“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 8/10.”

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 8/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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$$\int_C [f(q)+g(q)]\,dq = \int_C (f+g)(q)\,dq$$
$$= \int_a^b (f+g)(q(t))\frac{d}{dt}[q(t)]\,dt$$
$$= \int_a^b [f(q(t))+g(q(t))]\frac{d}{dt}[q(t)]\,dt$$
$$= \int_a^b \left[f(q(t))\frac{d}{dt}[q(t)] + g(q(t))\frac{d}{dt}[q(t)]\right]dt$$
$$= \int_a^b f(q(t))\frac{d}{dt}[q(t)]\,dt + \int_a^b g(q(t))\frac{d}{dt}[q(t)]\,dt,$$

bearing in mind the provisions of Theorem TII-15. However, since the definite integrals,

$$\int_a^b f(q(t))\frac{d}{dt}[q(t)]\,dt = \int_C f(q)\,dq,$$
$$\int_a^b g(q(t))\frac{d}{dt}[q(t)]\,dt = \int_C g(q)\,dq,$$

by virtue of Eq. (iii), it automatically follows that the integral formula,

$$\int_C [f(q)+g(q)]\,dq = \int_C f(q)\,dq + \int_C g(q)\,dq,$$

is likewise valid, $\forall t \in [a,b]$, as required. Q.E.D.

Theorem TII-20.

Let there exist a single-valued quaternion hypercomplex function; $f(q)$, which is accordingly restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t) \;, \quad \forall t \in [a,b].$$

Subsequently, we may prove that, for any arbitrary quaternion constant,

$$q_0 = c_1 + ic_2 + jc_3 + kc_4 \;, \quad \forall c_1, c_2, c_3, c_4 \in \mathbb{R},$$

the integral formulae,

(a) $q_0 \int f(q)\,(dq)_C = \int q_0 f(q)\,(dq)_C$,

(b) $q_0 \int_C f(q)\,dq = \int_C q_0 f(q)\,dq$,

are always valid, provided that the function, $f(q(t))\frac{d}{dt}[q(t)]$, is likewise integrable with respect to the real parameter 't'.

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<u>PROOF</u>:-

(a) Since, by definition, the indefinite integral,

$$\int f(q)\,(dq)_C = \int f(q(t))\frac{d}{dt}[q(t)]\,dt \;, \quad \forall t \in [a,b],$$

then, in accordance with Theorem TII-16, we deduce that

$$q_0 \int f(q)\,(dq)_C = q_0 \int f(q(t))\frac{d}{dt}[q(t)]\,dt$$

$$= \int q_0 f(q(t))\frac{d}{dt}[q(t)]\,dt \;.$$

Now let there exist a quaternion function,

$$F(q) = q_0 f(q) \Longrightarrow F(q(t)) = q_0 f(q(t)),$$

such that we obtain the indefinite integral,

$$\int F(q)(dq)_C = \int F(q(t)) \tfrac{d}{dt}[q(t)]\,dt \Longrightarrow$$

$$\int q_0 f(q)(dq)_C = \int q_0 f(q(t)) \tfrac{d}{dt}[q(t)]\,dt$$

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$$= q_0 \int f(q)(dq)_C, \quad \forall t \in [a, b], \text{ as required. } \underline{\text{Q.E.D.}}$$

(b) Allied to the foregoing, we also recall that, by definition, the definite integral,

$$\int_C f(q)\,dq = \int_a^b f(q(t)) \tfrac{d}{dt}[q(t)]\,dt, \quad \forall t \in [a, b],$$

whereupon, in accordance with Theorem TII-17, we likewise deduce that

$$\begin{aligned} q_0 \int_C f(q)\,dq &= q_0 \int_a^b f(q(t)) \tfrac{d}{dt}[q(t)]\,dt \\ &= \int_a^b q_0 f(q(t)) \tfrac{d}{dt}[q(t)]\,dt. \end{aligned}$$

Now let there exist a quaternion function,

$$F(q) = q_0 f(q) \Longrightarrow F(q(t)) = q_0 f(q(t)),$$

such that we obtain the definite integral,

$$\int_C F(q)\,dq = \int_a^b F(q(t))\frac{d}{dt}[q(t)]\,dt \Longrightarrow$$

$$\int_C q_0 f(q)\,dq = \int_a^b q_0 f(q(t))\frac{d}{dt}[q(t)]\,dt$$

$$= q_0 \int_C f(q)\,dq, \quad \forall t \in [a,b], \text{ as required. } \underline{\text{Q.E.D.}}$$

