = A$ because $A \cup \emptyset = A = A \cup A$. Therefore, the elements in the union are in correspondence to the elements of one of the sets.

As suggested previously this does not imply that sets are fully specified by their elements nor that elements exist in the first place. For example, assume that $\mathcal{S}$ consists of $\emptyset, \{1\}, \{1, 2\}$, then $\{1\}$ is an atom, but $\{1, 2\}$ does not merely consist out of atoms. Standard set theory makes the assumption that

$$B = \{\widehat{A} | \widehat{A} \in B\}$$

meaning that a set equals a collection of its elements. In this case, we have just proved that $\cap$ and $\cup$ coincide with the usual operations of intersection and union. The reader might think this is all a bit abstract and utter "well, can I just not assume this without all these rules?". The simple answer is "no"; mathematicians are very scarce on their assumptions indeed! Why writing an extra sentence into the constitution when the latter is already a consequence of the former rules?! The next question one could pose then is "well on then, but how do you make up for all these theorems as well as the formal proofs?". The simple answer is that the results have to be in your mind prior to making up the concepts! A proof is no more as a logical confirmation of a kind of naturalistic observation in a way. Henceforth, it is merely an exercise to verify that the concepts lead to the appropriate results. This applies in the case of set theory due to the existence of the natural concept of an atom being equivalent to an element.

These are by far not the only rules of set theory which we shall slowly expand upon by means of more complicated objects and operations. Let us now deviate a bit and reflect further upon the commutation and associative properties of the intersection as well as union. We imagined that a set can be thought of as items in a bag; however, in reality our bag is a phantom bag given that the operations of emptying and resorting do not matter in taking the intersection or union. This would lead to complications involving the order of operations leading to a non-commutative logic which we shall study later on in this book. A true Frenchman would expect such rule to emerge in a way from the simple ones and indeed this is the case. Another field where such a thing happens is

Riemannian geometry which is a generalization of flat Euclidean geometry.

We define the natural numbers n by means of the sum operation $n = 1 + 1 + 1 + 1 + \ldots + 1$ by means of the following prescription:

$$\begin{aligned} 0 &= \{\emptyset\} \\ n+1 &= \{n, \emptyset\}. \end{aligned}$$

Hence, $1 = \{\{\emptyset\}, \emptyset\}$, $2 = \{\{\{\emptyset\}, \emptyset\}, \emptyset\}$ etcetera; this is a partial dictionary made out the symbols $\emptyset, \{, \}$ which are part of any set theory. I have warned the reader that symbolic notation often is the most difficult part of set theory and the latter notation allows for a definition comprehensible by a computer albeit the latter uses binary representations. We define in the same way $n + m$ by means of the prescription $n + (m + 1) = \{(n + m), \emptyset\}$ where $n + 0 = 0 + n = n$. The reader shows that $n + m = m + n$ for every natural number m which is true by definition for $m = 0$. Indeed, suppose it is true for $m = k$, then we show it holds for $m = k + 1$. Indeed, $n + (k + 1) = \{n + k, \emptyset\} = \{k + n, \emptyset\} = \{k + (1 + (n - 1)), \emptyset\} = \{(k + 1) + (n - 1), \emptyset\} = (k + 1) + n$ where, in the first step, we have used the definition of the natural numbers, in the second the assumption that $k + n = n + k$ and finally, in the third step, the associativity of $+$. We pose that $\mathbb{N}$ is the set of all natural numbers, something which defines a set theory by means of taking all subsets of $\mathbb{N}$.

The operation $+$ maps two natural numbers onto a natural number; it is associative, commutative and has 0 as a neutral element implying that $0+n = n+0 = n$. For any n, it is possible to define an inverse $-n$ satisfying $n + (-n) = 0 = (-n) + n$ something we denote by $n - n = 0$; $n + (-m) = n - m$ is a natural number $n > m$ and minus a natural number if $n < m$. The set of natural numbers taken together with their inverse is called the entire numbers and is universally denoted by $\mathbb{Z}$. $\mathbb{Z}, +$ is called a commutative group given that the operation $+$ is interior, associative, has a neutral element and inverse.

As previously stated, one starts by making a distinction between elements of a set and sets themselves; we departed from the concept of an empty set $\emptyset$, the intersection and union and therefrom we deducted the first three axioms of set theory. The approach taken here is somewhat more general as we defined an element as a primitive set. Zermelo-Frankel set theory has plenty of more assumptions which have to do with infinity culminating into the axiom of choice which we shall prove to be incompatible with the other, much more plausible axioms resulting into the real number system. A fifth axiom deals with taking set theoretical differences

$$B \setminus C = \{\widehat{A} | \widehat{A} \in B \wedge \widehat{A} \notin C\}$$

and we shall always assume the difference set to exist. In the field of geometry, it is not only possible to take the union of two lines or the intersection theirof but we can also take the so called Cartesian product, defining a two dimensional sheet. More in particular, given two sets B, C, we define the Cartesian product $B \times C$ as the *set* of all tuples (x, y) such that $x \in B$ and $y \in C$ giving a six'th axiom in $\mathcal{S}$ and henceforth is this last one closed with respect to $\times$ from which holds

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

and

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

The existence of Cartesian products allows us to define relations where a *relationship* R between sets B and C constitutes a subset of $B \times C$. In case $B = C$ we can demand plenty of criteria. With the notation xRy we intend to say that x has a relation of type R to y if and only if $(x, y) \in R$; we call R reflexive if xRx for all $x \in B$, symmetric if xRy implies that yRx for all $x, y \in B$ and finally transitive if xRy and yRz imply that xRz. A reflexive, anti-symmetric, transitive relation is called to be a partial order and is noted by $\prec$ or $\leq$. A reflexive, symmetric and transitive relation is called an equivalence relation and is usually denoted by $\equiv$. One should think of an equivalence relation as a generalization of the equality sign given that it concerns objects with similar properties. One should prove that an equivalence relationship defined on a set A pulverizes it in equivalence classes $\overline{x}$ where

$$\overline{x} = \{y \in A | x \equiv y\}.$$

The reader verifies that $\overline{x} = \overline{y}$ if and only if $x \equiv y$ and therefore the intersection $\overline{x} \cap \overline{y} = \emptyset$ if they are not equivalent. A partial order is a generalization of a total order such as "Jon is larger as Elsa". A partial order allows for two objects to be not related at all.

We have defined the natural numbers by means of the operation $+$; $\mathbb{N}$ has a natural *total* order $\leq$ defined by $n \leq n$ and $n \leq n + 1$ and one takes the *transitive closure* therefrom which is defined by imposing transitivity on the existing relationship. This can be compared with lacing a chain. From the natural numbers we constructed the entire numbers $\mathbb{Z}$ and the definition of $\leq$ has a natural extension towards $\mathbb{Z}$. We construct now the rational numbers starting from $\mathbb{Z} \times \mathbb{N}_0$ and imposing the equivalence relationship $(m, n) \equiv (m', n')$ if and only if there exist a $k, l \in \mathbb{N}_0$ such that $km = lm'$, $kn = ln'$ where $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$. De *rational* numbers are henceforth defined as the equivalence classes defined by means of this *equivalence* relation.

The six axioms discussed are by far the most important ones of set theory and allow one to construct the rational numbers; the remaining two axioms concern infinity and are in general added to generalize aspects of the rational numbers to the real ones. We shall be very cautious here with the kind of infinity we shall allow for culminating into a thorough discussion of the axiom of choice. In fact, we shall demonstrate that this highly contested axiom is wrongly chosen in the sense that it contradicts the existence of the real numbers. The kind of mathematics required here is not at its place in this chapter; however, it is presented here for matters of completeness and the reader is invited to thoroughly check the details later on. The seven'th axiom allows one to define subsets of sets: given a set D, the power set 2^D of all nontrivial subsets of D is a set and belongs to $\mathcal{S}$. This axiom leads to the construction of the ordinary numbers by Cantor. The definition the Cartesian product is extended to so called "index" sets something which requires a partial order $\prec$. An index set I is a set equipped with a partial order $\prec$ such that for any $x, y \in I$ there exists a $z \in I$ such that $x, y \prec z$. This condition is required and sufficient if we want to take unique limits such any reader should check. If this is not valid, then several sub limits could exist; hence, we denote by

$$\times_{i \in I} A_i = \{(x_i)_{i \in I} | x_i \in A_i\}$$

where all I-tuples are partially ordered by $\prec$. Finally, we have the so called axiom of choice which can be formulated as follows: given sets A_i, $i \in I$, with I an index set, then the Cartesian product is nonempty. Another, but equivalent formulation is that there exists a function f from I to $\cup_{i \in I} A_i$ such that $f(i) \in A_i$. So, one can constitute a set by drawing an element from each set. This axiom has plenty of ramifications in some parts of mathematics, in particular functional analysis although some mathematicians have refuted it because some results appear too strong and give the transfinite an equal status to the finite situation. I have stated it already a few times: mathematics as such is not open to proof; it is a language and we have to make some grammatical choices. The reader has to reflect about these rules en be conscious of the fact that commutativity, associativity as well as the formation of a power set are the most simple of all symmetrical rules. An example which does not obey these rules has been constructed from this ideal situation; for example, we shall study later on non commutative or associative operations and construct those from the simple commutative situation. This leads to non commutative groups, quantum groups etcetera. This reminds us about the Egyptian architectural art followed by the Roman and French symmetrical ones: super simple, magnificent and logical.

To clarify, the axiom of choice supports the idea that the Cartesian product is non empty whereas the Cartesian product axiom presupposes that the product is a set. We now show that the axiom is in contradiction with our previous ones; to this purpose, consider two rotations a, b with as angle $r2\pi$ where r is an irrational number, around the x and z axes respectively. One considers the free group F_2 constructed from a, b which can be split into five parts

$$S(a), S(a^{-1}), S(b), S(b^{-1}), e$$

where $S(a)$ contains all irreducible words starting with a. Clearly, one obtains that $S(a) \sim S(b)$ from a geometrical perspective applying the rotation ab^{-1}. The *axiom of choice* allows for the construction M of a set which contains exactly one representant from any F_2 orbit on the sphere S^2. The construction goes as follows: consider the set of all equivalence classes $\widetilde{M}$ of S^2 under F_2 and denote by $p : S^2 \to \widetilde{M}$ the associated projection. If one equips $\widetilde{M}$ with a trivial partial order $\prec$ by picking one element of $\widetilde{M}$ and putting it on top of all others which remain unrelated, then one arrives at an index set $(\widetilde{M}, \prec)$ and the axiom of choice is applied to $\times_{m \in \widetilde{M}} p^{-1}(m)$ giving rise to an element F. Consider the subsets

$$A = S(a)M, B = S(a^{-1})M, C = S(b)M, D = S(b^{-1})M, M$$

and observe that $bD = A \cup B \cup D \cup M$. The reader notices that $b^n D \subseteq b^{n+m} D$ for $n, m > 0$ and subsequently $\lim_{n \to \infty} b^n D = S^2$ giving rise to $D = S^2$ because b is continuous and given that $a^{n_1} b^{m_1} \ldots a^{n_k} b^{m_k}$ is not reducible to the unity unless all $n_i = m_j = 0$ we arrive at a contradiction given that D cannot be the entire S^2. Consequently, M does not exist.

In order to complete the proof; consider the z axis v_z and notice that $b^n(v_z) = v_z$ as well as $a^m(v_z) = \cos(rm2\pi)v_z - \sin(rm2\pi)v_y$. v_y is rotated by means of a^m into v_y and v_z whereas the action of b^n results in v_y and v_x. In general, we obtain for any sequence $n_1, \ldots, n_k, m_1, \ldots, m_k$ a formula of the form $\sum_l \pm \prod_{j=1}^k g_j^l f_j^l = 1$ where the sum is finite and each $f_j^l = \sin(rm_j 2\pi)$, $f_j^l = \cos(rm_j 2\pi)$ or $f_j^l = 1$ as well as $g_j^l = \sin(rn_j 2\pi)$, $g_j^l = \cos(rn_j 2\pi)$ of $g_j^l = 1$. Given that there exists a countable number of words and the real number system is not of such nature, we arrive at the conclusion that there exists an over countable number numbers r such that the condition above is violated. I refer to further explanations regarding the words countable and over countable. Although the axiom of choice is not valid in general, it holds for a countable number of Cartesian products invalidating plenty of results in functional analysis.

The crucial lesson here is that writing down a set of consistent rules for a mathematical language is far from obvious. Few persons would anticipate any trouble with the axiom of choice and argue that the transfinite situation equals that one of the natural numbers. This is however not so from the point of abstract set theory.

One has to contemplate about topology as a refinement of set theory; it is to say, we limit ourselves to special sets being the so called open sets. In nature, an open set is an abstraction, an imaginary concept which has no real existence. An open surrounding has to be thought of as a voluminous object: for example, a straight line segment is the set of all real numbers between two extremal values denoted by $(a, b) = \{x | a < x < b\}$ with a natural length of $b - a$. A point is an example of a closed set and has vanishing volume or length. We now consider some properties regarding the set theoretical operations on the open segments (a, b): the union of two open segments is declared open by fiat whereas the intersection of two open segments is an open segment anew. Note that the union of open segments can be written as a disjoint union. Given a set D, we call a set $\tau(D)$ of subsets of D a topology if and only if

- $\emptyset \in \tau$,
- $A, B \in \tau$ implies that $A \cap B \in \tau$,
- $A_i \in \tau$ implies that $\cup_{i \in I} A_i \in \tau$ for every second countable index set I.

I stress again that this definition depends upon the commutativity as well as associativity of the intersection and union; it is possible to define a non-associative and non commutative topology by means of deformations. We shall study this from the viewpoint of logic further on and the reader may repeat these constructions almost ad verbatim here. In this chapter, we start pedestrian by studying the classical case where taking the union can be seen as putting landscape maps together; typically such charts overlap and all we demand is that the intersection of two charts is again a chart and that arbitrary many of them can be put together. There exist special subsets $E \subseteq D$ such that it is

- *closed* if and only if $E^c := D \setminus E \in \tau(D)$,
- *compact* if and only if for any coverage by means of open sets O_α of E there exists a finite sub coverage $O_i; i = 1 \ldots n$ such that $E \subseteq \cup_{i=1}^{n} O_i$.

Henceforth, the compact sets are those which can always be covered by means of a finite sub cover such as for example a globe: irrespectful of how small you make the charts, the globe is covered by a finite number of them. Given a point $p \in D$, we say O is an open environment of p if and only if $p \in O$. Given a point p, a basis of open environments is given by a countable collection of open neighborhoods O_i of p, such that for any open V encompassing p it holds that there exists an index i such that $O_i \subseteq V$. One could moreover demand that $O_{i+1} \subseteq O_i$ by taking intersections but this is not mandatory however. Regarding the closed sets X, Y one has to verify the following truisms: (a) $\emptyset, D$ are closed (b) $X \cup Y$ is closed (c) $\cap_{i \in I} X_i$ is closed if and only if all X_i are as such. Sets such as $\emptyset, D$ which are open and closed at the same time are dubbed cloped. Given $B \subseteq D$, the intersection of all closed sets X encompassing B is closed and called the closure of B which we denote as $\overline{B}$. The closure of a set is therefore the smallest closed set encompassing the latter itself. In other words, one adds elements or points which are limits of elements in B. More concretely, we call x a limit point of a sequence $(x_i)_{i \in I}$ if and only if for every open neighborhood $\mathcal{O}$ of x holds that there exists an index j such that $\forall j \prec i$ it holds that $x_i \in \mathcal{O}$. Now one shows that, using the properties of an index set, if y were another limit point then the open neighborhoods of x and y coincide. This motivates the following definition: a topology is Hausdorff if and only if all disjunct points x and y have open neighborhoods each with empty mutual intersection. It is to say that $x \in \mathcal{O}, y \in \mathcal{V}$ and $\mathcal{O} \cap \mathcal{V} = \emptyset$. For Hausdorff topologies holds that the limit point of a sequence is unique. We now prove the following result for topologies with a countable basis: a set is closed if and only if it contains all its limit points. Indeed, suppose that B is closed, and $(x_i)_{i \in I}$ is a sequence in B with limit point $x \in D$, then it holds that $x \in B$ otherwise one can find an open neighborhood B^c of x which is disjoint with $(x_i)_{i \in I}$, something which contradicts the definition of a limit point. Reversely, suppose that any limit point of B belongs to B, then we show that B is closed; suppose it is not, then we find an $x \in \overline{B} \setminus B$ such that for any basis-open neighborhood $\mathcal{O}_n$ of x we find an element $x_n \in B \cap \mathcal{O}_n$ and as such it holds that x is a limit point of $(x_n)_{n \in \mathbb{N}} \in B$ and henceforth, by assumption, an element of B which leads to a logical contradiction. Later on, we give an example of a compact set in a non-Hausdorff topology with a sequence containing no subsequence with a limit point (in case you want to think about this; find an example in an infinite number of dimensions). We shall study further characteristics of compactness in the so called metric topologies, which are determined by a distance function d.

So far, the treatment of topology appears to be very abstract and not very useful at all, one can think of any topology one wants to and indeed, all subsets of the real number system for example constitute a topology called the discrete topology. Indeed, all sets are cloped there which suggests a huge triviality. The physical reality we live in appears by very close inspection much more peculiar given that we speak about distance functions and spheres such as for example the circle with radius of 10 kilometer around Brussels measured from the Grand Place in bird flight. On earth this procedure only goes wrong when one traverses half of the circumference; one step further in the same direction would replace that journey by a different one where one originally departs in the opposite direction. Therefore, at large distances, one can expect problems of this global nature and in quantum geometry, one suspects those issues can occur at small distances too. Typical scales here are much smaller as those of an atom. By definition, a distance function $d : X \times X \to \mathbb{R}^+$ defined on a set X satisfies

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ for each $x, y \in X$,
- $d(x, z) \leq d(x, y) + d(y, z)$ the so called triangle inequality.

A distance function defines a so called Hausdorff topology with countable basis by means of the open balls

$$B(x, \epsilon) = \{z | d(x, z) < \epsilon\}$$

giving rise to a countable basis defined by $B(x, \frac{1}{n})$ where $n \in \mathbb{N}_0$. Two points x, y separated by means of a distance $d(x, y) > 2\epsilon$ can be surrounded by means of two disjoint balls $B(x, \epsilon), B(y, \epsilon)$ respectively. This representation of affairs is still a bit abstract given that one wants to measure angles as well contemplate a notion of orthogonality which is not so simple in this formalism. In other words, we require further specialization extending beyond the distance function only. Nevertheless, one can prove plenty of theorems in this primitive language relying solely upon those three axioms. A generalization consists in specifying that the distance function has a local origin; it is to say that the distance between two points can be chopped into arbitrarily small pieces. This leads to the notion of a path metric: d is a path metric if and only if the property holds that for any two points x, y there exists a z such that

$$d(x,z) = d(y,z) = \frac{d(x,y)}{2}.$$

In other words, every two points define at least one midpoint. We shall later on give a better representation of those facts.

We will study now an equivalence relationship between two topological spaces; in other words, when are two topological spaces the same? To determine that, we shall study topological mappings between two topological spaces X, Y. A mapping $f : X \to Y$ is defined by means of a subset F of the Cartesian product $X \times Y$; F obeys the law that for any $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in F$. y is then denoted as $f(x)$ and F is the graph of f. In human language, this signifies that each element chosen from X has precisely one image in Y. Concerning mappings, we formulate still the following extremal properties: (a) f is injective if and only if $f(x) = f(x')$ implies that $x = x'$ or each x has a different image (b) f is surjective if and only if for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$ or, in other words, every potential image is realized effectively. Finally, we say that f is a bijection if and only if it is injective as well as surjective; bijective mappings are equivalences between sets as we shall see now. Suppose $f : X \to Y$ and $g : Y \to Z$ then $g \circ f : X \to Z : x \to g(f(x))$ is the composition of these two mappings. Show that $g \circ f$ is injective if and only if g has this property on $f(X)$ and f obeys this law on X. Show that $g \circ f$ is surjective if and only if g is on $f(X)$; finally, show that $g \circ f$ is a bijection if and only if g, f are. In case $f : X \to Y$ is a bijection, it becomes possible to define a unique inverse $f^{-1} : Y \to X$ by means of

$$f^{-1}(f(x)) = x$$

or $f^{-1} \circ f = \mathrm{id}_X$ where id_X constitutes the identity mapping on X. Derive here from that

$$f \circ f^{-1} = \mathrm{id}_Y$$

using the surjectivity of f. Finally, one shows that f^{-1} also is a bijection; we say henceforth that X and Y are equivalent if and only if there exists a bijection from X onto Y. Using the previous properties, one shows that this relation is reflexive, symmetric and transitive. Now, we are in position to define topological equivalences $f : X \to Y$; f is continuous if and only if the inverse of each open set O in Y, denoted by $f^{-1}(O)$, is open in X. For a continuous bijection, one has that f^{-1} is continuous if and only $f(V)$ open is in Y for any V. In case a function f satisfies this property, we call it an open mapping. An example of a continuous bijection for which the inverse is not continuous, is given by $f : (-1, 1) \to (-1, 0] \times \mathbb{Z}_2 : x \to (-|x|, \theta(x))$ where $|x| = -x$ if $x < 0$ and x if $x \geq 0$. $\theta(x) = 0$ for $x \leq 0$ and 1 otherwise; finally, $\mathbb{Z}_2 = \{0, 1\}$. The topology defined on $(-1,0] \times \mathbb{Z}_2$ is the natural one of $(-1,0]$ and is henceforth not Hausdorff on $\{0,1\}$. One has that $f((-1,0)) = (-1,0) \times \{0\}$ is not open whereas $(-1,0) \times \mathbb{Z}_2$ is. A topological equivalence is given by means of a bijection f which is continuous and open. Such mappings are called homeomorphisms and the reader verifies that this definition obeys all requirements of an equivalence relationship indeed.