<u>Definition DII-19.</u>

Let there exist a <u>contour</u>, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent <u>smooth arcs</u>, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + i y_n(t) + j \hat{x}_n(t) + k \hat{y}_n(t), \quad \forall t \in [a_n, b_n],$$

whereupon we make the additional stipulation that the <u>endpoints</u>,

$$q_{m+1}(a_{m+1}) = q_m(b_m), \quad \forall m \in \{1, 2, 3, \ldots\ldots\ldots, N-1\}$$

(N is some finite positive integer).

In the circumstances, we postulate the existence of the definite integral,

$$\int_C f(q)\,dq = \int_{K_1} f(q)\,dq + \int_{K_2} f(q)\,dq + \ldots\ldots\ldots\ldots + \int_{K_N} f(q)\,dq$$

$$= \sum_{n=1}^{N} \int_{K_n} f(q)\,dq,$$

insofar as the functions, $f(q_n(t))\frac{d}{dt}[q_n(t)]$, are likewise integrable with respect to the real parameter 't'.

Theorem TII-21.

Let there exist a contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent smooth arcs, K_n, is accordingly represented by the equation,

$q_n(t) = x_n(t) + iy_n(t) + j\hat{x}_n(t) + k\hat{y}_n(t)$, $\forall t \in [a_n, b_n]$,

and whose endpoints,

$q_{m+1}(a_{m+1}) = q_m(b_m)$, $\forall m \in \{1, 2, 3, \ldots\ldots, N-1\}$

(N is some finite positive integer).

Subsequently, it may be shown that the integral formulae,

(a) $\int_C [f(q) + g(q)]\,dq = \int_C f(q)\,dq + \int_C g(q)\,dq$,

(b) $q_0 \int_C f(q)\,dq = \int_C q_0 f(q)\,dq$,

where the arbitrary quaternion constant,

$q_0 = c_1 + ic_2 + jc_3 + kc_4$, $\forall c_1, c_2, c_3, c_4 \in \mathbb{R}$, are always valid, provided that the functions, $f(q_n(t))\frac{d}{dt}[q_n(t)]$ and $g(q_n(t))\frac{d}{dt}[q_n(t)]$, are likewise integrable with respect to the real parameter 't'.

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PROOF:-

(a) In accordance with the criteria specified in both Definition DII-19 and Theorem TII-19, we perceive that the definite integral,

$$\int_C [f(q) + g(q)]\,dq = \sum_{n=1}^{N} \int_{K_n} [f(q) + g(q)]\,dq$$

$$= \sum_{n=1}^{N} \left(\int_{K_n} f(q)\,dq + \int_{K_n} g(q)\,dq \right)$$

$$= \sum_{n=1}^{N} \int_{K_n} f(q)\,dq + \sum_{n=1}^{N} \int_{K_n} g(q)\,dq$$

$$= \int_C f(q)\,dq + \int_C g(q)\,dq,$$ as required. Q.E.D.

(b) Similarly, in accordance with the criteria specified in both Definition DII-19 and Theorem TII-20, we deduce that the definite integral,

$$q_0 \int_C f(q)\,dq = q_0 \sum_{n=1}^{N} \int_{K_n} f(q)\,dq$$

$$= \sum_{n=1}^{N} q_0 \int_{K_n} f(q)\,dq$$

$$= \sum_{n=1}^{N} \int_{K_n} q_0 f(q)\, dq$$

$$= \int_C q_0 f(q)\, dq \text{ , as required. } \underline{\underline{Q.E.D.}}$$

Theorem TII-22.

Let there exist a single-valued quaternion hypercomplex function, $f(q)$, which is accordingly restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t) \text{ , } \forall t \in [a,b].$$

Subsequently, we may prove that the modular inequality,

$$\left| \int_C f(q)\, dq \right| \leq \int_a^b \left| f(q(t)) \tfrac{d}{dt}[q(t)] \right| dt,$$

is always valid, provided that the function, $f(q(t)) \frac{d}{dt}[q(t)]$, is likewise integrable with respect to the real parameter 't'.