We return to our study of metric topologies and in particular alternative characterizations of compactness. A sequence $(x_i)_{i \in I}$ is called Cauchy if and only if for each $\epsilon > 0$, there exists an i, such that for all $i \prec j, k$ one has that $d(x_j, x_k) < \epsilon$. In human language, this reads: if one proceeds sufficiently far in the sequence then the points reside arbitrarily small together. Such a property suggests the existence of a unique limit point x; a metrical space (X, d) for which any Cauchy sequence has a limit point is called *complete*. In case K is a compact set, then one shows that any sequence $(x_i)_{i \in I}$ has a subsequence with a limit point in K. The proof is simple, consider arbitrary finite (due to compactness) covers with balls of radius $\frac{1}{n}$; then one finds a sequence of balls $B(y_n, \frac{1}{n})$ such that finite intersections $\cap_{n=1}^{m} B(y_n, \frac{1}{n})$ contain an infinite number of $x_i \in K$. This defines a subsequence with as limit point

$$x = \cap_{n=1}^{\infty} B(y_n, \frac{1}{n})$$

in K. Reversely, suppose that any sequence in K has a Cauchy subsequence with a limit point in K, then K is compact. Choose a cover of K of open balls - without limitation of validity- $B(y_n, \epsilon_n)$ where $n \in \mathbb{N}$ and suppose that no finite sub cover exists. Define then $B_m = \cup_{n=1}^{m} B(y_n, \epsilon_n)$, we henceforth arrive at the conclusion that for any m there exists an $m' > m$ such that $B_k \cap \overline{B_m}^c \cap K \neq \emptyset$ for each $k > m'$. In particular, we construct a sequence (x_m) with the property that for any m there is an $m' > m$ such that $x_k \in \overline{B_m}^c$ for $k \geq m'$. This sequence cannot contain a Cauchy subsequence with some limit point x because $x \in B_m$ for m sufficiently large which is a contradiction. We just proved that a set is compact in a metric topology if and only if any sequence contains a Cauchy subsequence with limit point in K. Prove the following properties:

- define on $\mathbb{R}$ the function $d(x, y) = |y - x|$, show that this defines a metric (easy exercise),
- prove that in the metric topology on $\mathbb{R}$, the closed interval $[a, b]$ is compact (hint: use the decimal representation of real numbers) (difficult),
- suppose two topological sets X, Y, then the product topology $\tau(X \times Y)$ is the smallest topology containing $\tau(X) \times \tau(Y)$, where the last contains elements $U \times V$ with $U \in \tau(X)$ and $V \in \tau(V)$,
- show that the Cartesian product $K_1 \times K_2$ of two compact sets is compact in the product topology (average),

- a metrical space (X, d) is bounded if and only if there exists an $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in X$; show that a compact space is closed and bounded (easy).

Again, the reader might utter that this kind of considerations are far too general and that our world is much more detailed in the sense that light rays bend and twist around one and another and that this behavior is geometrical and continuous in nature. To describe these features in detail, one needs the notion of a local scalar product which we shall study further on in chapter six giving further rise to analytical geometry. Note the following: suppose that $\gamma : [a, b] \to X$ is a continuous curve joining x and y and define the length functional as $L(\gamma)$ van γ where

$$L(\gamma) = \sup_{a=t_0<t_1<t_2<\ldots<t_{n+1}=b} \sum_{k=0}^{n} d(\gamma(t_k), \gamma(t_{k+1}))$$

and sup means taking the supremum of this sum over all finite partitions $a = t_0 < t_1 < t_2 < \ldots < t_{n+1} = b$ of the closed interval $[a, b]$. The supremum of a set of real numbers A is the smallest number larger or equal as any number $x \in A$. The supremum is also called the upper bound and the reader shows that by definition the supremum always exists and is unique by means of addition of the number $+\infty$. Likewise, one defines the infimum or under bound and one shows again it exists and is unique. Concerning the sum, one notices that breaking up an interval $[t_k, t_{k+1}]$ into two disjoint pieces by means of addition of an intermediate point $t_k < t_{k+\frac{1}{2}} < t_{k+1}$ the sum increases by means of the triangle inequality. Henceforth, splitting up an interval $[a, b]$ leads to a higher sum by means of the triangle inequality.

Now, we will formulate our main result; a complete metric space (X, d) defines a path metric d if and only if

$$d(x, y) = \min_{\gamma:[a,b]\to X, \gamma(a)=x, \gamma(b)=y} L(\gamma).$$

In other words, when the distance between two points equals the minimal length of a curve joining x to y we speak about a path metric space. The reader is advised to show this by means of using the midpoint property in order to construct such curve using that $L(\gamma) \geq d(x, y)$. Reversely, in case such a curve exists, one automatically finds a midpoint. A curve minimizing length is called a geodesic and in a path metric space, the length of a geodesic equals the distance between two points. Later on, we shall arrive at a more detailed characterization of geodesics when imposing more structure. gain, those primitive notions allow one to obtain a substantial amount of results some of which have been obtained by Mikhail Gromov and Peter Anderson. Studying those primitive metric spaces further on requires consultation of their work.

As one notices, our language is not rich enough to speak about notions such as perpendicularity, angles etcetera. One gradually learns that this book will become more and more specific, that the language gets more rich and complex allowing for stronger connections and results. Compactness or local compactness is an important notion because the (local) topology is finite in a way. Structures which are not locally compact often do not allow for certain mathematical structures to exist because there is too much "room" or space such as is the case for integrals. We now arrive at very special building blocks: line segments, triangles and pyramids as well as higher dimensional generalizations thereof. We shall use those to describe certain topological spaces and characterize those: a central element herein is the concept of homology which leads to further categorical abstractions.

Whereas the previous topics were very abstract, we shall now continue to work with more tangible objects, things we know from everyday life. We shall use abstraction of these objects to deal with them in a more appropriate way. This has its advantages because it allows us to *calculate* with them; this actually is the main miracle of abstraction, that it allows us to *do* things. The topological spaces to be studied here are those which are modelled by means of the n-dimensional real space

$$\mathbb{R}^n = \times_{i=1}^n \mathbb{R} = \{(x_i)_{i=1}^n | x_i \in \mathbb{R}\}$$

which is the set of n-tuples of real numbers equipped with the product metrical topology of $\mathbb{R}$. One can extend the notion of a sum by means of the definition

$$(x_i) + (y_i) = (x_i + y_i)$$

and likewise can one define the *scalar* multiplication of a real number with an n-tuple *vector* by means of

$$r.(x_i) = (rx_i).$$

More in general, let R be a field and $G, +$ a commutative group, then we say that G is an R module in case there exists a scalar multiplication such that

$$1.g = g;\ (rs).g = r.(s.g);\ (r+s).g = r.g + s.g;\ r.(g_1 + g_2) = r.g_1 + r.g_2$$

for all $r, s \in R$ and $g, g_1, g_2 \in G$. In case $R = \mathbb{R}$ we call the module a real vector space. In $\mathbb{R}^n, +$, we have special vectors e_i, defined by the number 1 on the i'th digit and zero elsewhere; here fore, it holds that

$$\sum_{i=1}^n r_i.e_i = 0$$

if and only if it holds that all $r_i = 0$ and moreover all vectors can be written uniquely as

$$\sum_{i=1}^n r_i.e_i.$$

In case these properties hold for a set of vectors $\{v_i | i = 1 \ldots m\}$, then we call $\{v_i | i = 1 \ldots m\}$ a *basis*. One notices that we have used two integer numbers here, n for the e_i and m for all v_j; it is now a piece of cake to show that $n = m$. The reason is the following, because e_i is a basis, one can write the v_j uniquely as

$$v_j = \sum_{i=1}^{n} v_j^i e_i$$

and reversely

$$e_i = \sum_{j=1}^{m} e_i^j v_j.$$

Henceforth,

$$\sum_{i=1}^{n} v_j^i e_i^k = \delta_j^k;\ j,k : 1 \ldots m$$

and

$$\sum_{j=1}^{m} e_i^j v_j^l = \delta_i^l;\ i,l : 1 \ldots n$$

where $\delta_j^k = 1$ if and only if $j = k$ and zero otherwise. This system of equations is symmetrical in e and v and therefore $m = n$ given that both mappings are injective. Henceforth n is a basis *invariant* and called the dimension of $\mathbb{R}^n, +$. Now, we have a sufficient grasp upon real vector spaces and we proceed by defining special building blocks mandatory for the construction of simplicial manifolds.

What follows is a generalization of simple cutting and pasting of higher dimensional triangles and pyramids. We may construct so called Euclidean bodies in this way and the old fashioned approach towards a classification of topological spaces upon a homeomorphism has been made as such. However, different lines of argumentation which are less constructivist can lead towards such classification too. Consider the space $\mathbb{R}^{n+1}$ and consider a basis $v_i; i = 0 \ldots n$, then the n simplex $(v_0 v_1 \ldots v_n)$ is defined by means of the closed space

$$(v_0 v_1 \ldots v_n) = \{\sum_{i=0}^{n} \lambda_i v_i | \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1\}.$$

This is all a bit abstract and in order to get a picture of how such space looks like, one imagines the $0, 1, 2, 3$ dimensional cases. A zero dimensional simplex (v_0) is simply a point, a one dimensional simplex is given by the line segment $(v_0 v_1)$ which may be embedded into the plane $\mathbb{R}^2$. A two dimensional simplex $(v_0 v_1 v_2)$ is given by a triangle which can be embedded into $\mathbb{R}^2$ whereas finally $(v_0 v_1 v_2 v_3)$ describes a pyramid in $\mathbb{R}^3$. In general, the simplex $(v_0 v_1 \ldots v_n)$ is a convex space meaning that the line segment between two points $x, y \in (v_0 v_1 \ldots v_n)$ completely belongs to $(v_0 v_1 \ldots v_n)$. The line segment between two points x, y is the set

$$\{\lambda x + (1 - \lambda) y | 0 \leq \lambda \leq 1\}.$$

Points of the simplex which do not belong to the interior of a line segment belonging entirely to the simplex are called extremal. Show by means of exercise that the only extremal points of $(v_0 v_1 \ldots v_n)$ are given by v_i. One calls the simplex the convex hull of the extremal points $\{v_i | i = 0 \ldots n\}$. We know now how a module is defined as well as a simplex which allows us for the definition of a linear operator. A mapping $A : V \to W$ between two R modules V, W is linear if and only if

$$A(rv_1 + sv_2) = rA(v_1) + sA(v_2)$$

for all $r, s \in R$ and $v_i \in V$. Show that A is injective if and only if $A(v) = 0$ implies that $v = 0$. Prior to proceeding, we study some very special rings $\mathbb{Z}_n$. These are defined by means of $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, in other words the rest equivalence classes defined by division through n:

$$k \equiv l \Leftrightarrow k - l = rn \,\text{for some}\, r \in \mathbb{Z}.$$

The sum and product are henceforth defined by taking the equivalence class associated to the standard sum and product in $\mathbb{Z}$. Show that these operations are well defined and that $\mathbb{Z}_n, +, .$ is a field if and only if n equals a prime number. For example, in $\mathbb{Z}_3$ it holds that $1 + 2 = 0, 2 + 2 = 1$; $\mathbb{Z}_3$ classes which can equally be represented by $\{-1, 0, 1\}$. Now, we shall work a bit more abstractly: actually, we do not need the property that $v_i \in \mathbb{R}^{n+1}$, something which was only needed for matters of representation. From now on, the v_i, w_j are merely points which do not need to be associated to vectors in some linear space. We define the $\mathbb{Z}$ module Z_n by means of taking entire multiples of simplicial complexes, defined by means of the simplexes $(v_0, \ldots, v_n)$ where swapping v_i and v_j causes a minus sign to arise. It is to say, the orientation reverses. A simplicial complex is defined by means of the topological space associated to a sum of simplices wherein every simplex in the sum, or its orientation reverse, can occur multiple times. We define the boundary operator $\partial_n : Z_n \to Z_{n-1}$ as the linear operator over $\mathbb{Z}$ mapping a simplex $(v_0 v_1 \ldots v_n)$ to

$$\partial_n (v_0 v_1 \ldots v_n) = \sum_{i=0}^{n} (-1)^i (v_0 \ldots v_{i-1} v_{i+1} \ldots v_n).$$

One verifies next that $\partial_{n-1} \partial_n S_n = 0$ given a sum of simplexes S_n defining a simplicial complex. One easily verifies that the boundary of a simplex is an oriented simplicial complex. In principle, it agrees with taking the boundary of an oriented body; for a bottle of milk this is the glass body whereas for a disk, this is an oriented circle. We consider a sum T_k of $k = 0 \ldots n$ simplices closed if and only if $\partial_k T_k = 0$ and exact if and only if $T_k = \partial_{k+1} T_{k+1}$ for $k = 0 \ldots n-1$. It is clear that exact simplicial complexes are closed using the crucial property of a boundary operator and we define accordingly the $\mathbb{Z}$ modules $G_k(S_n)$ and $E_k(S_n)$, where $E_n(S_n) = \{0\}$ and $G_0(S_n)$ equals $\mathbb{Z}^V$ with V the number of points or vertices in S_n. Clearly it holds that $E_k(S_n) \subseteq G_k(S_n)$ and we define the homology classes $H_k(S_n)$ as the quotient module

$$H_k(S_n) = \frac{G_k(S_n)}{E_k(S_n)}$$

being the $\mathbb{Z}$ module of $E_k(S_n)$ equivalence classes in $G_k(S_n)$. We say that two closed indices T_k, Y_k are equivalent if and only if $T_k - Y_k \in E_k(S_n)$ and the reader notices that $H_k(S_n)$ does not depend upon the nonzero coefficients of the n-simplices in S_n and henceforth depends merely on the simplicial space defined by S_n. So far for the general theory of simplical complexes, we now arrive to the very important sub theory of topological spaces A homeomorphic to a simplicial complex S_n; the important step herein consists in proving that $H_k(A)$ is well defined because homeomorphic simplicial sums define the same homology module. The reader may try to show this fact by him or herself as a kind of difficult exercise but it it clear that the statement is rather obvious. Indeed, the boundary operator is defined independently of the simplicial decomposition. The dimension of $H_k(S_n)$ is called the k-th Betti number b_k of the simplicial complex S_n. The reader now makes the following exercises: take a two dimensional spherical surface and show that $b_2 = 1, b_1 = 0, b_0 = 1$. The two torus T_2 is defined by taking a square and glue opposite sides to one and another; show that $b_2 = 1, b_1 = 2, b_0 = 1$. In general, one defines the Euler number of a two dimensional simplicial complex S_2 as

$$\chi(S_2) = D - L + V$$

where D is the number of triangles and L the number of line segments. One can show that the Euler number is a topological invariant; calculate that the Euler number of a two sphere is given by $2 = b_2 - b_0 + b_1 = 1 - 0 + 1$ and that of a torus by $0 = 1 - 2 + 1$. In general, one shows that

$$\chi(S_n) := \sum_{i=0}^{n} (-1)^i V_{n-i} = \sum_{i=0}^{n} (-1)^i b_{n-i}$$

where V_i equals the number of i dimensional sub-simplices. To start with the calculation of the dimension of a homology class, note that an element of $H_k(S_n)$ corresponds to a closed k dimensional connected surface which cannot be contracted to a point. Concerning the calculation of b_1 on the two sphere, it is clear that any closed curve can be reduced to a point whereas on the two torus two fundamental circles do exist. Consider two closed surfaces A_2 and B_2 and remove a two disk from both of them; now, paste each of the remainders along the circular boundaries resulting in a new closed surface denoted by $A_2 \diamond B_2$. Show that the operation $\diamond$ is associative as well as commutative with as identity element the two dimensional surface S^2. Calculate that the Euler number of the n-fold crossproduct of T_2 equals $2 - 2n$; more in particular, it holds that

$$\chi(A_2 \diamond B_2) = \chi(A_2) + \chi(B_2) - 2.$$

Later on, we shall study the notion of a manifold and one of the most important results is that any closed, compact, connected and oriented two dimensional manifold is homeomorphic to S^2 or an n-fold product $T_2 \diamond T_2 \diamond \ldots \diamond T_2$. This implies that closed, compact, connected as well as orientable two dimensional manifolds are completely characterized topologically by means of the Euler number. For closed manifolds, one shows that $b_{n-i} = b_i$ something which is called Betti duality, a result which may be proved by definition of a duality operator $\star$ on the simplicial complexes such that $S_n^\star$ is homeomorphic to S_n and $H_k(S_n)$ is mapped bijectively to $H_{n-k}(S_n^\star)$. One can imagine $\star$ as a natural generalization of the following operation on a one dimensional simplicial complex S_1: it maps every line segment r to a point $r^\star$ and each point p to a line segment $p^\star$ such that $\star$ interchanges the operation $\subseteq$ meaning $r^\star \subseteq p^\star$ if and only if $p \subseteq r$. $S_1^\star$ is a closed simplicial complex if and only if S_1; however, the Euler number may change in case S_1 is not a manifold. Henceforth, the manifold condition is mandatory and Betti duality does not hold for general closed simplicial complexes. The reader should prove that two circles having a common point show bad behavior under the duality transformation. The notion of a variety is henceforth really special and our result, that closed two dimensional and oriented varieties are classified by the Euler number only does not hold in higher dimensions. Here ends our discussion of simplicial homology which can be summarized by a chain of operations $\partial_k : Z_k(S_n) \to Z_{k-1}(S_n)$ met $\partial_0 : Z_0(S_n) \to 0$ en $\partial_{k+1}\partial_k = 0$. Such a structure is called a chain and those objects enjoy plenty of beautiful characteristics which are much more primitive as the topological point of departure. An initial point for higher mathematics therefore!

It is clear, from the simplicial point of view, that topological spaces of dimension n cannot be classified by means of the Betti numbers. The reader is invited to show this by means of braiding three closed surfaces in different ways. Later on, we shall study Betti numbers from the viewpoint of vectorfields, akin Morse theory, as well as closed differential forms determined by the homology classes.

Exercise: the Poincaré conjecture.
The conjecture of Poincaré is that every n dimensional compact, closed topological space $\mathcal{M}$ which is path connected and has trivial first homology class, is homeomorphic to the n dimensional sphere.

- Show that $\mathcal{M}$ allows for a path metric d.
- Consider an arbitrary point p and show that for sufficiently small r, the surface $L_r := \{x | d(p,x) = r\}$ is homeomorphic to the $n-1$ dimensional sphere S^{n-1}.
- Show that there exists a critical point r_0 such that L_{r_0} is no longer a sphere.
- In case L_{r_0} is a point, the theorem is proved; otherwise we have a compact $n-1$ dimensional topological space obtained from the sphere by means of identification of k dimensional subspaces where k can range from 0 to $n-1$ subsequently followed by pinching.

- Show that the subsequent connected components obtained by means of pinching are not contractible to a point unless they are homeomorphic to $n-1$ dimensional spheres.

- Subsequently, to close the topological space, all components different from a S^{n-1} must be pasted together leading to a nontrivial first homology class which is forbidden.

- Consequently, all branches must be equal to S^{n-1} and pasting is forbidden leading to a blown up graphical tree. Conclusion: $\mathcal{M}$ is a n dimensional sphere.

Simplicial gravitation.
Simplicial metric spaces are very simple and entirely characterized by means of distances $d(v_0 v_1)$ defined on the line segments $(v_0 v_1)$. One defines the following operators: $x_w(v_0 \ldots v_i) = (w v_0 \ldots v_i)$ and $\partial_w(w v_0 \ldots v_i) = (v_0 \ldots v_i)$ in case none of the v_j equals w. The remaining cases where this last condition is violated lead to the null simplex with as boundary conditions $\partial_w(w) = \mathbf{1}$, $x_w \mathbf{1} = (w)$ where $\mathbf{1} = ()$ is the empty simplex. From this, it follows that $(x_w)^2 = 0$ as well as $(\partial_w)^2 = 0$. One verifies that the operator $\partial = \sum_{w \in S} \partial_w$ is the usual boundary operator what shows that ∂_w constitutes the appropriate derivative operator defined by means of the boundary operator ∂. The empty simplex constitutes the neutral element regarding the cross product $*$ defined by means of

$$(v_0 \ldots v_i) * (w_0 \ldots w_j) = (v_0 \ldots v_i w_0 \ldots w_j).$$

One simply verifies that $x_w x_v = -x_v x_w$ and likewise for the operators ∂_v, ∂_w. Henceforth, the creation operators associated to a vertex generate a Grassmann algebra; moreover, it holds on the vector space of simplices that

$$\partial_v x_w + x_w \partial_v = \delta(v, w)$$

such that the ∂_v represent Grassmann annihilation operators. Bosonic line segment operators are consequently defined by means of

$$\partial_{(vw)} = \partial_w \partial_v$$

and such operators satisfy

$$\partial_{(vw)}(yz) = \delta(v, y)\delta(w, z) - \delta(v, z)\delta(w, y)$$

giving rise to an oriented derivative. The simplex algebra is henceforth defined by means of polynomials spanned by monomials which are formal products of simplices $(v_0 \ldots v_j)$ for all $j : 0 \ldots n$. Mind that this formal product does not equal the crossproduct implying that $\mathbf{1}$ does not constitute the neutral element. Given that on general spaces bi relations carry an evaluation by means of the metric d it is natural to limit the function algebra to two simplices $(v_0 v_1)$ given that other simplices do not procure for independent variables. The bosonic character of $\mathbf{1}$ implies that the ∂_v, x_w constitute Fermionic Leibniz operators on the function algebra. Indeed, one has that

$$\partial_v((w)Q) = \partial_v((x_w\mathbf{1})Q) = \partial_v x_w(\mathbf{1}Q) - \partial_v(\mathbf{1}x_w Q) =$$

$$(k+1)\delta(v,w)\mathbf{1}Q - x_w(\mathbf{1}\partial_v Q) - \partial_v(\mathbf{1}x_w Q)$$

which reduces to

$$(k+1)\delta(v,w)\mathbf{1}Q - (x_w)\partial_v Q - \mathbf{1}x_w\partial_v Q - \mathbf{1}\partial_v x_w Q = \delta(v,w)\mathbf{1}Q - (x_w)\partial_v Q$$

where k denotes the degree of the monomial Q given by the number of factors. This follows immediately from the Leibniz rule given that the operator

$$x_w\partial_v + \partial_v x_w = \delta(v,w)$$

is bosonic. Henceforth, the even simplex variables behave bosonically whereas the odd ones fermionic. Indeed,

$$\partial_v((wz)Q) = \partial_v((x_w(z))Q) = \partial_v(x_w((z)Q)+((z)x_w Q)) = -x_w\partial_v((z)Q)-(z)(\partial_v x_w Q)$$

which reduces to

$$= x_w((z)\partial_v Q) - (z)(\partial_v x_w Q) = (wz)\partial_v Q.$$

Given that the usual derivatives of a function are defined by means of the infinitesimal intervals $(x - |\epsilon|, x + |\epsilon|)$ where $f(v+\epsilon, v-\epsilon)$ gets identified with the coordinate function $f(x)$. This is logical given that the $v \pm \epsilon$ are fermionic and independent such that the intervals $(v-\epsilon, v+\epsilon) \sim x$ are bosonic. Note that products of the form $(v-\epsilon)(v+\epsilon)$ can be further derived such that

$$\partial_x f(x) = \mathbf{L}\left[\partial_{(v-\epsilon,v+\epsilon)} f(v-\epsilon, v+\epsilon)\right]$$

where $\mathbf{L}$ merely retains the monomials depending exclusively of the line segments. This phenomenon clearly occurs in $(vw)^2$ whose (vw) derivative equals

$$2(vw) - 2(v)(w).$$

To obtain the standard commutation-relations on the function algebra generated by (vw) we define

$$\widehat{x}_{(vw)}Q := x_{(vw)}x_{\mathbf{1}}Q$$

where Q is a polynomial defined on the edges (r,s) and $x_{(vw)}$ is a bosonic Leibniz operator defined by

$$x_{(vw)}(v_0 \ldots v_j) = (vwv_0 \ldots v_j).$$

By definition, one has that

$$x_{(vw)}(rs) = 0$$

if and only if r or s equals v, w and moreover

$$(x_{(vw)} + x_{(rs)})((vw) + (rs)) = 2(vwrs)$$

which vanishes unless (r, s) is the opposite side of a pyramid which we shall forbid from now on. In particular, this does not apply to geodesics

$$\gamma(v_0 v_i) := (v_0 v_1) + (v_1 v_2) + \ldots (v_{i-1} v_i)$$

which satisfy

$$x_{\gamma(v_0 v_i)} := \sum_{j=1}^{i} x_{(v_{j-1} v_j)}$$

and therefore

$$x_{\gamma(v_0 v_i)} \gamma(v_0, v_i) = 0.$$

Next, we define the derivatives

$$\partial_{\gamma(v_0, v_i)} := \sum_{j=1}^{i} \partial_{(v_{j-1} v_j)}$$

and consider the operator

$$\widehat{\partial}_{\gamma(v_0, v_i)} = \mathbf{L} \circ \partial_{\gamma(v_0, v_i)}$$

and one calculates that

$$\widehat{\partial}_{\gamma(v_0, v_i)} \widehat{x}_{\gamma(v_0, v_i)} - \widehat{x}_{\gamma(v_0, v_i)} \widehat{\partial}_{\gamma(v_0, v_i)} = 1$$

on the function algebra generated by the monomials Q of the form $(\gamma(v_0, v_i))^k$ where $k > 0$. We have now a tool to do physics; in particular,generated by the monomials Q of the form $(\gamma(v_0, v_i))^k$ where $k > 0$. We have now a tool to do physics; in particular,

$$\mathbf{E} P(\gamma(v_0, v_i)) = P(\sum_{j=1}^{i} d(v_{j-1} v_j))$$

is the evaluation function. The reader is invited to expand this theory further as well as to implement the Fourier transformation from chapter fourteen on conic tangent spaces. Hint: integrate in "hyperbolic" or "spherical" coordinates by replacing the $n-1$ sphere with the level surface $H^{n-1}(\epsilon, v_0) = \{x | d(v_0, x) = \epsilon\}$ for ϵ sufficiently small such that $H^{n-1}(\epsilon, v_0)$ belongs to the star neighborhood of v_0. See chapter thirteen for more information.

Betti numbers.
Give an example of two oriented spaces with the same Betti numbers and develop the homology concept further on with the purpose of distinguishing both (very difficult).

This is all there is to say basically at the classical level regarding topology for general enough spaces; further specialization is obviously always possible and can lead to very rich results such as is the case for the de Rahm theorem connecting the exterior derivative to the boundary operator by means of the Hodge theorem. Another step away then consists in abstraction of this duality from the point of category theory and in particular long left and right exact sequences attached to the exterior derivative and boundary operator respectively. In my opinion, this topic is too specialist to be treated here and philosophically, the very gist of classicality has been treated and resides in the axioms of Boolean logic or the algebraic structure generated by $\wedge, \cap$ and $\times$. This author has recently suggested an interesting extension of this formalism by extending those operations to semi-group ones where the semi reflects the fact that the inverse is not necessarily unique. That is, given a set A, an anti-set obeys

$$A \times A^{\times} = \{1\}$$

where the last one is a set with one element 1 and henceforth serves as the identity element for $\times$. To represent an anti-set in the set-like fashion; denote that if $A = \{x | x \in A\}$ and $A^{\times} = \{\omega_A^{\star}\}$ where $\omega_A : A \to \{1\}$ is the constant mapping onto 1 and $\star$ is the associated duality relation, then

$$A \times \{\omega_A^{\star}\} = \omega_A(A) = \{1\}.$$

So, taking inverses regarding the Cartesian product naturally leads to a notion of duality which may be interpreted as an anti-event. Likewise, we can demand

$$A \cup A^{\cup} = \{\emptyset\}$$

as well as

$$A \cap A^{\cap} = \Omega$$

where Ω is supposed to be a maximal set. $A^{\cup}$ can be seen to exist out of the negative events $-x$ where $x \in A$. The latter induces the following natural rules

$$-(A \cup B) = (-A) \cup (-B),\ -(A \cap B) = (-A) \cap (-B)$$

and henceforth it is a Boolean algebra isomorphism. In logic, the notion of a negative primitive sentence has no obvious philosophical meaning whereas this is the case for the negative of an event as one absorbing the other. The Boolean $\neg$ operation satisfies

$$A \cap \neg A = 0,\ A \cup \neg A = 1,\ \neg(A \cup B) = \neg A \cap \neg B$$

and has as set theoretical counterpart the complementation operation. However, Boolean logic does possess another operation called xor instead of or which does allow for these things to happen; A xor B is true if and only if exactly one of them is true and the other is false. In set theory, the equivalent is given by the disjoint union

$$A \sqcup B = (A \cup B) \setminus (A \cap B)$$

and in such a case $A^{\sqcup} = A$. However, such a thing is rather mundaine and $-x$ should really be seen as eating x meaning

$$\{x, -x\} = \{\emptyset\}$$

which is a serious departure from the negative of integer numbers given that the equivalence $\{5, -5\} = \{\emptyset\}$ does not exist. This brings along some subtleties with the complementation operation A^c given that $(-\Omega) \cup \Omega = \{\emptyset\}$ and henceforth $\{\emptyset\}^c = \{\emptyset\}$ in the enlarged setting of negative events breaking hereby the equivalence with the logical operator $\neg$. This is logical from a philosophical point of view given that $-$ is associated to death and therefore pressuposes creation whereas $\neg$ pertains to an eternal truism. This gives problematic aspects regarding the intersection operation

$$A \cap \{\emptyset\} = B$$

where, in last instance, B is any subset of A. This is obviously not desirable and is resolved by insisting that

$$(-A) \cap B = -(A \cap B)$$

where A, B are ordinary sets. This implies that the notion of element becomes superfluous given that

$$\{x\} \cap \{-x\} = \{-x\}$$

which is a situation intermediate between the classical and quantum; there is no contradiction with our previous treatment of set theory given that $\{x\}$ is no longer a primitive set but $\{-x\}$ is (so x is no longer an element). So, even if we start out with a standard set theory, adding all anti sets to it leads to a new theory where the elements of the old sets are no longer elements in the full set theory but their anti elements are. It is natural to posit that the boundary operator ∂ commutes with $-$ meaning $\partial(-A) = -(\partial A)$ and obviously it holds as well that

$$\partial A^c = -\partial A$$

assuming Ω has no boundary and the minus sign in this case refers to the opposite orientation (and has nothing to do with negative events). We leave such exotisms for future exploration.

Chapter 2

Quantum logic and topology.

We now treat quantal set theory from an axiomatic point of view and connect this with the subject of quantal logic developed by Von Neumann a century ago. The central idea in quantum theory is that a proposition is associated to a linear space and the mathematics of linear spaces is provided for by the (set theoretical) intersection $\cap$, the (direct) sum $+ (\oplus)$ (replacing the union) and the tensor product $\otimes$ as a substitute for the Cartesian product. Given two Hilbert spaces $\mathcal{H}_i$, the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ constitutes again a Hilbert space spanned by pure vectors $v_1 \otimes v_2$ where $v_i \in \mathcal{H}_i$. Regarding sums $\sum_{i=1}^{n} z_i v^i \otimes w^i$, the following equivalences are in place

$$\begin{aligned} z(v \otimes w) &\equiv (zv) \otimes w \equiv v \otimes (zw) \\ v \otimes w_1 + v \otimes w_2 &\equiv v \otimes (w_1 + w_2). \end{aligned}$$

We define $\mathcal{H}$ as the linear space of such equivalence classes and make a completion in the metric topology defined by means of the scalar product

$$\langle v_1 \otimes w_1 | v_2 \otimes w_2 \rangle := \langle v_1 | v_2 \rangle \langle w_1 | w_2 \rangle.$$

In a similar vein, the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ is defined by means of the equivalences

$$\begin{aligned} z(v \oplus w) &\equiv (zv) \oplus (zw) \\ v_1 \oplus w_1 + v_2 \oplus w_2 &\equiv (v_1 + v_2) \oplus (w_1 + w_2) \end{aligned}$$

with as scalar product

$$\langle v_1 \oplus w_1 | v_2 \oplus w_2 \rangle := \langle v_1 | v_2 \rangle + \langle w_1 | w_2 \rangle.$$

One verifies that a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$ is provided by means of $v_i \otimes w_j$ where the v_i constitute a basis of $\mathcal{H}_1$ and w_j of $\mathcal{H}_2$. A basis for $\mathcal{H}_1 \oplus \mathcal{H}_2$ is provided by $v_i \oplus 0, 0 \oplus w_j$. Linear subspaces of Hilbert spaces are characterized by means of Hermitian projection operators. The reader is advised to make the following exercises.

- Let P, Q be two Hermitian projection operators meaning that $P^2 = P$, $Q^2 = Q$, $P^\dagger = P, Q^\dagger = Q$. Show that $P + Q$ constitutes a Hermitian projection operator if and only if $PQ = QP = 0$. Show that the same holds for PQ if and only if $PQ = QP$.
- Two Hermitian projection operators P, Q are orthogonal if and only if $PQ = 0$; we define the partial order $\leq$ by means of $P \leq Q$ if and only if $QP = PQ = P$. Prove explicitly that $\leq$ defines a partial order on the set of Hermitian projection operators. In particular, it holds that $P \leq Q$ and $Q \leq P$ implies that $P = Q$. Also, $P \leq Q$ and $Q \leq R$ leads to $P \leq R$.
- We call the set of Hermitian projection operators on a vector space, equipped with $\leq$, a raster. Show that for any P, Q there exists a minimal projection operator $P \vee Q$ such that $P, Q \leq P \vee Q$ and any R such that $P, Q \leq R$ satisfies $P \vee Q \leq R$. On the other hand, one may construct a maximal projection operator $P \wedge Q \leq P, Q$. Show that $\vee, \wedge$ do not in general obey the rule of de Morgan:

$$P \wedge (R \vee Q) \neq (P \wedge R) \vee (P \wedge Q).$$

- In terms of subspaces, $P \vee Q$ is the projection operator on $V + W$, whereas $P \wedge Q$ on $V \cap W$ where V (W) is the image of P (Q).
- Show that the raster possesses a unique minimum as well as maximum provided by 0 and 1 respectively.
- Show that there exist minimal Hermitian projection operators, called atoms. Every Hermitian projection operator may be written as a sum of orthogonal atoms.

Quantum logic.
Given that in the previous exercise $\vee$ and $\wedge$ may be conceived as "or" and "and" respectively, it becomes possible to understand quantal logic by means of using Hermitian projection operators as propositions. Reflect on this and retrieve classical pointer propositions by considering a complete set of orthogonal projection operators. In such case, $P \vee Q = P + Q$ and $P \wedge Q = PQ$.

Quantum set theory.
Sets are given by objects P, Q and we have again $\wedge, \vee$ where $P \wedge P = P = P \vee P$ with minimal and maximal elements $0, 1$ replacing the empty set and the entire universe. The distinction with classical set theory is to be found in the de-Morgan rule which needs a substitute; in a way, it would be nice if we could find a logical rule in terms of $\wedge$, $\vee$ which would deliver us with the Hilbert space setting where P, Q may be seen as Hermitian projection operators. This has been the topic of research of the Geneva school for plenty of years and was rather extensively documented for by Piron.

In particular, the set of propositions must give rise to a so called orthomodular lattice defined by

- sets P, Q,
- a minimal 0 and maximal element 1,
- commutative and associative operations $\vee, \wedge$ satisfying $P \vee P = P \wedge P = P$ as well as $P \wedge 0 = 0$, $P \wedge 1 = P = P \vee 0$, $P \vee 1 = 1$. As we have seen before when studying deformed quantum logic, it is possible to drop the first two aspects (commutativity and associativity) and we shall provide for an example of that later on.
- a partial order $\leq$ defined by $P \leq Q$ if and only if $P \wedge Q = P = Q \wedge P$ with 0 as unique minimal element and 1 as maximal one where $P \vee Q$ is the supremum of P, Q and $P \wedge Q$ the infimum (one could possibly go further here and deal with a left and right order in case commutativity fails; in case associativity is dropped for, the formulae for the supremum and infimum are not any longer correct),
- **linearity**: meaning there exists a binary relation $+$ such that $\mathcal{S}$ can be extended to a real vector space with 0 as neutral element obeying $P \wedge (1 - P) = 0$ where $-P$ (which always exists) denotes the inverse for P. To obtain an orthogonality property, we need a multiplication; notice that $\wedge$ could serve for this on the algebra generated by two elements $1, P$ with the operations $+, \wedge$ leading to $P^2 = P$. However, if one takes Q, P then $PQ \neq QP$ and therefore it does not hold that $(PQ)^2 = PQ$ assuming the product satisfies the standard requirements of commutativity and associativity. To compensate for this, one should consider something like

  $$P \wedge Q = \frac{1}{2}(PQ + QP)$$

  which still does not lead to $(P \wedge Q)^2 = P \wedge Q$. However, there is a unique natural fix given by

  $$P \wedge Q = \lim_{n \to \infty} \left(\frac{1}{2}(PQ + QP) \right)^n$$

  which we will take as a standard formula. We moreover demand that $P + Q \in \mathcal{S}$ if and only if $PQ = 0$ in which case it coincides with $P \vee Q$,
- atomisticity, meaning every set P can be written as $P = \vee_\alpha Q_\alpha$ with the $Q_\alpha \neq 0$ orthogonal primitive propositions $Q_\alpha Q_\beta = 0$ for $\alpha \neq \beta$,
- a reality notion given by an involution $\dagger$ such that $P^\dagger = P$.

The reader notices that the first four axioms could give rise to classical set theory and that the linearity really decides upon the case being “quantal”. We have also used the Cartesian product which belongs to classical set theory and which we shall not repeat here. The reader should note that our axioms do not force for a Hilbert space interpretation.

Non commutative Quantum logic or quantum topology.
We generalize the operations $\wedge$ and $\vee$ to a context in which they are no longer commutative; this procedure holds as well for the classical Boolean logic or the quantual logic explained above where the de Morgan rule gets a minor blow. It is natural to interpret $\wedge$ as well as $\vee$ as mappings $\wedge, \vee : P \times P \to P : (x,y) \to x \wedge y$, $(x,y) \to x \vee y$ where P denotes the lattice of propositions defined by means of a linear Euclidean space in the quantal case. Define the mapping $S : P \times P \to P \times P : (x,y) \to (y,x)$ and consider $\wedge^{'(V,W)} := W \circ \wedge \circ S \circ V$ as well as $\vee^{'(V,W)} = W \circ \vee \circ S \circ V$ where $V : P \times P \to P \times P$ is required to be invertible as well as is the case for $W : P \to P$. Requiring ${}^{'(V,W)}$ to satisfy $\left(\wedge^{'(V,W)}\right)^{'(V,W)} = \wedge$ it is sufficient and mandatory that $W^2 = 1$ as well as $S \circ V \circ S \circ V = 1$. This demand is of a special algebraic nature which we dub by the name of an involution; so we are going to study involutive deviations from quantal logic. An involution gives rise to a notion of duality; in particular self-duality is defined by the condition that

$$\wedge^{'(V,W)} = \wedge, \vee^{'(V,W)} = \vee.$$

It is natural to propose first S symmetrical logics; these are given by

$$\wedge^{'(V,W)} \circ S = \wedge^{'(V,W)}, \vee^{'(V,W)} \circ S = \vee^{'(V,W)}.$$

This can only happen by choosing V such that

$$V \circ S = S \circ V$$

reducing a previous condition to

$$V^2 = 1$$

whereas it still holds that

$$\wedge^{'(V,W)} = W \circ \wedge \circ S \circ V.$$

In case $\wedge$, $\vee$ coincide with the standard Boolean or Quantal operations denoted by $\wedge_d$, $\vee_d$ where $d = c, q$ one has that

$$\wedge_d \circ S = \wedge_d, \; \vee_d \circ S = \vee_d.$$

In such a case,

$$\wedge := \wedge_d^{'(V,W)} = W \circ \wedge_d \circ V$$

a small simplification of the previous formula and $\vee$ is defined in a similar way. Now, to remain entirely clear, it is so that the d index should be the same in $\wedge$, $\vee$ but (V,W) becomes (R,T) for $\vee$ whereas the former pertains to $\wedge$. We now isolate the "de Morgan expression" $a \wedge (b \vee c)$:

$$\wedge \circ (1 \times \vee)(a,b,c) = W \wedge_q V(1 \times T \vee_q R)(a,b,c).$$

It is subsequently natural to call T - $(\wedge_q, V)$ compatible if and only if $\wedge_q V(1 \times T) = T' \wedge_q V$ for some $T' : P \to P$. Likewise, it is natural to call V - $\vee_q$ compatible if and only if $V(1 \times \vee_q) = (1 \times \vee_q)V'$ for some $V' : P^3 \to P^3$. Under these assumptions, the previous expression reduces to

$$WT'(\wedge_q(1 \times \vee_q))V'(1 \times R)$$

which was the desirable separation. It is furthermore natural to suggest further restrictions

$$WT' = 1,\ V'(1 \times R) = 1_3.$$

Truth evaluators ω

The material presented below constitutes an extension of the notes I have received once from Rafael Dolnick Sorkin; in classical Boolean logic one disposes of truth evaluator ω of logical sentences which constitutes a homomorphism from the set of propositions $P, \vee_c, \wedge_c$ to $\mathbb{Z}_2, +, .$ where 0 is interpreted as false and 1 as true and $\vee_c$ is the so called exclusive *or* in the sense that $a \vee_c b$ is true if and only if exactly one of them is true. It is to say that

$$\omega(a \vee_c b) = \omega(a) + \omega(b),\ \omega(a \wedge_c b) = \omega(a)\omega(b).$$

To get an idea of what more general, quantal truth evaluators are about, let us describe a classical system in a quantum mechanical fashion. An example is give by means of the weather, "the sun shines", modelled by $|l\rangle$, or "it is dark" given by $|d\rangle$. Quantum mechanically, one disposes of a complex two dimensional Euclidean space spanned by the extremal vectors $|l\rangle, |d\rangle$. Consider now a general state

$$|\psi\rangle = \alpha|l\rangle + \beta|d\rangle$$

and study the class of truth functionals ω which merely depend upon

$$\frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2}, \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2}$$

something which reduces to a parameter $0 \leq \lambda \leq 1$ due to

$$\frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2} + \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2} = 1.$$

When all truth evaluators merely depend upon this parameter only, the complex plane may be reduced to the line segment connecting both extremal vectors to one and another. An example of such an evaluator is provided by

$$\omega^l_\epsilon : [0,1] \to \mathbb{Z}_3$$

given by means of the prescription

$$\omega^l_\epsilon(\lambda|l\rangle + (1-\lambda)|d\rangle) = \chi(\lambda + \epsilon - 1) + 2\chi(\lambda - \epsilon)\chi(1 - \epsilon - \lambda).$$

ω^l and is henceforth connected to the question whether the light shines and ϵ is the tolerance of the observer. This truth evaluator says "yes", given by means of 1, in case $1-\epsilon \leq \lambda \leq 1$, under determined or "vague" when $\epsilon \leq \lambda \leq 1-\epsilon$ and no, given by 0, when $0 \leq \lambda \leq \epsilon$. We have that χ is the so called characteristic function defined on the real numbers by means of $\chi(x) = 1$ in case $x \geq 0$ and zero otherwise. The issue is that we departed from a quantum mechanical description of the weather and by reduction of the allowed questions arrived to a classical system where, moreover, ω^l_ϵ is nonlinear. This suggests that the distinction between quantum and classical largely is hidden in the questions asked and not as much in the Schroedinger dynamics.

Most physicists would suggest at this moment that we did not make a sufficient distinction between classical and quantum logic as yet because $\wedge_q, \vee_q$ are commutative, assiociative but $\wedge_q$ is not distributive with regard to $\vee_q$ which is the case for $\wedge_c$, $\vee_c$. In our most general setting, one has that $\wedge$ and $\vee$ are neither commutative, nor associative

$$\vee(1\times\vee)(a,b,c) = T\vee_d R(1\times T\vee_d R)(a,b,c) \neq T\vee_d R(T\vee_d R\times 1)(a,b,c) = \vee(\vee\times 1)(a,b,c)$$

and likewise so for $\wedge$. The main distinction between classical and quantum logic resides in the fact that the set of propositions constitutes a distributive lattice in the former case whereas it does not in the latter; this results in the statement that the classical rule

$$\mu(a|b)\mu(b) = \mu(b|a)\mu(a)$$

is no longer true in the quantal case. Here, μ is the probability measure that a is true; in other words, the truth determinations of a and b depend upon the order in which they occur. This has so far not been accounted given that a homomorphism $\vee_{c,q}, \wedge_{c,q}$ does not make any distinction in the order of the factors. Therefore, classically, the task is to determine $\omega_c(a \wedge_c b)$ *given* the unordered tuple $\{\omega_c(a), \omega_c(b)\}$. Quantum mechanically, it is as such that the *reality* $\omega_q(a|b)$ is *not* provided by means of the ordered couple $(\omega_q(a), \omega_q(b))$ as elements of $\mathbb{Z}_2$ but also depends upon a, b them self. Therefore, in quantum theory, the correct question is, "what is the probability of the reality ω_q?". Moreover, it is *not* so that

$$\mu_{|v\rangle}(a|b) = \frac{\mu_{|v\rangle}(a \wedge_q b)}{\mu_{|v\rangle}(b)}$$

due to commutativity of $\wedge_q$ as well as $a \wedge_q b = 0$ for distinct one dimensional Hermitian projection operators a, b on a Hilbert space $\mathcal{H}$. The exact formula is given by

$$\mu_{|v\rangle}(a|b) = \frac{\mathrm{Tr}(|v\rangle\langle v|bab)}{\mathrm{Tr}(|v\rangle\langle v|b)}$$

and the reader notices that the non-commutativity of a and b is of vital importance. Henceforth, the ontological mapping defined in quantum theory is given by $\kappa : P \to \mathbf{L}(\mathcal{H})$ where P is the set of prepositions with a yes or no answer onto the lattice of Hermitian projection operators defined on the Hilbert space of states of the system. The classical Lagrange formula

$$\mu(a|b)\mu(b) = \mu(b|a)\mu(a)$$

where μ is determined by the state of the system is abandoned upon provided that $\wedge_q$ a la Von Neumann offers no alternative. The natural question henceforth is whether we may find a natural $\wedge$ as well as a consistent set of realities

$$\omega_q^\rho : P \to \mathbb{Z}_2 \times [0,1]$$

attached to density matrices ρ defined on $\mathcal{H}$, such that

$$\omega_q^\rho(a) = (1, \lambda)$$

and

$$\omega_q'^\rho(a) := (0, 1 - \lambda)$$

is defined as the complementary observation. It is clear that ω_q is not always given by a homomorphism; prior to proceeding, it is important to understand $\vee_q$. It is clearly so that in quantum theory, we have an extended ontology; we do not only pose the question "what is the probability that $a \wedge_c b$ holds given that a as well as b are true" such as the case in classical logic, but we insist on the formulation "what is the chance that $a \wedge_q b$ holds given that a after b has been experimentally established". There is general no experiment attached to $a \wedge_q b$ something which was assumed to be true in the setting of Von Neumann setting which invalidates an appropriate description. The right answer is easy if $a \wedge b$ is represented by the non Hermitian operator ab which is logical given that the order of measurements matters. In general, one shows that

$$a \wedge_q b = \lim_{n \to \infty} \left(\frac{1}{2}(ab + ba) \right)^n$$

and in the framework of our deformation theory $\wedge$ is given by means of

$$R(a,b) = (1, \frac{ab}{\mathrm{Tr(ab)}})$$

at least this is so for atomistic elements a, b. T is henceforth determined on the rank 1 matrices by means of

$$T(|v\rangle\langle w|) = \frac{\mathrm{Tr}(|v\rangle\langle w|)}{\mathrm{Tr}(|v\rangle\langle v|)\mathrm{Tr}(|w\rangle\langle w|)} |v\rangle\langle w|$$

and obeys

$$T(\lambda|v\rangle\langle w|) = T(|v\rangle\langle w|)$$

indicating conformal invariance. Hence, for rank one projectors a, b it holds that

$$a \wedge b = T \circ \wedge_q \circ R(ab) = ab.$$

Subsequently, one has that

$$\omega_q^\rho(a) = (1, \mathrm{Tr}(a^\dagger \rho a))$$

or

$$\omega_q^\rho(a) = (0, 1 - \mathrm{Tr}(a^\dagger \rho a))$$

for a of rank one. Clearly, by definition

$$\omega_{q,1}^\rho(a|b) := \frac{\pi_2(\omega_{q,1}^\rho(a \wedge b))}{\pi_2(\omega_{q,1}^\rho(b))}$$

equals the probability that a is measured after b. Here π_j equals the projection on the j'th factor and $\omega_{q,1}^\rho$ is the extremal functional given by $\pi_1(\omega_{q,1}^\rho) = 1$. Elaborate further on this theory and determine a suitable $\vee$ operation. Hint: the latter is *cannot* be given by $a \vee b = a + b$ in the deformation framework provided that $\vee_q$ does not allow to determine the projection of a on b as is given by $\mathrm{Tr}(ab)$, something which is mandatory to extract the sum operation. Henceforth, any quantum deformed logic violates the de Morgan rule given that it is difficult to assign a physical meaning to not measuring an observable b because this violates the completeness assumption of quantum theory. One measures or one does not which vehemently contradicts the *being* of things. To define $\vee$ it is advised to use the classical rule

$$\neg(a \vee_c b) = (\neg a) \wedge_c (\neg b)$$

and using $\neg\neg = 1$, it holds that

$$a \vee b = \neg((\neg a) \wedge (\neg b)).$$

In quantum theory, $\neg(a)$ is provided by $1 - a$ and henceforth, we arrive at

$$a \vee b = 1 - (1 - a) \wedge (1 - b)$$

which leads to a violation of the de Morgan rule given that

$$a \wedge (b \vee c) = a.(1 - (1 - b).(1 - c)) = -ab - ac + abc$$

whereas

$$(a \wedge b) \vee (a \wedge c) = 1 - (1 - ab).(1 - ac) = -ab - ac + abac.$$

So far the abstract theory; we may now proceed by defining quantal simplices with associated quantal orientation as well as boundary operator. In general, we want to associate a geometrical significance to an object such as

$$(P_1, \ldots, P_{n+1})$$

where the P_i are Hermitian projection operators defined on a Hilbert space. The problem is that such lattice is not convex and therefore more information is required. We shall see that the classical situation can be modelled by means of the complex (pseudo) Hilbert space $\mathcal{H}(v_i; \mathcal{V}, h)$ of square integrable complex valued functions on a compact subset $\mathcal{V} \subset \mathbb{R}^n$ which are linear combinations of functions f perpendicular to a real positive function $h \in \mathcal{H}(v_i; \mathcal{V}, h)$ meaning

$$\int_{\mathcal{V}} d^n x f(x) \overline{h(x)} = 0$$

where, moreover, $h(\sum_i \lambda_i v_i) = 1$ for $\lambda_i \geq 0$ given $\sum_i \lambda_i = 1$ and h decreases towards 0 at the boundary of $\mathcal{V}$. Here, a vertex $\sum_i \lambda_i v_i$ is a distributional element from the functional point of view and associated to the operator $\widehat{\delta}(\sum_i \lambda_i v_i) : (f + \lambda h) \to [f(\sum_i \lambda_i v_i) + \lambda]h$ such that

$$\widehat{\delta}^2(\sum_i \lambda_i v_i)[f + \lambda h] = \widehat{\delta}(\sum_i \lambda_i v_i)[f + \lambda h] = [f(\sum_i \lambda_i v_i) + \lambda]h$$

as well as

$$\langle f + \lambda h | \widehat{\delta}(\sum_i \lambda_i v_i)[g + \delta h] \rangle = (g(\sum_i \lambda_i v_i) + \delta)\overline{\lambda} ||h||^2$$

whereas

$$\langle \widehat{\delta}(\sum_i \lambda_i v_i)[f + \lambda h] | g + \delta h \rangle = \overline{f(\sum_i \lambda_i v_i) + \lambda} \delta ||h||^2.$$

To make the operator $\widehat{\delta}(\sum_i \lambda_i v_i)$ self adjoint, it is necessary for $||h||$ to be zero; that is, to adapt the scalar product as such that h is a ghost. Another option would be to take the adjoint $\widehat{\delta}(\sum_i \lambda_i v_i)^\dagger$ in the standard in-product and to proceed with half the sum of $\widehat{\delta}$ with its adjoint. We proceed in the first manner; this can be achieved by means of a simple redefinition

$$\langle \psi | \phi \rangle_h = \langle \psi | \phi \rangle - \frac{\langle \psi | h \rangle \langle h | \phi \rangle}{\langle h | h \rangle}$$

so that everything is made indefinite and $\mathcal{H}(v_i; \mathcal{V}, h)$ can be taken as the full function space $L^2(\mathcal{V}, d^n x \sqrt{\eta(x)})$ but then equipped with the scalar product $\langle \cdot | \cdot \rangle_h$. There is a canonical mapping from $(v_1, \ldots, v_{n+1})$ to $(\widehat{\delta}(v_1), \ldots, \widehat{\delta}(v_{n+1}))$ where the latter is defined by means of the operators

$$\sum_i \lambda_i \widehat{\delta}(v_i), \; \sum_i \lambda_i = 1, \; \lambda_i \geq 0$$

which coincides with

$$\widehat{\delta}(\sum_i \lambda_i v_i)$$

on the *invariant* $n + 1$ dimensional pseudo-Hilbert subspace of affine functions

$$f(\vec{x}) = \vec{a}.\vec{x} + bh(\vec{x}).$$

Physically, it might be desirable to demand that the subspace is of positive norm only so that $b = 0$ leaving for an n dimensional Hilbert space of linear functions;

to make that one invariant under the action of the operator $\widehat{\delta}(\sum_i \lambda_i v_i)$, it is necessary that

$$\vec{a}.(\sum_i \lambda_i \vec{v}_i) = 0$$

leaving for a vertex dependent $n-1$ dimensional subspace of linear functions. Note furthermore that in general

$$\widehat{\delta}(\sum_i \lambda_i v_i)\widehat{\delta}(\sum_i \mu_i v_i) = \widehat{\delta}(\sum_i \mu_i v_i)$$

so that the vertex operators themselves constitute a closed algebra. More abstract, we have the following structure: a classical simplex is functionally represented by means of a mapping

$$\widehat{\delta} : (v_1, \ldots, v_{n+1}) \to \mathcal{P}(\mathcal{H}(v_i; \mathcal{V}, h))$$

where $\mathcal{H}(v_i; \mathcal{V}, h)$ constitutes a pseudo Hilbert space with an invariant $n+1$ dimensional function space $\mathcal{H}_{\text{affine}}(v_i; \mathcal{V}, h)$ (containing one ghost) on which the action of $\widehat{\delta}$ is a linear one. Moreover, in general, we have that the algebra closes by means of the property

$$\widehat{\delta}(\sum_i \lambda_i v_i)\widehat{\delta}(\sum_i \mu_i v_i) = \widehat{\delta}(\sum_i \mu_i v_i).$$

Aditionally, all Hermitian projectors in $\mathcal{P}(\mathcal{H}(v_i; \mathcal{V}, h))$ have rank one and are therefore atomistic. The reader notices furthermore that in this setting, the Hahn-Banach theorem does not hold and it is impossible to write

$$\widehat{\delta}(\sum_i \mu_i v_i)(f) = \langle K(\sum_i \mu_i v_i)|f\rangle_h K(\sum_i \mu_i v_i)$$

for some $K(\sum_i \mu_i v_i)$ because the latter has zero norm. It is however, possible to find a conjugate distributional vector $\delta(\sum_i \mu_i v_i)$ such that

$$\widehat{\delta}(\sum_i \mu_i v_i)(f) = \langle \delta(\sum_i \mu_i v_i)|f\rangle h.$$

Remark furthermore that the symmetry of the $\vee$ operation imposes

$$\widehat{\delta}(\sum_i \mu_i v_i) \vee \widehat{\delta}(\sum_i \lambda_i v_i) = \widehat{\delta}(\sum_i \frac{1}{2}(\lambda_i + \mu_i)v_i)$$

whereas the standard prescription for the $\wedge$ product leads to

$$\widehat{\delta}(\sum_i \mu_i v_i) \wedge \widehat{\delta}(\sum_i \lambda_i v_i) = \frac{1}{2}(\widehat{\delta}(\sum_i \mu_i v_i) + \widehat{\delta}(\lambda_i v_i))$$

which is not of the appropriate form. Both expressions violate associativity which is not the case for standard Hilbert spaces (the reason is the rescaling freedom of the null vectors). Henceforth, it is logical to *impose* (in contradiction to the standard requirement of quantal logic) that

$$\widehat{\delta}(\sum_i \mu_i v_i) \vee \widehat{\delta}(\sum_i \lambda_i v_i) = \widehat{\delta}(\sum_i \mu_i v_i) \wedge \widehat{\delta}(\sum_i \lambda_i v_i) = \widehat{\delta}(\sum_i \frac{1}{2}(\lambda_i + \mu_i) v_i)$$

so that the failure of the de-Morgan on the full pseudo-Hilbert space is due to the failure of the associativity rule of $\vee = \wedge$. Notice that we have a very subtle mid-position here between the demand that the projection operators commute and that they constitute a lattice with a rule as close as possible to the de-Morgan rule. When looking at the affine $n+1$ dimensional subspace, one has associativity of three operators $\widehat{\delta}(\sum_i \lambda_i v_i)$, $\widehat{\delta}(\sum_i \mu_i v_i)$, $\widehat{\delta}(\sum_i \kappa_i v_i)$ on subspace $\mathcal{H}_{\text{affine}}(v_i; \mathcal{V}, h, \lambda_i, \mu_i, \kappa_i)$ of dimension n. These regard subtleties of the classical limit and are really the best one could do.

The standard approach going the other way to distributional states instead of null vectors, emerging from the spectral theorem, has as salient feature that the associated distributional vertex operators do commute. However, they do not constitute an algebra, neither are they projection operators in any sense and albeit the standard definition for $\wedge$ does not function, it is reasonable to put it equal to zero so that the de-Morgan rule is satisfied in a way. Physically, however, it is nonsensical as we cannot ask for the probability that the point $\sum_i \lambda_i v_i$ is probed after $\sum_i \mu_i v_i$ is given a certain state. It is kind of undesirable that probing the simplex at a sharp location completely crumbles it to a point (this problem still holds for finite disjoint regions in the standard approach too given that $P_{\mathcal{O}} \wedge P_{\mathcal{V}} = P_{\mathcal{O} \cap \mathcal{V}}$). Even in standard quantum theory, one can legitimately question this procedure as one would expect a pointlike measurement by some apparatus of a particle to cause for a state of the particle which is initially democratically spread over the entire span of the apparatus as well. Indeed, when spiritually looking for a new place in a city, you are in a way spiritually in the entire city at once prior to further encounters with specific inhabitants. The fringe effects associated to h *outside* of the simplex are then merely due to an incomplete description separating the simplex from all others and leaving for a hard boundary. The price to pay is that h is a ghost and this is rather normal given that stuff is not measured without consequence. In this vein, h is a gauge function meaning that after measurement, the apparatus is in a ground state and the subject is gone from our description and associated to a union with the apparatus instead of being in an individual description. So, after measurement, only a ghost remains of the individual subject state.

We now proceed to define the quantal rules. The key trick to obtain for the classical limit $\epsilon \to 0$ where ϵ is *dimensionless* is to take

$$P^{\epsilon,h}(\mathcal{O}) := \widehat{h} \circ \widehat{R(\epsilon^{-1}\mathcal{O})} \circ P_{\mathcal{O}} \circ \widehat{R(\epsilon\mathcal{O})}$$

where $\widehat{R(\epsilon^{-1}\mathcal{O})}$ is the adjoint and inverse of $\widehat{R(\epsilon\mathcal{O})}$ in the scalar product

$$\langle f|g\rangle := \int_{\mathbb{R}^n} d^n x \sqrt{\eta(x)}\ \overline{f(x)}g(x)$$

and

$$R(\epsilon\mathcal{O})(x) = \epsilon(x - x_{\mathcal{O}}) + x_{\mathcal{O}}$$

with $x_{\mathcal{O}}$ the barycenter of $\mathcal{O}$. Henceforth, $P^{\epsilon,h}(\mathcal{O})$ always remains a Hermitian projection operator apart from fringe effects which only show up when the region $\epsilon^{-1}\mathcal{O}$ exceeds the simplex within $\mathcal{V}$. Furthermore, $\widehat{h}$ is the multiplication operator with the function h whereas $\widehat{R(\lambda\mathcal{O})}$ is defined by means of

$$[\widehat{R(\lambda\mathcal{O})}(f)](x) = f(R(\lambda\mathcal{O})(x)).$$

To obtain the pointlike versions, one first takes the ϵ to zero limit and then notices that only $x_{\mathcal{O}}$ remains important and not the details of its shape. In order to select the particular class of affine functions, we choose the maximal one such that the action of $\widehat{\delta}^{\epsilon,h}$ becomes linear in the limit for ϵ to zero. As it turns out, this space also has the property that it is orthogonal to h and invariant under the scaling procedure as long as one adapts $\mathcal{O}(\epsilon)$ such that $\epsilon^{-1}\mathcal{O}(\epsilon)$ is given by the simplex $(v_1, \ldots, v_{n+1}) \subset \mathcal{V}$ for ϵ sufficiently close to zero. Given, moreover, that the image of $\widehat{\delta}(\sum_i \lambda_i v_i)$ is given by h up to a (λ_i) dependent constant, all operators can be simultaneously brought into standard form

$$\widehat{\delta}(\sum_i \lambda_i v_i)(f) :=$$

$$\left[\int_{\mu_j \geq 0; j:1\ldots n+1} d\mu_1 \ldots d_{\mu_{n+1}} \delta(1 - \sum_{i=1}^{n+1} \mu_i)\delta^n(\vec{\lambda} - \vec{\mu})\langle\delta(\sum_{i=1}^{n+1} \mu_i v_i)|f\rangle\right] h$$

which is a continuous generalization of the Cayley-Hamilton classification scheme for finite dimensional matrices in terms of singular commuting nilpotent matrices. This is all we need and the sober conclusion is that the commuation property

$$\left[\widehat{\delta}(\sum_i \lambda_i v_i), \widehat{\delta}(\sum_i \mu_i v_i)\right] = 0$$

one would normally insist upon, is too strong for our purposes.

We now will formulate general principles of a quantal simplex; we start with a classical vector space $\mathbb{R}^n$ with a compact set $\mathcal{V}$ and a simplex $(v_1, \ldots, v_n)$ in it. We take the flat Euclidean metric η and consider the scalar product

$$\langle f|g\rangle = \int_{\mathbb{R}^n} d^x \sqrt{\eta(x)}\,\overline{f(x)}g(x)$$

as well as the function algebra of square integrable functions with support in $\mathcal{V}$. Take any real positive $h \leq 1$ satisfying the master conditions $h(\sum_i \lambda_i v_i) = 1$ for $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$ and $h = 0$ outside $\mathcal{V}$. Define then

$$\mathcal{H}(v_i; \mathcal{V}, h)$$

as previously and likewise so for

$$\mathcal{H}_{\text{affine}}(v_i; \mathcal{V}, h).$$

Given the scale ϵ of deformation and starting from the quantal case $\widehat{\delta}$ one defines

$$\widehat{\delta}^{\epsilon,h}(\sum_{i=1}^{n+1} \lambda_i v_i) = \widehat{h} \circ R(\epsilon^{-1}\widehat{\mathcal{O}_\epsilon(\sum_{i=1}^{n+1}} \lambda_i v_i)) \circ \widehat{\delta}(\sum_{i=1}^{n+1} \lambda_i v_i) \circ R(\epsilon \widehat{\mathcal{O}_\epsilon(\sum_{i=1}^{n+1}} \lambda_i v_i))$$

where $\epsilon^{-1}\mathcal{O}_\epsilon(\sum_{i=1}^{n+1} \lambda_i v_i)$ satisfies the technical requirement that it becomes the simplex $(v_1, \ldots, v_{n+1})$ for ϵ sufficiently small. Then we demand that for all $0 \leq \epsilon \leq 1$,

$$\widehat{\delta}^{\epsilon,h}(\sum_{i=1}^{n+1} \lambda_i v_i)$$

provides for an orthomodular lattice when the action is restricted to functions with support within the simplex (in either boundary effects are negated). This is equivalent to stating that this is the case for the quantal action $\widehat{\delta}$, the scaling procedure respecting that requirement. Moreover, we also demand that $\widehat{\delta}^{0,h}$ coincides with the classical theory defined by h explained before. Moreover, the function

$$\epsilon \to \widehat{\delta}^{\epsilon,h}$$

must be strongly continuous in the $|| \cdot ||_{\text{sup}}$ norm with boundary conditions

$$||\widehat{\delta}^{0,h}||_{\text{sup},\, h} = 0, \; ||\widehat{\delta}^{1,h}||_{\text{sup}} = 1$$

where the operator or supremum (h) norm is defined by

$$||A||_{\text{sup},\, h} = \sup_{||f||=1} ||A(f)||_h$$

and

$$||A||_{\text{sup}} = \sup_{||f||=1} ||A(f)||$$

and f is understood to be of support within $(v_1, \ldots, v_{n+1})$.

Chapter 3

Classical and quantal particle and simplex dynamics.

Consider a particle moving in a bundle $\mathcal{E}$ over a Lorentzian spacetime $(\mathcal{M}, g)$ where the fibers are equipped with a metric field and the associated connection preserves the total metric (which is usually a product metric). Regard the word line as an immersion $\gamma : \mathbb{R} \rightarrow \mathcal{E}$ and the momentum as its the push forward of ∂_t with equals

$$\frac{D}{dt} := \nabla_{\frac{d}{dt}\gamma(t)}$$

where ∇ is the bundle connection. Given that we shall only work with functions $f : \mathcal{E} \rightarrow \mathbb{R}$, the latter expression can be taken for $(\partial_t)_\star$ as an ordinary vectorfield instead of a general derivative operator. To every curve γ and function f we can attach a function $\gamma_f : \mathbb{R} \rightarrow \mathbb{R} : t \rightarrow f(\gamma(t))$. We can now define a $C^\infty(\mathbb{R})$ algebra of operators $\mathbb{L}$ on the function space $f : \mathcal{E} \rightarrow \mathbb{R}$ mapping them to functions from $\mathbb{R}$ to $\mathbb{R}$. Concretely

$$[(\gamma_f)(g)](t) := f(\gamma(t))g(\gamma(t))$$

and

$$[p_\gamma f](t) := \frac{d}{dt} f(\gamma(t)).$$

We have moreover,

$$\gamma_f(g + h) = \gamma_f(g) + \gamma_f(h)$$

and

$$[(\partial_t)(\gamma_f g)](t) := [(\partial_t)_\star f](t)g(\gamma(t)) + f(\gamma(t))[(\partial_t)_\star(g)](t).$$

This suggests to extend the definition of the momentum in this way to functions $\mathbb{R} \rightarrow \mathbb{R}$. The same comment holds for γ_f. In this vein,

$$[\gamma_g \gamma_f h](t) = g(\gamma(t))f(\gamma(t))h(\gamma(t))$$

and

$$[p_\gamma \gamma_f h](t) := \partial_t(f(\gamma(t))h(\gamma(t)))$$

as well as

$$[\gamma_f p_\gamma h](t) := f(\gamma(t))\partial_t h(\gamma(t)).$$

Finally,

$$[p_\gamma p_\gamma h](t) = (\partial_t)^2 h(\gamma(t))$$

which induces a real algebra generated by

$$\gamma_g, p_\gamma$$

where γ varies over all immersions. This algebra is represented by means of linear operators on the function algebra

$$\mathcal{B} := C^\infty(\mathbb{R}) \otimes C^\infty(\mathcal{E})$$

which may be given the structure of an Hilbert algebra in the usual L^2 sense by introducing an einbein on the "time line" $\mathbb{R}$. Concretely

$$\left[\gamma_f, \gamma_h\right](g) = 0 = \left[p_\gamma, p_\gamma\right](g), \left[p_\gamma, \gamma_f\right](g) = p_\gamma(f)\gamma_\star(g) = \gamma_{p_\gamma(f)}(g)$$

where $\gamma_\star$ is the pull back defined by the immersion γ. Here, the commutation relations employ the full $\mathcal{B}$ action but are understood to apply on $f, g, h \in C^\infty(\mathcal{E})$ and result in an element of $C^\infty(\mathbb{R})$.

Covariant dynamics requires dynamics without potential energy terms; therefore, any force has to be implemented in the momentum what explains the bundle $\mathcal{E}$. Moreover, according to Einstein himself, every force, including the gravitational one, can be gauged away in some point so that locally and physically every particle is a free one meaning that the correct equation is the geodesic bundle equation. Therefore, the classical Hamiltonian is a constraint and moreover, commuting it with a vector defines for a covector if it were an invariant energy so that

$$[\mathcal{H}(\gamma_f, p_\gamma), p_\gamma]$$

cannot represent $\frac{D}{dt}p_\gamma$ unless we would make an extra metric contraction. Actually, the whole Hamiltonian edifice is kind of meaningless as we shall see now. Indeed, taking $\mathcal{H}(\gamma_f, p_\gamma)$ to be p_γ with equations of motion given by

$$[\frac{D}{dt}\triangle\gamma_f](g) := [p_\gamma, \gamma_f](g) = \gamma_{p_\gamma(f)}(g)$$

and

$$[\frac{D}{dt}(\partial_t)_\star](g) = [\frac{D}{dt}\triangle p_\gamma](dg) := [p_\gamma, p_\gamma](dg) = [p_\gamma, p_\gamma](g) = 0$$

where

$$[\frac{D}{dt}\triangle\zeta](g) = [\frac{D}{dt}, \zeta](g).$$

There is nothing more to say really apart from the constraint $g(p_\gamma, p_\gamma) = \frac{m^2c^4}{\hbar^2}$ which is the mass energy relation. This is all what is allowed in classical physics of point particles really and we now proceed to quantum theory. Notice that the dynamical content is completely implied by the commutator algebra which constitutes a total unison between dynamics and kinematics. Physically, this is entirely trivial and completely justified given that the momentum just corresponds to the energy in a rest frame. Note also the presence of $\hbar$ in the latter formula which is there for dimensional reasons; alas, it does not do anything else apart from setting a time scale given that the covariant derivative does not depend upon it.

As we have just shown, the Poisson Bracket really is a commutator and the Hamiltonian formulation is rather void given that the total free momentum, constrained by the quantum mechanical mass formula is the only real quantity of interest. Unlike in classical physics, quantum mechanics cannot use an external time in a sense given that a particle is not specified anymore by a world line but by a wave. In a way, it is the *complex dual* of the classical situation where "world lines" correspond to functions $\psi : \mathcal{E} \to \mathbb{C}$ which are C^∞. The operators γ_f and p_γ are replaced then by x_f and $i\nabla_V$ where V is a real vectorfield over $\mathcal{E}$ and f is a real valued function over $\mathcal{E}$. Here, $[x_f](g)(x) = f(x)g(x)$ and

$$P(V)(g) := i\nabla_V(g) = iV(g).$$

They obey the algebra

$$[x_f, x_h] = 0, [i\nabla_V, i\nabla_W] = -R(V, W)(\cdot) - \nabla_{[V,W]}$$

and finally

$$[i\nabla_V, x_f] = x_{iV(f)}.$$

The momentum commutation relations have been put in this exotic form because the covariant derivative can work on vectorfields and higher objects too. The i is just there to ensure that the momentum operator is real given that the commutator of two real operators is imaginary. The situation here is very different as one cannot just pick a Hamiltonian linear in the momenta given that one would as thus preselect a non dynamical arrow of time. Hence our only choice is given by

$$H = \sum_{i,j=1}^{n} \eta^{ij} \nabla_{E_i} \nabla_{E_j}$$

where the E_i correspond to local vielbeins and η^{ij} is the inverse of the standard flat metric. In order for this to work ∇ must be extended to the spin connection to digest local boost transformations. Furthermore, one has

$$H = m^2$$

as constraint. It is clear one has no Heisenberg type dynamics here as the vectorfields really are spacetime vectorfields; hence, the entire theory is encapsulated by the constraint and the geometry of the bundle $\mathcal{E}$. It has been shown by Ashtekar and Magnon that this theory only works out fine in stationary space times with Minkowski as the prime example due to the existence of scalar products on leafs of a foliation for which the latter is preserved in "time".

In the next section, we generalize classical geometry and subsequently construct a quantum geometry by taking the bi-dual case.

We now return to the issue of classical and quantal constraints on the natural simplex algebra from a dynamical point of view. Note that, so far, we have only used the $\mathbb{R}^n$ metric regarding the volume form and further details are so far absent. We shall adapt the convention here that η is Lorentzian and that $v_i - v_1$ are all spacelike for $i : 2 \ldots n$ and that $v_{n+1} - v_1$ is timelike and therefore determining a natural notion of time in the simplex. The convex sum

$$\sum_{i=1}^{n+1} \lambda_i v_i = \sum_{i=2}^{n+1} \lambda_i (v_i - v_1) + v_1$$

where $\sum_{i=2}^{n+1} \lambda_i < 1$ and all $\lambda_i \geq 0$. λ_1 is then by definition equal to $1 - \sum_{i=2}^{n+1} \lambda_i$. Given this, a natural congruence of vectors exists given by

$$\lambda(s, \sum_{i=2}^{n} \lambda_i (v_i - v_1) + v_1; \sum_{i=2}^{n} \lambda_i < 1) := \sum_{i=2}^{n} (1-s)\lambda_i (v_i - v_1) + s(v_{n+1} - v_1) + v_1.$$

Those are lines with as tangent vector $(v_{n+1} - v_1) - \sum_{i=2}^{n} \lambda_i (v_i - v_1)$ with norm

$$(1, \lambda_i) S (1, \lambda_i)$$

where

$$S_{ij} = \eta_{ab} (v_i - v_1)_a (v_j - v_1)_b$$

which is a $n \times n$ matrix in the index $i, j : 2 \ldots n+1$. The latter determines a Lorentzian Kac-Moody algebra which we shall explicitate later on. Suffice it to say that all vectors $(1, \lambda_i)$ are timelike if and only if

$$(1, \lambda_i) S (1, \lambda_i) > 0$$

which we shall assume the case given that the set $(1, \lambda_i)$ is a convex one and therefore determines a cone which could coincide with the lightcone of the Lorentzian metric η. Therefore we are interested in the time translation operators

$$T(s)(\sum_{i=2}^{n+1} \lambda_i (v_i - v_1) + v_1) = \sum_{i=2}^{n+1} (1-s)\lambda_i (v_i - v_1) + s(v_{n+1} - v_1) + v_1$$

and one notices that

$$T(s)T(t)(\sum_{i=2}^{n+1} \lambda_i (v_i - v_1) + v_1) = T(s)(\sum_{i=2}^{n+1} (1-t)\lambda_i (v_i - v_1) + t(v_{n+1} - v_1) + v_1)$$

$$= T(\sum_{i=2}^{n+1}(1-s)(1-t)\lambda_i(v_i - v_1) + (1-t)s(v_{n+1} - v_1) + t(v_{n+1} - v_1) + v_1)$$

$$= \sum_{i=2}^{n+1}(1-s-t+st)(v_i - v_1) + (s+t-ts)(v_{n+1} - v_1) + v_1$$

giving rise to

$$T(s)T(t) = T(s+t-st)$$

which defines an abelian group but in a rather non-linear parameter. This was to be expected given that time must slow down near the top given by v_{n+1} (there is a huge compression there). The inverse for s is for example given by $t + s - st = 0$ or $t = -\frac{s}{1-s}$. Likewise, we can define

$$\widehat{T}(s)f(\sum_{i=2}^{n+1}\lambda_i(v_i - v_1) + v_1) = f(\sum_{i=2}^{n+1}(1-s)\lambda_i(v_i - v_1) + s(v_{n+1} - v_0) + v_1)$$

$$= f(T(s)(\sum_{i=2}^{n+1}\lambda_i(v_i - v_1) + v_1))$$

and we notice that the operator $\widehat{T}(s)$ has the following property regarding $\widehat{\delta}(v)$

$$([\widehat{\delta}(v)]f)(x) = ([\widehat{\delta}(T(-\frac{s}{1-s})(v))\widehat{T}(s)]f)(x)$$

or

$$\widehat{\delta}(T(s)(v)) = \widehat{\delta}(v)\widehat{T}(-\frac{s}{1-s})$$

and expression which we only demand to be well defined near $s \sim 0$ and it holds as a kind of analycity property. These operators have good commutation properties with the dilations $R(\lambda\mathcal{O})$ so that it is reasonable to demand that in the quantal case the dynamical master formula

$$\widehat{\delta}(T(s)(v)) = \widehat{\delta}(v)\widehat{T}(-\frac{s}{1-s})$$

holds for $s \sim 0$. This is a twisted commutator relation of the kind

$$\left[\widehat{\delta} \circ T(s)\right](v) = (\widehat{T}(s) \circ \widehat{\delta}(v))^{\dagger}$$

given that $T(s)^{\dagger} = T(-\frac{s}{1-s})$ and $\widehat{\delta}$ is Hermitian. The last fact implies that

$$\widehat{\delta} \circ T(s) = \widehat{T}(s) \circ \widehat{\delta}$$

which is normal commuator relation. Henceforth, we have obtained a natural generalization of Heisenberg dynamics. indeed, taking the time derivative

$$\frac{d}{ds}\widehat{\delta} \circ T(s) = \lim_{\epsilon \to 0} \frac{\widehat{\delta} \circ T(s+\epsilon) - \widehat{\delta} \circ T(s)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\widehat{T}(s+\epsilon) - \widehat{T}(s)}{\epsilon} \widehat{\delta}$$

$$= \lim_{\epsilon \to 0} \frac{\widehat{T}(\frac{\epsilon}{1-s}) - 1}{\epsilon} \widehat{\delta} \circ T(s) = \frac{1}{1-s} iH \widehat{\delta} \circ T(s)$$

where

$$iH = \lim_{\epsilon \to 0} \frac{\widehat{T}(\epsilon) - 1}{\epsilon}$$

is anti-Hermitian due to $\widehat{T}^{\dagger}(\epsilon) = \widehat{T}(-\epsilon - \epsilon^2 + \ldots)$ and H anti-commutes with $\widehat{\delta}$. This has the same shape of the covariant Schroedinger equation in relativistic quantum theory where unitary operators are replaced by Hermitian projection operators. Due to the anti-commutation, it is also equivalent to the standard Heisenberg dynamics

$$\frac{d}{ds} \widehat{\delta} \circ T(s) = \frac{i}{2(1-s)} \left[H, \widehat{\delta} \circ T(s) \right]$$

with dynamical Planck constant $2(1-s)$. Note that it concerns a theory with a limited scaling (in either conformal) symmetry which provides for the notion of time. This scaling symmetry actually is a conformal diffeomorphism of the metric and at the same time a symmetry of the associate measure. Actually, all one needs is a one parameter group of diffeomorphisms which are symmetries of the measure, so called ergodic tranformations of the *space-time* volume form instead of the microcanonical one used in statistical physics where any symplectic transformation provides for such a symmetry (and the dynamics provides for such a symplectic transformation by Liouville theorem). Specifically, we search for a vectorfield $\mathbf{V}$ such that

$$\mathcal{L}_{\mathbf{V}} (\sqrt{\eta(x)} dx^1 \wedge \ldots \wedge dx^n) = 0$$

an equation which reduces to $\mathbf{V}$ having zero divergency

$$\nabla_{\mu} V^{\mu} = 0$$

whereas the demand of a conformal symmetry leads to

$$\mathcal{L}_{\mathbf{V}} g = \Omega g$$

or equivalently

$$\nabla_{(\mu} V_{\nu)} = \Omega g_{\mu\nu}.$$

It is easily verified that the correct group parameter is given by $\tilde{s}$ with $s = 1 - e^{-\alpha \tilde{s}}$ so that the correct vectofield is provided by

$$\mathbf{V}(\lambda_i, s) = (-\sum_{i=2}^{n} \lambda_i (v_i - v_1) + (v_{n+1} - v_1)) \alpha (1-s) = \alpha(-\lambda_i(1-s), 1-s)$$

whereas the metric in the (λ_i, s) coordinate system reads

$$(d[(1-s)\lambda_i], ds)S(d[(1-s)\lambda_i], ds) = (-\lambda_i ds + (1-s)d\lambda_i, ds)S(-\lambda_i ds + (1-s)d\lambda_i, ds)$$

$$= (ds)^2(-\lambda_i, 1)S(-\lambda_i, 1) + \sum_{i,j=2}^{n}(1-s)^2 S_{ij}d\lambda_i d\lambda_j + 2(1-s)\sum_{i=2}^{n} d\lambda_i ds S_i(-\lambda_k, 1)$$

$$= e^{-2\alpha\tilde{s}}\left[(d\tilde{s})^2(-\lambda_i, 1)S(-\lambda_i, 1) + \sum_{i,j=2}^{n} S_{ij}d\lambda_i d\lambda_j - 2\alpha\sum_{i=2}^{n} d\lambda_i d\tilde{s} S_i(-\lambda_k, 1)\right].$$

The conformal symmetry under $U(\tilde{s}) = T(1 - e^{\alpha\tilde{s}})$ is now obvious as well is the invariance of the measure.

Chapter 4

Classical metric spaces and connection theory thereupon.

This chapter is entirely new and contains novel means of reflection about geometry. To summarize again, we started from the edifice of set theory, followed by number theory, theory of linear spaces and operators, differential geometry based upon the notion of a perfect exterior derivative d obeying $d^2 = 0$. We were somewhat extravagant is using coordinate dependent methods associated to local linear properties. We shall now forget about perfect exterior derivatives as well as local arthmetic properties inherited from function theory on $\mathbb{R}^n$. This implies we have to abandon a naive definition of the Lie bracket and we shall try to restitute it later on from the viewpoint of generalized connection theory.

Topological differentials.
Crucial here it a *geometrical* notion of transport: consider two sets X, Y with reflexive and symmetrical relations $R \subset X \times X, T \subset Y \times Y$ which are topologically open, then we call

$$\nabla_X : \{(x, y, z) : y, z \in R(x, \cdot)\} \to R : (x, y, z) \to \nabla_{(x,y)}(x, z) \in R(y, \cdot)$$

the transported relation (x, z) over the relation (x, y) from x to y. ∇_X obeys the following property: one can find an open neighborhood $\mathcal{Q} \subset R$ obeying $\pi_1(\mathcal{Q}) = X$ such that for each $(x, w) \in \mathcal{Q}$ there exists an open set $\mathcal{W}$ of w as well as an open set $\mathcal{O} \subset R(x, \cdot) \subset X$ such that the mapping

$$T_{\mathcal{O}} : \{(x, y, z) : y, z \in \mathcal{O}\} \to X : (x, y, z) \to \pi_2(\nabla_{(x,z)}(x, y))$$

is continuous and surjective on $\mathcal{W}$. Here, π_j equals the projection on the j'th factor. Moreover, we define the *commutator* between two elements of $R(x, \cdot)$ as

$$[(x, w), (x, v)] = ((v, x) \circ P(\nabla_{(x,v)}(x, w)))\sharp(\nabla_{(x,w)}(x, v) \circ (x, w))$$

where $(w,z)\sharp(x,v) = \big(\nabla_{(w,v)}(w,z)\big) \circ (x,v)$ and $P(x,w) = (w,x)$. Notice that, in a certain way, we have by definition that the connection is torsion free which is logical given that the standard definition of the commutator follows from the vector space structure of $\mathbb{R}^n$. Given that we do not dispose of such a thing, we have no choice for now but to define the commutator by means of the connection. Later on, by means of different techniques of "Riemannization" shall we single out a torsion free connection which will enable us to define torsion as well as the appropriate commutator in the end. We call a function $F : X \to Y$ differentiable in an open neighborhood of $x \in X$ in case for any open $\mathcal{V} \subset T(F(x), \cdot)$ there exists an open $\mathcal{O} \subset R(x, \cdot)$ as well as a bi-continuous mapping $DF_w(w,v) : (w,v) \in \mathcal{O}^2 \to \mathcal{V}^2$ defined by $(F(v), F(w)) = DF_v(v,w)$ and henceforth having the property that

$$DF_x\left(\big(\nabla_{(x,w)}(x,y)\big) \circ (x,w)\right) = DF_w(\nabla_{(x,w)}(x,y)) \circ DF_x(x,w).$$

To complete the definition in a non trivial way and to impose a suitable substitute for linearity, we introduce scale functions $h, g : R, T \to \mathbb{R}^+$ satisfying $g(x,x) = h(x',x') = 0$ and nonzero otherwise. Concretely, the *existence* of the differential is expressed by means of the condition

$$\frac{g(P(DF_x(x,w))\sharp DF_x(w,w))}{h(v,w)} < C$$

for a certain positive constant C; *linearity* of the D mapping requires g to be a metric on Y. One notices that the composition $\circ$ on R (T) is defined by means of $(w,z) \circ (x,w) = (x,z)$. The reader is invited to further expand upon these ideas and develop a notion of Riemann curvature (which we shall do later on). We expand now on our arsenal of definitions: a connection is "metrically constant" if and only if

$$d(\nabla_{(xy)}(xz)) = d((xz)).$$

One can extend now the entire edifice of the theory of linear functionals in our setting of topological differentials and torsion less connections ∇_X. A functional ω_X is an anti-symmetric continuous function on the displacements (x,y) satisfying

$$\omega_X((y,z) \circ (x,y)) = \omega_X((x,y)) + \omega_X((y,z))$$

as well as

$$\omega_X((x,y)) = -\omega_X((y,x)).$$

A curve is a one dimensional object without holes in X; more in particular $\gamma \subset X$ is a curve if and only if there exists a homeomorphism ψ of γ to a connected subset $A \subset \mathbb{R}$ such that for each $r < s \in A$ the displacement $(\psi^{-1}(r), \psi^{-1}(s))$ is irreducible in the limit for s to r in A. A displacement is irreducible if and only if

$$\lim_{s \to_{>r} r} \left| \frac{\omega_X(D\psi_r^{-1}(r,s))}{s-r} \right| < C(\psi^{-1}, \omega_X)$$

for any continuous functional ω_X.

All this leads to the definition of forward differential equations in the sense that

$$\frac{d}{ds}F(\psi^{-1}(s)) = g(\psi^{-1}(s))$$

for all continuous functions $F, g : X \to \mathbb{C}$ where $\frac{d}{ds}$ stands for

$$\lim_{r \to_> s} \frac{f(r) - f(s)}{r - s}$$

and everything is supposed to be ψ independent. This means that the whole expression must be invariant under order preserving diffeormorphisms $\phi : A \to A$ in the sense that the equation is invariant under the substitution $\psi \to \psi \circ \phi$.

In the sequel, we take notice of the scaling functions and introduce a scale ϵ such that all pairs $(x, y) \in R$ satisfy $h(x, y) \geq \epsilon$. Given X, then we define the maximal directions set TX^2 by means of couples $(x, y) \in R$ such that there exist no z, w such that

$$(x, y) = \nabla_{(x,w)}(x, z) \circ (x, w)$$

with exception of the zero displacement. This definition can be generalized towards TX^l where a displacement cannot be written as a composition of l displacements in x which are suitably transported. In what follows, we work with $TX_m = \cap_{m \geq l \geq 2} TX^l$ and we demand moreover that $\nabla_{(x,y)}(x, z) \in TX^l$ for $(x, y), (x, z) \in TX^l$ and that TX^l is maximal; in other words ∇ constitutes an internal operation and TX^l is chosen as large as possible. The local tangent space is then given by $TX^l_x = \{(x, y) | (x, y) \in TX^l, y \in X\}$ and is as such a local construct which has to obey global conditions. In particular, we are interested in the following notions: we say that two maximal direction sets TX^l_1 and TX^l_2 obey $TX^l_2 < TX^l_1$ if and only if any $(x, y) \in TX^l_1$ may be written as the limit of finite compositions of elements in TX^l_2. We consider moreover infinite sequences $TX^l_{k+1} < TX^l_k$ and inductive limits $\wedge_{k,l} TX^l_k$ thereof. These limits of maximal direction sets are called super maximal direction sets and those carry away our interest. These super maximal direction sets do not necesarily contain minimal elements although displacements of elements are clearly elements again. For manifolds, where the local tangent spaces are equal to one and another, there is no obstruction or limitation. In general, given that m scales appropriately in function of k in TX_{k,m_k} all global obstructions dissapear in the definition of $\wedge_{k,l} TX^l_k$ which allows one to speak of a "local" bundle.

4.1 Riemannian geometry.

We continue our study of complete path metric spaces; as a recollection, the latter are denoted by (X, d). By definition, we have that for any $x, y \in X$ there exists a $z \in X$ such that

$$d(x,z) = d(y,z) = \frac{d(x,y)}{2}.$$

The existence of a midpoint is equivalent to the existence of a geodesic $\gamma : [0,1] \to X$ minimizing the length functional L for paths with fixed enpoints x, y and moreover, it holds that $L(\gamma) = d(x,y)$. The functional is defined by means of

$$L(\gamma) = \sup_{0=t_0<t_1\ldots<t_n=1, n>0} \sum_{j=0}^{n-1} d(\gamma(t_j), \gamma(t_{j+1}))$$

and γ may be reparametrized in arclength by means of the Radon Nikodym derivative. In the framework of the definition of sectional curvature, we did study the quantity

$$R_n(y,z) = \frac{-2d(x,y_n)^2 - 2d(x,z_n)^2 + d(y_n,z_n)^2 + 4d(x,r_n)^2}{d(x,y_n)^2 d(x,z_n)^2 \sin^2(\theta_x(y_n,z_n))}$$

having a dimension of m^{-2}. For metric and torsionless theories, the teller is given up till fourth order by

$$d^2(x+w, x+v) = g_x(v-w, v-w) + \gamma(N) g_x(R_x(v,w)v, w) + \delta(N) g_x(C_x(v,w)v, w) +$$

$$\kappa(N) R_x(g_x(v,v)g_x(w,w) - g(v,w)^2) + \text{ higher order terms}$$

where R_x constitutes the Ricci scalar and C_x the Weyl tensor. Terms of the form $\zeta(N)(R_x(v,v)g_x(w,w) + R_x(w,w)g_x(v,v))$ are forbidden given that they do not vanish in the limit $v = w$. From the geodesic follows that $\delta(N) = \kappa(N) = 0$ and $\gamma(N) = \gamma$ a constant independent of N. This shows that we work with the correct definition; given the lack, at this point, of a coordinate independent definition of torsion, we work towards a natural definition for the Riemann curvature based upon symmetry demands

$$T_n(a,c,b,d) + T_n(c,b,a,d) + T_n(b,a,c,d) = 0$$

as well as

$$-T_n(a,b,d,c) = T_n(a,b,c,d) = -T_n(b,a,c,d) = T_n(c,d,a,b)$$

and request moreover that T_b is quadratic in the metric and produces the correct sectional curvature. In the framework of the first Bianchi type symmetry properties which partially reflect linearity one considers the permutations $\alpha = (12), \beta = (34)$ and $\tau = (13)(24)$ where α, β, τ constitute a subgroup K of the permutation group S_4 keeping the following relations in mind $\alpha^2 = \beta^2 = \tau^2 = 1$ en $\alpha \circ \beta = \beta \circ \alpha, \alpha \circ \tau = \tau \circ \beta, \beta \circ \tau = \tau \circ \alpha$. Consider a functional

$$H(a_i; i = 1 \ldots 4) = \sum_{\sigma \in K} \text{Sign}(\sigma) F(a_{\sigma(i)})$$

where F remains invariant under a permutation $\rho \in S_4$ not belonging to K (otherwise $H(a_i; i = 1 \dots 4)$ vanishes), then $H(a_i; i = 1 \dots 4)$ obeys the first Bianchi identity

$$\sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) H(a_0, a_{\sigma(i)}) = 0.$$

Define $G(a_0, a_i) = \sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) H(a_0, a_{\sigma(i)})$ and remark that S_3 is generated by $r = (123), t = (23)$ where

$$(123)(23) = (12), (23)(123) = (13)$$

and $(123)^3 = e$. Each element is therefore of the form $r^j t^p$ with $j = 0 \dots 2; p = 0, 1$; consequently

$$G(a_0, a_i) = \sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) \sum_{\kappa \in K} \mathrm{Sign}(\kappa) F(a_{\sigma(\kappa(i))})$$

which allows for further reduction to

$$2 \sum_{j=0}^{2} \sum_{\kappa \in K} \mathrm{Sign}(\kappa) F(a_{r^j(\kappa(i))}).$$

Group theoretically, one arrives at $r \kappa r^2 \in K_+$ for $\kappa \in K_+$ even and $r \kappa r \in K_-$ for $\kappa \in K_-$ odd. Consequently

$$\sum_{\kappa \in K_+} \mathrm{Sign}(\kappa) F(a_{\kappa(i)}) + \sum_{\kappa \in K_-} \mathrm{Sign}(\kappa) F(a_{\kappa(i)})$$

$$+ \sum_{\kappa \in K_+} \mathrm{Sign}(\kappa) F(a_{\kappa(r(i))}) + \sum_{\kappa \in K_-} \mathrm{Sign}(\kappa) F(a_{\kappa(r^2(i))}) +$$

$$\sum_{\kappa \in K_+} \mathrm{Sign}(\kappa) F(a_{\kappa(r^2(i))}) + \sum_{\kappa \in K_-} \mathrm{Sign}(\kappa) F(a_{\kappa(r(i))}).$$

So one arrives at

$$G(a_0, a_i) = 2 \sum_{j=0}^{2} \sum_{\kappa \in K} \mathrm{Sign}(\kappa) F(a_{\kappa(r^j(i))}) = 2 \sum_{\rho \in S_4} \mathrm{Sign}(\rho) F(a_{\rho(i)})$$

which shows that the sum procedures over K and S_3 commute. Note that $(13) = (12)(23)(12)$ and therefore $\rho = (13)$ may substituted by means of a conjugation with (12) in K to (23); the same holds for (14) or (24) which may be conjugated in K to (23) what demonstrates that $\rho = s(23)s^{-1}$ with $s \in K$. Evidently, it follows that

$$G(a_0, a_{(23)(i)}) = G(a_0, a_i) = -G(a_0, a_i) = 0$$

where we used the properties of Sign as well as the F invariance under ρ.

We now return to the general analysis of the Riemann tensor using the previous theorem as well as the quadratic nature of the expression. Mind, we also add terms which break the first Bianchi identity and argue from a distinct perspective why those should vanish:

$$T_x^{\epsilon,\alpha,\beta,\kappa,\delta,\lambda,\gamma,\mu,\zeta,\rho,\nu,\sigma,\psi,\phi,\pi}(a,b,c,d) = \frac{1}{\epsilon^4}\left(\alpha(d(\widehat{bd},\widehat{ac})^2 - d(\widehat{bc},\widehat{ad})^2) + \beta(d(x,\widehat{\widehat{ac}\widehat{bd}})^2 - d(x,\widehat{\widehat{ad}\widehat{bc}})^2)\right)$$
$$+\frac{1}{\epsilon^4}\left(\kappa(d(a,c)^2 - d(a,d)^2 - d(b,c)^2 + d(b,d)^2) + \delta(d(a,\widehat{bd})^2 - d(a,\widehat{bc})^2 - d(b,\widehat{ad})^2 + d(b,\widehat{ac})^2)\right)$$
$$+\frac{1}{\epsilon^4}\left(\lambda(d(x,\widehat{ac})^2 - d(x,\widehat{ad})^2 - d(x,\widehat{bc})^2 + d(x,\widehat{bd})^2) + \gamma(d(x,\widehat{ac})d(x,\widehat{bd}) - d(x,\widehat{ad})d(x,\widehat{bc}))\right)$$
$$+\frac{\mu}{\epsilon^4}\left(d(x,a)d(x,c) - d(x,a)d(x,d) - d(x,b)d(x,c) + d(x,b)d(x,d)\right)$$
$$+\frac{\rho}{\epsilon^4}(d(x,\widehat{ad})d(x,\widehat{bc}) - d(x,\widehat{ac})d(x,\widehat{bd}))$$
$$+\frac{\zeta}{\epsilon^4}(d(x,a)d(x,\widehat{ac}) - d(x,a)d(x,\widehat{ad}) - d(x,b)d(x,\widehat{bc}) + d(x,b)d(x,\widehat{bd}) + d(x,c)d(x,\widehat{ac}))$$
$$+\frac{\zeta}{\epsilon^4}(-d(x,c)d(x,\widehat{cb}) - d(x,d)d(x,\widehat{da}) + d(x,d)d(x,\widehat{db}))$$
$$+\frac{\nu}{\epsilon^4}(d(x,\widehat{a\widehat{ac}})d(x,\widehat{bc}) - d(x,\widehat{b\widehat{bc}})d(x,\widehat{ac}) - d(x,\widehat{a\widehat{ad}})d(x,\widehat{bd}) + d(x,\widehat{b\widehat{bd}})d(x,\widehat{ad}))$$
$$+\frac{\nu}{\epsilon^4}(d(x,\widehat{c\widehat{ac}})d(x,\widehat{da}) - d(x,\widehat{d\widehat{ad}})d(x,\widehat{ac}) - d(x,\widehat{c\widehat{bc}})d(x,\widehat{bd}) + d(x,\widehat{d\widehat{bd}})d(x,\widehat{bc}))$$
$$+\frac{\sigma}{\epsilon^4}(d(x,a)d(x,\widehat{d\widehat{cb}}) - d(x,a)d(x,\widehat{c\widehat{db}}) - d(x,b)d(x,\widehat{d\widehat{ca}}) + d(x,b)d(x,\widehat{c\widehat{da}}))$$
$$+\frac{\sigma}{\epsilon^4}(d(x,c)d(x,\widehat{b\widehat{ad}}) - d(x,c)d(x,\widehat{a\widehat{db}}) - d(x,d)d(x,\widehat{b\widehat{ca}}) + d(x,d)d(x,\widehat{a\widehat{bc}}))$$
$$+\frac{\psi}{\epsilon^4}(d(x,a)d(x,\widehat{d\widehat{ab}}) - d(x,a)d(x,\widehat{c\widehat{ab}}) - d(x,b)d(x,\widehat{d\widehat{ba}}) + d(x,b)d(x,\widehat{d\widehat{db}}))$$
$$+\frac{\psi}{\epsilon^4}(d(x,c)d(x,\widehat{b\widehat{cd}}) - d(x,c)d(x,\widehat{a\widehat{cd}}) - d(x,d)d(x,\widehat{d\widehat{cd}}) + d(x,d)d(x,\widehat{b\widehat{db}}))$$
$$+\frac{\phi}{\epsilon^4}(d(x,a)d(x,\widehat{b\widehat{cb}}) - d(x,a)d(x,\widehat{b\widehat{db}}) - d(x,b)d(x,\widehat{a\widehat{ca}}) + d(x,b)d(x,\widehat{a\widehat{da}}))$$
$$+\frac{\phi}{\epsilon^4}(d(x,c)d(x,\widehat{d\widehat{ad}}) - d(x,c)d(x,\widehat{d\widehat{bd}}) - d(x,d)d(x,\widehat{c\widehat{ca}}) + d(x,d)d(x,\widehat{c\widehat{cb}}))$$
$$+\frac{\pi}{\epsilon^4}(d(x,a)d(x,\widehat{a\widehat{cb}}) - d(x,a)d(x,\widehat{a\widehat{db}}) - d(x,b)d(x,\widehat{b\widehat{ca}}) + d(x,b)d(x,\widehat{b\widehat{da}}))$$
$$+\frac{\pi}{\epsilon^4}(d(x,c)d(x,\widehat{c\widehat{ad}}) - d(x,c)d(x,\widehat{c\widehat{bd}}) - d(x,d)d(x,\widehat{d\widehat{ca}}) + d(x,d)d(x,\widehat{d\widehat{cb}})).$$

All expressions prior to the ρ terms satisfy our criterion regarding the first Bianchi identity whereas ths subsequent ones do not. Explicit verification shows indeed that this identiy is violated. “Ricci-Alexandrov” contraction gives

$$T_x^{\epsilon,\alpha,\beta,\kappa,\delta,\lambda,\gamma,\mu,\zeta,\rho,\nu,\sigma,\psi,\phi,\pi}(a,b,a,b) = \frac{1}{\epsilon^4}\left((2\delta - 2\kappa + \alpha)\, d(a,b)^2 + (\lambda + \mu + 2\zeta)(d(x,a)^2 + d(x,b)^2)\right)$$

$$+\frac{1}{\epsilon^4}\left(-(2\lambda+\gamma-\rho)d(x,\widehat{ab})^2+(\gamma-2\mu-\rho-4\phi)d(x,a)d(x,b)-\delta(d(a,\widehat{ab})+d(b,\widehat{ab}))\right)$$

$$+\frac{2((\nu-\sigma)-\zeta)}{\epsilon^4}\left(d(x,a)d(x,\widehat{ab})+d(x,b)d(x,\widehat{ab})\right)+$$

$$\frac{2((\sigma-\nu)+\phi-(\pi-\psi))}{\epsilon^4}\left(d(x,a)d(x,\widehat{b\widehat{ab}})+d(x,b)d(x,\widehat{a\widehat{ab}})\right)$$

$$+\frac{2(\pi-\psi)}{\epsilon^4}\left(d(x,a)d(x,\widehat{a\widehat{ab}})+d(x,b)d(x,\widehat{b\widehat{ab}})\right).$$

Consequently, $-2\lambda-\gamma+\rho=4, \lambda+\mu+2\zeta=-2, \zeta-\nu+\sigma=0$ and

$$2\delta-2\kappa+\alpha=1, \gamma-2\mu-\rho-4\phi=0, \delta=0=\pi-\psi=\zeta-\phi$$

leaving for seven free parameters from in total fourteen given that the system has a single degeneracy. It is logical that $\alpha, \beta, \nu, \delta$ must vanish given that the number of d-arguments in the associated terms is larger than four which goes beyond the linearity of the Riemann tensor. This leaves for four free parameters. In differential geometry, we have two tensors of second order in the metric with the symmetries of the Riemann tensor which do not contribute to the sectional curvature. They are given by $R\sqrt{g}\epsilon_{\mu_1\mu_2\mu_3\mu_4}, \sqrt{g}\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ what suggests that only two parameters have to be fixated.

Let us look at the Bianchi violating terms from our "Ricci-contraction" which are given by

$$(\pi-\psi), (\sigma+\phi)-(\pi-\psi), -(\sigma+\phi)$$

and note that those contain a cycle in the sense that they sum up to zero. This formula has a homological interpretation in the sense that the boundary of the boundary of the triangle defined by

$$0, (\pi-\psi), (\sigma+\phi)$$

vanishes which strongly suggests the existence of a second cohomology class associated to those terms (more precisely one would expect the Gauss Bonnet term or Ricci density over the triangle (x, a, b)). This is precisely the case given that the vanishing of those terms leaves for a ϕ term

$$(d(x,a)-d(x,b))^2$$

in the above formula which is a boundary contribution indeed. We conclude henceforth that only the four parameters ψ, ϕ, μ, ρ remain and moreover holds that

$$\delta=0, \pi=\psi, \zeta=\phi, \lambda=-2-\mu-2\phi, \gamma=2\mu+\rho+4\phi, \kappa=-\frac{1}{2}.$$

Further fixation of three parameters occurs by noticing that μ, γ, ρ terms do not represent real quadratic forms and therefore are not analytical in the x, a, b, c, d

variables. This implies they behave badly under "rescalings" of $(xa), (xb), \ldots$ and consequently $\mu = \gamma = \rho = 0$. We remain with the "tensor"

$$T_x^{\epsilon,\alpha,\beta,\kappa,\delta,\lambda,\gamma,\mu,\zeta,\rho,\nu,\sigma,\psi,\phi,\pi}(a,b,c,d) = T_x^{\epsilon,0,0,-\frac{1}{2},0,-2,0,0,0,0,0,0,\pi,0,\pi}(a,b,c,d)$$

and the reader verifies that the sum of coefficients is given by $-\frac{5}{2} + 2\pi$. Because the π terms violate the first Bianchi identity, it is natural to eliminate them; on the other hand π must at least depend in a linear fashion upon ϵ such that it vanishes in the limit for epsilon to zero. In that case, it provides for a finite correction to the first Bianchi identity depending upon the localization scale.

We define now the one parameter family of "metrics"

$$g_x^{\epsilon}(a,b) = \frac{d(x,a)d(x,b)\cos(\theta_x(a,b))}{\epsilon^2}$$

as well as the inverses $g_x^{\epsilon}(\widehat{a},\widehat{b})$ by means of

$$\int_{B(x,\epsilon)} d\mu_d(b) g_x^{\epsilon}(\widehat{a},\widehat{b}) g_x^{\epsilon}(b,c) = \delta(a,c)$$

where μ_d constitutes the Hausdorff measure. The construction of a δ function is entirely trivial and the existence of a unique inverse $g_x^{\epsilon}(b,c)$ follows from the fat that the former defines a Toeplitz operator with trivial kernel. It is to say, $g_x^{\epsilon}(\widehat{a},\widehat{b})$ is the standard Green's function of the metric regarding the Hausdorff measure. To construct the proof, note that $g_x^{\epsilon}(b,c)$ separates the points meaning that

$$g_x^{\epsilon}(b,\cdot) : B(x,\epsilon) \to \mathbb{R} : c \to g_x^{\epsilon}(b,c)$$

constitutes a basis with regard to the standard Hilbert inproduct.

Prior to defining contractions with the metric tensor, remark that

$$\int_{B(x,\epsilon)} \int_{B(x,\epsilon)} d\mu_d(b) d\mu_d(a) g_x^{\epsilon}(\widehat{a},\widehat{b}) g_x^{\epsilon}(b,a)$$

is ill defined and requires "a point splitting" procedure to obtain a well defined answer. Concretely, we consider

$$\int_{B(x,\epsilon)} \int_{B(x,\epsilon)} d\mu_d(b) d\mu_d(a) \int_{B(a,\delta)} \int_{B(b,\delta)} d\mu_d(d) d\mu_d(c) g_x^{\epsilon}(\widehat{a},\widehat{b}) g_x^{\epsilon}(c,d) \sim$$

$$\alpha^2(x)\delta^{2d_{H,x}}(1 + \text{epsilon, delta correcties})$$

what suggests to take the δ-derivative and divide by

$$2\alpha^2(x)\delta^{2d_{H,x}-1}.$$

Subsequently, one considers the limit for $\delta \to 0$ leading to

$$T_x^{\epsilon,\pi}(a,b) := \lim_{\delta\to 0} \frac{1}{2\alpha^2(x)\delta^{2d_{H,x}-1}} \frac{d}{d\delta} \int_{B(x,\epsilon)} d\mu_d(k) \int_{B(x,\epsilon)} d\mu_d(l) \int_{B(k,\delta)} d\mu_d(r) \int_{B(l,\delta)} d\mu_d(s) g_x^{\epsilon}(\widehat{k},\widehat{l})$$

$$T_x^{\epsilon,0,0,-\frac{1}{2},0,-2,0,0,0,0,0,0,\pi,0,\pi}(r,a,s,b)$$

as well as a Ricci scalar $T_x^{\epsilon,\pi}$ by means of a similar procedure. The Einstein tensor is subsequently defined as

$$G_x^{\epsilon,\pi}(a,b) = T_x^{\epsilon,\pi}(a,b) - \frac{1}{2} g_x^{\epsilon}(a,b) T_x^{\epsilon,\pi}$$

and the trace reads

$$G_x^{\epsilon,\pi} = -\frac{d_{H,x} - 2}{2} T_x^{\epsilon,\pi}$$

in case $d_{H,x}$ is a continuous function; otherwise, we define π as a function of x and ϵ such that the above equation holds. One may hope that the Bianchi violating π terms are a consequence of a bizarre topological structure and should vanish in the limit $\epsilon \to 0$.

Define g^{ϵ} geodesics as curves γ minimizing the length functional

$$L(\gamma) = \sup_{0=t_0,\ldots t_n=1, n>0} \sum_{j=0}^{n-1} \sqrt{g^{\epsilon}_{\gamma(t_j)}(\gamma(t_{j+1}), \gamma(t_{j+1}))}\, \epsilon.$$

In order to define torsion, we continue our study of the Levi-Civita transporter. We say that $\nabla^{\epsilon,\delta}_{(x,a)}$ is metric compatible if and only if

$$\nabla^{\epsilon,\delta}_{(x,a)}\left(g_x^{\epsilon}(b,c)\right) = g_a^{\epsilon}(\pi_2(\nabla^{\epsilon,\delta}_{(x,a)}(x,b)), \pi_2(\nabla^{\epsilon,\delta}_{(x,a)}(x,c))) = g_x^{\epsilon}(b,c)$$

for each x, a and $b, c \in B(x,\epsilon)$ with $0 \leq \theta_x(a,c), \theta_x(a,b) < \frac{\pi-\delta}{2}$. The ∇-ϵ dependency resides in the condition that a, b, c are sufficiently close to x and δ allows for conical singularities. Evidently, $d(x,a) = d(\nabla^{\epsilon,\delta}_{(x,b)}(x,a))$ for all x, a, b such that $\nabla^{\epsilon,\delta}$ is well defined. There exist plenty of $\nabla^{\epsilon,\delta}$ on a conical space where 2δ is larger as the difference or opening angle.

A ∇ geodesic is an autoparallel curve in arclength parametrization: that is

$$\nabla_{(\gamma(t),\gamma(t+\epsilon))}(\gamma(t), \gamma(t+\epsilon)) \circ (\gamma(t), \gamma(t+\epsilon)) = (\gamma(t), \gamma(t+2\epsilon)).$$

Give an example that not every geodesic is a ∇-geodesic for a specific ∇ such as is the case on conical spaces. Reversely, one has that not every ∇-geodesic constitutes a geodesic in the sense that

$$d(\gamma(t), \gamma(t+2\delta)) < 2\delta$$

something which is known to be the case for Riemannian manifolds with a nontrivial topology or in case of conical spaces where ∇ geodesics can travel through the singularity. Clearly, it holds that a geodesic is a ∇-geodesic for *some* ∇; the fact that each ∇-geodesic defines locally a geodesic requires a condition such as

$$\theta_y(x, \pi_2(\nabla_{(x,y)}(x,z))) = \pi - \theta_x(y,z)$$

for $\theta_x(y,z) < \pi - \delta$ what constitutes a retro progressive condition associated to differentiability.

Define the scalar product

$$\langle T(\cdot)|g(\cdot)\rangle_{x,\epsilon} = \int_{B(x,\epsilon)} \ldots \int_{B(x,\epsilon)} d\mu_d(a_1)\ldots d\mu_d(a_k)d\mu_d(b_1)\ldots d\mu_d(b_k)\overline{T(a_i)}g(b_i)g^{\epsilon}_x(\kappa_i(a_i),\kappa_i(b_i))$$

where κ_i is given by the unit or duality operation. This scalar product is not positive definite in general, but it is so for spaces with positive Alexandrov curvature but not so for the negative case. For standard Euclidean vector spaces, the ϵ metric is a positive multiple, with factor $\frac{\mathrm{Vol}^2(SO(N))\epsilon^{2N}}{N^2\mathrm{Vol}^2(SO(N-1))}$ of the standard tensorial inproduct.

Now, we shall deal with the appropriate definition of torsion; using the notation

$$(x,v)\oplus(x,w) = \nabla_{(x,w)}(x,v)\circ(x,w)$$

as well as

$$(x,w)\ominus(x,v) = P(x,v)\sharp(x,w)$$

one arrives at

$$[(x,v),(x,w)] = (\nabla_{(x,v)}(x,w)\circ(x,v))\ominus(\nabla_{(x,w)}(x,v)\circ(x,w))$$

what lead to my previous statement of the vanishing of the torsion tensor. At least, this is the only definition possible using the connection alone; zero torsion

$$T^{\epsilon,\delta}_x(a,b)\in R(x,\cdot)$$

with $a,b\in B(x,\epsilon)$ is a gauge enforced by the fact that $R_x(a,b,c)$ defined by

$$\left(\nabla^{\epsilon,\delta}_{(x,a)}(\nabla^{\epsilon,\delta}_{(x,b)}(x,c)\circ(x,b))\circ(x,a)\ominus\nabla^{\epsilon,\delta}_{(x,b)}(\nabla^{\epsilon,\delta}_{(x,a)}(x,c)\circ(x,a))\circ(x,b)\right)\ominus\nabla_{[(x,a),(x,b)]}(x,c)\circ[(x,a),(x,b)]$$

obeys

$$\hat{d}(R_x(a,b,c)) = \int_{B(x,\epsilon)} d\mu_d(k)g^{\epsilon}_x(\hat{d},\hat{k})R_x(a,b,c,k)$$

for a metric compatible connection $\nabla^{\epsilon,\delta}$. This demand may somewhat be too strong and minimization of the associated quadratic expression may be desirable. Such a connection is called Levi-Civita and torsion of another metric compatible connection $\widetilde{\nabla}^{\epsilon,\delta}$ is defined by means of

$$T^{\epsilon,\delta}_x = \widetilde{\nabla}^{\epsilon,\delta}\ominus\nabla^{\epsilon,\delta}.$$

This does not imply however that the second Bianchi identity

$$\sum_{\sigma\in S_3}\mathrm{Sign}(\sigma)\nabla_{a_{\sigma(1)}}R_x(a_{\sigma(2)},a_{\sigma(3)},b,c) = 0$$

is obeyed.

4.2 The Lorentzian theory.

The matter now is how to generalize the above theory towards spaces with a Lorentz metric. We henceforth consider spaces (X, d) with a compact topology such that $d : X \times X \to \mathbb{R}^+$ is continuous and obeys

- $d(x, y) \geq 0$ and $d(x, x) = 0$
- $d(x, y) > 0$ implies that $d(y, x) = 0$
- $d(x, y) > 0$ and $d(y, z) > 0$ implies that $d(x, z) > 0$.

From d, we construct a chronology relation $y \in I^+(x)$ if and only if $d(x, y) > 0$ where $I^+(x)$ constitutes the set of all happenings in the chronological future of x. Likewise, one defines the chronological past $I^-(x)$ consisting out of events y such that $d(y, x) > 0$. Associated to this is a partial order $\prec$ defined by means of $x \prec y$ if and only if $d(x, y) > 0$. We suppose that for ope $\mathcal{O}$ around x one finds points y, z such that $y \prec x \prec z$ and $I^-(z) \cap I^+(y) \equiv A(y, z) \subset \mathcal{O}$. The sets $A(x, y)$ called the Alexandrov sets define a basis for the space time topology. Regarding the Riemann tensor, it is required that $a, b \in I^-(c) \cap I^-(d) \cap I^+(x)$ where timelike geodesics are defined by means of a maximization instead of minimization procedure. Likewise, it may be that $c, d \in I^-(a) \cap I^-(b) \cap I^+(x)$ or two similar options with a, b, c, d in the past of x. Because Lorentzian geometries define no canonical *local* compact neighborhoods, it is impossible to define a Hausdorff measure starting from Alexandrov neighborhoods. For example, on a piecewise linear Lorentzian manifold with conical singularitirs the volume of an Alexandrov set is direction dependent. Consequently, it is better to define an additional metric $\tilde{d}$ as well as a Lorentzian metric tensor $g^{\pm}_{\epsilon}(a, b)$ on pairs of points $(a, b) \in I^{\pm}(x)$ for which holds that $d(a, b) > 0$ or $d(b, a) > 0$ such that hyperbolic angles are well defined (replace cosine and sine by cosinehyperbolic and sinehyperbolic). Call these regions $Z^{\pm}(x)$, then we define an inverse $g^{\pm,\epsilon}(\hat{a}, \hat{b})$ by means of integration over $(B(x, \epsilon) \times B(x, \epsilon)) \cap Z^{\pm}(x)$.

Chapter 5

Quantum metrics.

Let $(\mathcal{S}, \wedge, \vee, +, .)$ be a quantum space represented possibly by means of trace-class Hermitian projection operators on a (pseudo) Hilbert space $\mathcal{H}$. Then, a metric geometry is characterized by a bi-function

$$d : \mathcal{S} \times \mathcal{S} \to \mathcal{S}^{+}$$

satisfying

$$d(P, P) = 0,\ d(P, Q) = d(Q, P) > 0,\ \tilde{d}(P, Q) + \tilde{d}(Q, R) \geq \tilde{d}(P, R)$$

where $\tilde{d}$ is a classical metric associated to d in a way explained below. Here $\mathcal{S}^{+}$ is defined by means of all *positive* Hermitian operators B satisfying $\langle \psi | B \psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$. It is clear that positive operators constitute a cone. One notices furthermore that bijective linear operators commuting with the multiplication on $\mathcal{H}$, the so-called algebra automorphisms, determine a diffeomorphism on the underlying space in case $\mathcal{H} = L^2(\mathcal{M}, \mu)$ for some compact manifold $\mathcal{M}$ and measure μ with the complex conjugation as involution. The proof is simple; note that an automorphism maps characteristic functions to characteristic functions preserving the entire algebra. Therefore, it induces a mapping on the points which must be bijective given that the automorphism is. In case the automorphism is unitary as a linear mapping, then it corresponds to an isometry of the measure. If, moreover, the distance function is preserved, then we recover the Killing fields. So, classically, the atomistic (Hermitian) projective elements χ of $\mathcal{H}$ which are of measure zero in the sense

$$||\chi|| = 0$$

correspond to points. Here, atomistic means that χ cannot be written as a sum of alike elements and we have put Hermitian between brackets because in the classical situation projective elements are automatically Hermitian. This point of view is not really exact and we better speak about a sequence of decreasing Hermitian projective elements $(\chi_k)_{k \in \mathbb{N}}$ such that $\chi_k < \chi_l$ for $k > l$ and

$$\lim_{k \to \infty} ||\chi_k|| = 0.$$

Generalized δ functions can be constructed with regard to a dense, unital, subalgebra $\mathcal{K}$ by demanding that there exists an increasing sequence of positive numbers a_k such that for any $\psi \in \mathcal{K}$

$$\lim_{k \to \infty} a_k \langle \chi_k | \psi \rangle := \widehat{\psi}_\chi$$

where the latter defines an automorphism on $\mathcal{K}$. The question now is how to suitably relax this structure; clearly if we keep the field of the complex numbers in the definition of the (pseudo) Hilbert algebra, as well as the standard commutatative, associative and distributive rules for the sum and product, then nothing is gained. We are stuck to classical spaces. On the other hand, if we throw away the Hilbert space character of $\mathcal{H}$, then there is very little structure in the operator algebra over it. Connes prefers to throw away the notion of multiplication on $\mathcal{H}$ (but keeps the Hilbert space character); unfortunately, that leaves us with no points or atoms. Hence, the most general setting which appears to be satisfying is one of a quaternionic (pseudo) Hilbert algebra with the standard abelian sum and scalar multiplication rules but with a multiplication between vectors which is generically non-abelian but satisfies the distributive law. We baptise these geometries to be quaternionic.

For example, on flat Minkowski compactified on a n-dimensional torus from $-L$ to L in every orthonormal direction, points are determined by distributional states $\delta(x - z) := |x\rangle$ and the unit operator is given by

$$1 = \int dx |x\rangle\langle x|.$$

Having said this, it must be clear to the reader that we shall take a different avenue here as we did in chapter three where we did away with the distributional aspects by means of ghost reference state. Notice that

$$0 \leq \pm(T(h) \pm T(-h))^2 = \pm(T(h)^2 + T(-h)^2 \pm 2)$$

and therefore

$$-2 \leq T(h)^2 + T(-h)^2 \leq 2$$

where T constitues a representation of the n dimensionial translation group which is nothing but the maximal abelian group of the isometry group of $\mathbb{R}^n$. Also, we know that

$$\int_{T_n^{[0,L]}} dr \int_{T_n^{[0,L]}} ds\, r.s\, T_{-r} T_s = \int_{T_n^{[0,L]}} dr \int_{T_n^{[-L,L]}} ds\, (||r||^2 - \frac{||s||^2}{4})\, T_s =$$

$$\left(\int_{T_n^{[0,L]}} dr ||r||^2 \right) \left(\int_{T_n^{[-L,L]}} ds\, T_s \right) - \frac{1}{4} \text{Vol}(T_n^{[0,L]}) \int_{T_n^{[-L,L]}} ds\, ||s||^2\, T_s$$

is a positive definite matrix. Moreover,

$$\int_{T_n^{[-L,L]}} ds\, T_s$$

usurpates (in the limit to maximal L) the action of T_t and therefore must equal the (distributional if $L = \infty$) state

$$|1\rangle\langle 1|$$

in the functional representation. Hence,

$$\left(\int_{T_n^{[0,L]}} dr||r||^2\right)|1\rangle\langle 1| - \frac{1}{4}\mathrm{Vol}(T_n^{[0,L]})\int_{T_n^{[-L,L]}} ds||s||^2 T_s$$

is the expression of our concern and the reader notices there is no way to regularize the operator in the limit for the compactification scale L to infinity. Therefore, a good definition of an operator valued distance d is determined by the "positive scalar product" operator

$$\langle A|B\rangle_{\mathrm{op}} = \int_{T_n^{[0,L]}} dr \int_{T_n^{[0,L]}} ds\, r.s\, A^\dagger T_{-r} T_s B.$$

Notice that the quantity

$$A(|x\rangle\langle x|, |y\rangle\langle y|) = \int_{T_n^{[-L,L]}} dh||h||^2 |x\rangle\langle x|T(h)|y\rangle\langle y|$$

equals

$$A(|x\rangle\langle x|, |y\rangle\langle y|) = d(x,y)^2\, |x\rangle\langle y|$$

where d is the distance on the n torus and therefore, $|x\rangle\langle x|$ is a distributional operator of norm squared $\frac{nL^{n+2}}{3}|x\rangle\langle x|$. The norm is then given by its square root which does not exist; it is however possible to construct quasi-roots by considering the operators

$$B^\epsilon(x) := \frac{1}{\epsilon^{\frac{5n}{2}}}\sqrt{\frac{nL^{n+2}}{3}}\int_{\times_n[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dh T_h |x\rangle\langle x| \int_{\times_n[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dz T_z$$

then

$$(B^\epsilon(x))^2 = \frac{nL^{n+2}}{3}\frac{1}{\epsilon^{2n}}\int_{\times_n[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dh T_h |x\rangle\langle x| \int_{\times_n[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dz T_z$$

which agrees in the limit for ϵ to zero with

$$\frac{nL^{n+2}}{3}|x\rangle\langle x|.$$

By definition, the distance formula equals

$$d(|x\rangle\langle x|, |y\rangle\langle y|)^2 = \int_{T_n^{[0,L]}} dr \int_{T_n^{[0,L]}} ds\, r.s\, \left(|y\rangle\langle y| - |x\rangle\langle x|\right) T_{r-s} \left(|y\rangle\langle y| - |x\rangle\langle x|\right) =$$

$$\frac{nL^{n+2}}{3}\left(|x\rangle\langle x| + |y\rangle\langle y| - |x\rangle\langle y| - |y\rangle\langle x|\right) + \frac{1}{4}L^n\, d(x,y)^2\, \left(|x\rangle\langle y| + |y\rangle\langle x|\right).$$

Hence, we obtain

$$\frac{nL^{n+2}}{3}\left(|x\rangle\langle x|+|y\rangle\langle y|\right)-\left(\frac{nL^{n+2}}{3}-\frac{L^n}{4}d(x,y)^2\right)\left(|x\rangle\langle y|+|y\rangle\langle x|\right).$$

We may again look for quasi-roots of the operator

$$\widehat{d}(x,y)^2 := a\left(|x\rangle\langle x|+|y\rangle\langle y|\right)-b(x,y)\left(|x\rangle\langle y|+|y\rangle\langle x|\right)$$

where $a=\frac{nL^{n+2}}{3}, b(x,y)=\left(\frac{nL^{n+2}}{3}-\frac{L^n}{4}d(x,y)^2\right)$ satisfying $0<b(x,y)<a$. They are all characterized by matrices of the type

$$B(x,y) := \begin{pmatrix} 1 & \epsilon d(x,y)-1 \\ \epsilon d(x,y)-1 & 1 \end{pmatrix}$$

with ϵ a very small number. Regarding a quantum triangle inequality

$$B(x,y)+B(y,z)-B(x,z)\sim\begin{pmatrix} 0 & \epsilon d(x,y)-1 & 1-\epsilon d(x,z) \\ \epsilon d(x,y)-1 & 2 & \epsilon d(y,z)-1 \\ \epsilon 1-d(x,z) & \epsilon d(y,z)-1 & 0 \end{pmatrix}$$

a Hermitian matrix with two positive and one negative eigenvalue due to the triangle inequality

$$d(x,y)+d(y,z)\geq d(x,z).$$

A reverse inequality, such as is the case in Lorentzian geometry, results in two negative eigenvalues and one positive one. It is therefore clear that no triangle inequality is satisfied at the level of $\widehat{d}(x,y)$ given that those operators do not commute and therefore Cauchy-Schwartz does not apply. These regard "quantum fluctuations" of metric geometry which are not present classicaly where the distance is positive real valued. We shall come back to this in the next chapter, but it must be clear that the quantity

$$\widetilde{d}(x,y)=\sqrt{-(\langle x|-\langle y|)\frac{1}{4}\mathrm{Vol}(T_n^{[0,L]})\int_{T_n^{[-L,L]}}ds\,||s||^2\,T_s(|x\rangle-|y\rangle)}$$

reproduces the classical distance and metric geometry. Indeed, notice that

$$\left(\langle x|-\langle y|\right)|1\rangle=0$$

and, moreover,

$$d(|x\rangle,|y\rangle)^2 := -(\langle x|-\langle y|)\frac{1}{4}\mathrm{Vol}(T_n^{[0,L]})\int_{T_n^{[-L,L]}}ds\,||s||^2\,T_s(|x\rangle-|y\rangle)=\frac{L^n}{2}d(x,y)^2$$

which is clearly a satisfying formula allowing for regularisation in the limit for L towards infinity. Clearly,

$$d(|x\rangle,|y\rangle)$$

restricted to those "atomic" states satisfies the full triangle inequality given that d does. For more general states;

$$d(|\psi\rangle, |\phi\rangle)^2 := \frac{nL^{n+2}}{3} \left| \int_{T_n^{[-L,L]}} dh\, (\psi(h) - \phi(h)) \right|^2 -$$

$$\frac{L^n}{4} \int_{T_n^{[-L,L]}} dx \int_{T_n^{[-L,L]}} dy\, d(x,y)^2 \overline{(\psi(x) - \phi(x))} (\psi(y) - \phi(y)).$$

We show this quantity is indeed positive; suppose ζ has only support in a region for which $d(x,y)^2 = (x-y)^2$, then

$$\int_{T_n^{[-L,L]}} dx \int_{T_n^{[-L,L]}} dy\, (x^2 - 2x.y + y^2) \overline{\zeta(x)} \zeta(y) =$$

$$\overline{\int_{T_n^{[-L,L]}} dx\, \zeta(x)} \int_{T_n^{[-L,L]}} dy\, y^2 \zeta(y) + cc - 2|| \int_{T_n^{[-L,L]}} dy\, y \zeta(y)||^2$$

which allows for explicit verification of positivity of

$$d(|\zeta\rangle, 0)^2.$$

Hence, the triangle inequality is satisfied and coincides with the usual one on distributional atomistic elements. It is clear that Lorentzian geometry and non-abelian generalizations thereof can be treated in an entirely similar manner.

5.1 Differential geometry.

In the previous section we has cast flat, compactified, Euclidean and Minkowskian geometry into a new jacket and the only task is to study the limit L to infinity in a very succint way which necessitates giving up on the concept of a function space hereby introducing the concept of "infinitesimal vectors" and operators by means of a Cauchy procedure. Quantum geometry obviously necessitates such thing given that "points", given by Hermitian projection operators, are atomistic in a much weaker sense than it is for classical vectors in the Hilbert algebra. This weaker notion is holistic given that it has non-zero measure whereas the classical one is limiting to the zero measure case or distributional if appropriate rescalings are applied for. The reader must have noticed by now that

$$d(x,y) \sim d(|x\rangle, |y\rangle) \sim (\langle x| + \langle y|) \int_{\times_n \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right]} dh\, T_h\, d(|x\rangle\langle x|, |y\rangle\langle y|) \int_{\times_n \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right]} dh\, T_h\, (|x\rangle + |y\rangle).$$

Therefore, regarding

$$\triangle(x,y;z) := d(|x\rangle\langle x|, |y\rangle\langle y|) + d(|y\rangle\langle y|, |z\rangle\langle z|) - d(|x\rangle\langle x|, |z\rangle\langle z|)$$

one obtains that

$$\lim_{\epsilon \to 0} (\langle x| + \langle y| + \langle z|) \int_{\times_n \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right]} dh\, T_h\, \triangle(x,y;z) \int_{\times_n \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right]} dh\, T_h\, (|x\rangle + |y\rangle + |z\rangle) \geq 0$$

due to the classical triangle inequality. So, this is our classical-quantum correspondance: from a democratic state over all points, such as is the state associated to the barycenter of the triangle, the triangle inequality is satisfied on a average. The reader may compute the second moment of a smearing of the operator $\triangle(x, y; z)$ over the points x, y, z by means of the translation operators in order to study "quantum" fluctuations.

The reader must correctly understand that underlying the quantum geometry is a fixed classical one just as is the case in this author's work on quantum gravity. We now generalize this work to a curved classical background by means of the exponential map which is after all immediately determined by the geodesic equation and vierbein and generalizes the idea of a translation group towards non-abelian bi-groups. That is, locally, we may write

$$T_{[T_x(v)]}(w) = T_x((w \oplus v)_x)$$

where $w \oplus v$ is uniquely given if we demand that geodesics do not leave a certain open region $\mathcal{O}$ around x and $T_x(v) = \exp_x(v)$. On the other hand $T_x(v)$ may be thought of as representing a translation on the tangent space at x in which case the usual law

$$T_x(w)T_x(v) = T_x(v + w)$$

holds. We shall be interested in the first representation which is isomorphic to the second in flat Minkowski with respect to a global inertial frame so that there, the x dependency can be dropped in T_x as well as $\oplus_x$. Specifically, the global action T is

$$(T(v)f)(x) := f(T_x(v(x)))$$

where $v(x)$ is a vectorfield on $\mathcal{M}$. The element $v(x)$, seen as an ultralocal vector, may also serve as $T_{v(x)}$ on the flat geometry modelled at x. It is the exponential map which connects both representations as we shall see soon. One also has

$$[T(w)(T(v)f)](x) := [T(v \oplus w)f](x) = f(T_x(v \oplus w)_x) = f(T_{T_x(w(x))}(v(T_x(w(x))))).$$

Therefore, the right framework for curved geometry is the one of the induced non-abelian sum on the vectorfields. This calls for an extension of our previous setting; one could work with the Hilbert-algebra $\mathcal{H}$ of functions on $\mathcal{M}$ where $\mathcal{M}$ is compact, equipped with the real *Leibniz* topological dual $\mathcal{H}^{\star,L}$ on it defined by the continuous, real linear functionals D satisfying

$$D(fg) = D(f)\tilde{g} + \tilde{f}D(g)$$

where

$$\tilde{f}(x) = \lim_{\epsilon \to 0} \frac{1}{\text{Vol}(B_\epsilon(x))} \int_{B_\epsilon(x)} f(y)dy\sqrt{h}_y.$$

The Leibniz rule is there to ensure the locality aspect and enables one to define $D(x)$ which is what we need; notice that the previous definition of $\mathcal{H}^{\star,L}$ does not depend upon the choice of H whereas quantum mechanically it might.

Given that $\mathcal{H}^{\star,L}$ is infinite dimensional, we cannot integrate over it; however, we restrict to *constant* elements D which are those satisfying some equation of constancy. Note that we have something as a pull back defined by

$$f_\star D$$

where $[(f_\star D)(g)](x) = [D(g \circ f^{-1})](f(x))$ for $f \in \mathrm{Diff}(\mathcal{M})$ which is an automorphism of $\mathcal{H}$. Formulated more algebraically, every automorphism χ of $\mathcal{H}$ induces a mapping $\chi_\star : \mathcal{H}^{\star,L} \to \mathcal{H}^{\star,L}$ by means of

$$[\chi_\star D](f) = \chi[D(\chi^{-1}(f))].$$

Indeed, one checks that

$$[\chi_\star D](fg) = \chi[D(\chi^{-1}[fg])] = \chi[D(\chi^{-1}(f))]g + f\chi[D(\chi^{-1}[g])] = [[\chi_\star D](f)]g + f[[\chi_\star D](g)]$$

which shows its sanity. In general, constancy of an element requires metric information to have a benchmark. Here, our translations might come in handy:

$$T(D)^2 = T(2D)$$

as an equality between automorphisms on $\mathcal{H}$. This restricts the field to be geodesic; however, that leaves plenty of freedom. It is better to fix a point x and drag $D(x)$ along the geodesics emanating from it. Concretely, we look for a mapping

$$(\exp_x)_\star : \mathbb{R}^n \to \mathcal{H}^{\star,L} : D(x) \to D$$

where

$$D(f)(\exp_x(v)) := \lim_{\epsilon \to 0} \frac{[T_{\epsilon D(x)}(f \circ T_x)(v) - (f \circ T_x)(v)]}{\epsilon}$$

and the reader verifies that the Leibniz rule is satisfied. To implement this idea in abstracto, we need to make use of the fact that the T map really connects the Leibniz dual $\mathcal{H}^{\star,L}$ with the Hilbert algebra $\mathcal{H}$ given that the specialization to a "stalk" of the Leibniz dual at a point provides for a local automorphism between the respective local Hilbert algebra in $\mathbb{R}^n$ and a part of the Hilbert algebra $\mathcal{H}$ by means of the associated local diffeomorphism T_x. Here, the local Hilbert algebra $\mathcal{H}^{x,\mathcal{O}}_{\mathrm{loc}}$ is canonically defined by

$$(T_x)_\star \chi_{\mathcal{O}}$$

where χ_A is the standard characteristic function on $A \subset \mathbb{R}^n$. In vector language,

$$\langle (T_x)_\star \chi_{\mathcal{O}} | y \rangle := \chi_{\mathcal{O}}((T_x)^{-1}(y))$$

which determines the mapping completely given that we assume disjunct atoms to ca. Therefore, we have to take into account that $(\exp_x)_\star$ is only defined on a neighborhood of the origin of $\mathbb{R}^n$ given that one meets serious problems *globally.* Here, locality is hiding in the classical distance on the natural Hilbert-algebra

$\mathcal{H}^{\text{flat}}(x)$ associated to the localized Leibniz topological dual $\mathcal{H}^{\star,L}$ at x. The formula for Df then reads

$$\lim_{\epsilon \to 0} \frac{f(T_x(v + \epsilon D(x))) - f(T_x(v))}{\epsilon}$$

and is to be understood in the usual way. $\mathcal{H}^{\text{flat}}(x)$ is defined by noticing that

$$(D(f\chi_{\mathcal{O}}))(x) = (Df)(x)$$

since

$$[D(\chi_{\mathcal{O}})](x) = [D(\chi_{\mathcal{O}}^2)](x) = 2[D(\chi_{\mathcal{O}})](x) = 0$$

and therefore D depends at x only on $f\chi_{\mathcal{O}}$ and not the entire f. Since $\mathcal{O}$ was arbitrary, the limit to zero size can be taken what justifies the notation $D(x)$. This requires an a priori input of a topological class of idempotent elements $\zeta^2 = \zeta$. More in particular, we demand the existance of a Boolean isomorphism from

$$\psi : I(\mathcal{H}) \to \tau(A(\mathcal{H}))$$

where $A(\mathcal{H})$ is the set of all atoms and τ is a topology on it locally homeomorphic with a $\mathbb{R}^n$ metric topology. ψ is defined by resorting to a notion of inclusion which is the restriction of the partial order $\prec$ on $I(\mathcal{H})$ where

$$\alpha \prec \beta \leftrightarrow \alpha\beta = \alpha.$$

Moreover, an idempotent ζ is an open neighborhood $\mathcal{O}$ of an atom x if and only if for any equivalence class $[\xi_n]_n$ of x, there exists an n such that for any $m \geq n$ holds that $\xi_m \prec \zeta$. $\psi(\zeta) = \chi_{\mathcal{O}}$ meaning that all other information about ζ is redundant with regard to the scalar product on $\mathcal{H}$. It is clear that ψ induces a mapping between atoms and points which allows one to speak about differentiable structure. We assume moreover a C^∞ atlas to exist on $A(\mathcal{H})$ equipped with its local topology homeomorphic to $\mathbb{R}^n$.

This prepares the setting for a generalization of the geometry defined in the previous section. The crucial part is to use the standard spectral theorem on $\mathcal{H}$ to know that every element can be written as a sum of complex multiples of Hermitian idempotents which in their turn can be written as an integral of distributional atomistic idempotents (a Hilbert algebra is a commutative $C^\star$ algebra as well as a Hilbert space, where the $C^\star$ algebra is represented on itself). Therefore, the position "basis" of atoms always is a basis of orthogonal elements in the general distributional sense. A classical metric is defined in the following way: pick a point x and a scalar product $h_x(v(x), w(x))$ on $\mathcal{H}^{\star,L}(x)$ which we assume to be isomorphic, as a vector space, to $\mathbb{R}^n$. The pull back of h_x is defined as

$$(\chi_\star h)_{\chi^{-1}(x)}((\chi_\star v)_{\chi^{-1}(x)}, (\chi_\star w)_{\chi^{-1}(x)}) := h_x(v(x), w(x)).$$

If one were to define the h field by means of

$$[(T_x^{-1})_\star h]_{T_x(v)} = h_x$$

or

$$h_{T_x(v)} = [(T_x)_\star h_x](v)$$

where $h_x(v) = h_x$ then that definition would be x dependent and result in a flat geometry. To rectify this, note that T defines the full connection and therefore the parallel transporter which we denote with $\widehat{T}$. T and $\widehat{T}$ satisfy

$$T_{T_x(v)}(-(\widehat{T}_v(v))(x)) = x$$

for all x and $v \in \mathcal{H}^{\star,L}(x)$. Moreover, locally,

$$(\epsilon v) \oplus (\epsilon w) = \epsilon(v + w) + O(\epsilon^3).$$

As is well known from differential geometry, this issue *does* depend upon the choice of h_x if the latter is nondegenerate and symmetric and of fixed signature. Indeed, take a matrix field $O(x)$, then the connection associated to $O(x)g(x)O^T(x)$ is given by

$$O(x)\gamma(x)O^T(x) \otimes O^T(x) + \frac{1}{2}(O^T)^{-1}g^{-1}O^{-1}(\text{first derivatives of O}).$$

There are in general $\frac{n^2(n+1)}{2}$ equations and n^2 variables so that inconsistencies arise. This issue is pretty easily solved by demanding that

$$\lim_{\epsilon \to 0} \frac{\widehat{T}_{\epsilon v} h - h}{\epsilon} = 0$$

for the appropriate metric h and any field $v \in \mathcal{H}^{\star,L}$. Consistency then implies that

$$\lim_{\epsilon \to 0} \frac{\widehat{T}_{\epsilon v}\widehat{T}_{\epsilon w} h - \widehat{T}_{\epsilon(v+w)} h}{\epsilon^2} = 0$$

for any fields v, w and the two conditions on T which define one parameter subgroups and restrict the coincidental behaviour of $\oplus$, together with the fact that $\widehat{T}$ must define an infinitesimal isometry of the metric field, fix the classical geometry entirely.

We now proceed towards the end of this short introduction which is by no means complete. Delta *densities* are defined by

$$\int_{A(\mathcal{H})} dx\, \delta(x, z) f(x) := f(z)$$

where the integral has been constructed by making use of ψ and the local charts at $A(\mathcal{H})$ and the vector f maps to a continuous function on atomic space by means of a Hilbert-space limiting procedure. This gives meaning to

$$\langle z | f \rangle := f(z)$$

and

$$f = \int_{A(\mathcal{H})} dx\, f(x)\sqrt{h_x}|x\rangle$$

with

$$\langle z|w\rangle = \frac{\delta(z,w)}{\sqrt{h_z}}.$$

To ensure that it is really $\sqrt{h_z}$ showing up, we demand that the $T(\epsilon v)$ are unitary in the limit ϵ to zero up to second order in ϵ for conformal vectorfields v satisfying

$$\lim_{\epsilon \to 0} \frac{(T(\epsilon v))_\star \sqrt{h} - \sqrt{h}}{\epsilon} = 0.$$

More in particular, for those Leibniz dual elements, we have that

$$\lim_{\epsilon \to 0} \frac{\langle T(\epsilon v)\alpha | T(\epsilon v)\beta\rangle - \langle \alpha|\beta\rangle}{\epsilon} = 0.$$

An alternative route consists in taking a point $z = [\chi_n]$ as a generalized vector satisfying

$$\langle z|f\rangle = \widehat{f}(z)$$

where $\widehat{f}(z)$ is an algebra homomorphism from $\mathcal{H}$ to $\mathbb{C}$. The reader should proof that the a_n accomplishing this are given by $\frac{1}{||\chi_n||^2}$. The vector sub-algebra spoken about before is then simply defined by demanding that this expression exists and is independent of the equivalence. $|z\rangle$ is then not a generalized density but a generalized function which is the better way to follow. Define the "formal" operator B with as prescription

$$B|f\rangle := \int dz f(z)\sqrt{h_z} \int_{\mathcal{O}_z} dh\, h\, |T_z(h)\rangle$$

where the integral in $\mathcal{H}^{\star,L}(z)$ is executed with respect to inertial coordinates associated to an orthonormal basis of h_z. The reader then sees that we are really interested in the expression

$$d(|\alpha\rangle, |\beta\rangle)^2 := \langle B(|\alpha\rangle - |\beta\rangle)|B(|\alpha\rangle - |\beta\rangle)\rangle$$

which reproduces, at least locally, the correct classical distance obeying the triangle inequality. Quantum distances are then constructed by means of the scalar product

$$\langle A|C\rangle_{\text{op}} := A^\dagger B^\dagger B C$$

for trace class operators A, C on $\mathcal{H}$.

Chapter 6

Afterword.

Long reflection about the essentials of geometry starting in my PhD time at Gent university as well as a thorough study of many rather ad-hoc quantization schemes have brought me to the ideas posited in this litte book. The latter can serve many researchers and be a source of inspiration for many PdD thesis. I have enjoyed many conversations with Norbert Van den Bergh and Frans Cantrijn regarding the essentials of rods and compasses. Also, Jan van Geel and Rafael Sorkin have been useful in this respect. Furthermore, I acknowledge financial and critical support from Renate Loll albeit we did not enjoy plenty of time discussing matters in more depth.

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