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PROOF:-

Since, by definition, the definite integral,

$$\int_C f(q)\, dq = \int_a^b f(q(t)) \tfrac{d}{dt}[q(t)]\, dt,$$

exists, $\forall t \in [a,b]$, with the proviso that the function, $f(q(t)) \frac{d}{dt}[q(t)]$, shall be integrable over the same domain, it therefore follows that the modulus,

$$\left|\int_C f(q)\,dq\right| = \left|\int_a^b f(q(t))\tfrac{d}{dt}[q(t)]\,dt\right|, \ \forall t \in [a,b].$$

However, in accordance with Theorem TII-18, we deduce that the modular inequality,

$$\left|\int_a^b f(q(t))\tfrac{d}{dt}[q(t)]\,dt\right| \leq \int_a^b \left|f(q(t))\tfrac{d}{dt}[q(t)]\right| dt,$$

also exists under the same conditions and hence we obtain the end result,

$$\left|\int_C f(q)\,dq\right| \leq \int_a^b \left|f(q(t))\tfrac{d}{dt}[q(t)]\right| dt, \ \forall t \in [a,b], \text{ as required. } \underline{\underline{Q.E.D.}}$$

<u>Theorem TII-23.</u>

Let there exist a <u>simple closed contour</u>, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent <u>smooth arcs, K_n</u>, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + iy_n(t) + j\hat{x}_n(t) + k\hat{y}_n(t), \ \forall t \in [a_n, b_n],$$

insofar as the endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m), \ \ \forall m \in \{1,2,3,\ldots\ldots\ldots\ldots,N-1\}$

(N is some finite positive integer)

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Subsequently, it may be proven that the integral formula,

$$\int_C dq = 0,$$

is always valid in terms of the conditions specified above.

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PROOF:-

In view of the criteria laid down in Definition DII-19, we deduce that the definite integral,

$$\int_C dq = \sum_{n=1}^{N} \int_{K_n} dq$$

$$= \int_{K_1} dq + \int_{K_2} dq + \ldots\ldots\ldots\ldots + \int_{K_n} dq$$

$$= \int_{a_1}^{b_1} \frac{d}{dt}[q_1(t)]\,dt + \int_{a_2}^{b_2} \frac{d}{dt}[q_2(t)]\,dt + \ldots\ldots\ldots + \int_{a_N}^{b_N} \frac{d}{dt}[q_N(t)]\,dt$$

$$= \int_{a_1}^{b_1} x_1'(t)\,dt + i\int_{a_1}^{b_1} y_1'(t)\,dt + j\int_{a_1}^{b_1} \hat{x}_1'(t)\,dt + k\int_{a_1}^{b_1} \hat{y}_1'(t)\,dt + \ldots\ldots\ldots\ldots +$$

$$\int_{a_N}^{b_N} x_N'(t)\,dt + i\int_{a_N}^{b_N} y_N'(t)\,dt + j\int_{a_N}^{b_N} \hat{x}_N'(t)\,dt + k\int_{a_N}^{b_N} \hat{y}_N'(t)\,dt$$

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$$= \left[x_1(t) + L_{11}\right]_{a_1}^{b_1} + i\left[y_1(t) + L_{12}\right]_{a_1}^{b_1} + j\left[\hat{x}_1(t) + L_{13}\right]_{a_1}^{b_1} +$$

$$k\left[\hat{y}_1(t) + L_{14}\right]_{a_1}^{b_1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$\left[x_N(t) + L_{N1}\right]_{a_N}^{b_N} + i\left[y_N(t) + L_{N2}\right]_{a_N}^{b_N} + j\left[\hat{x}_N(t) + L_{N3}\right]_{a_N}^{b_N} +$$

$$k\left[\hat{y}_N(t) + L_{N4}\right]_{a_N}^{b_N}$$

$$= x_1(b_1) - x_1(a_1) + i(y_1(b_1) - y_1(a_1)) + j(\hat{x}_1(b_1) - \hat{x}_1(a_1)) +$$

$$k(\hat{y}_1(b_1) - \hat{y}_1(a_1)) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$x_N(b_N) - x_N(a_N) + i(y_N(b_N) - y_N(a_N)) + j(\hat{x}_N(b_N) - \hat{x}_N(a_N)) +$$

$$k(\hat{y}_N(b_N) - \hat{y}_N(a_N))$$

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$$= q_1(b_1) - q_1(a_1) + \ldots\ldots\ldots\ldots\ldots\ldots + q_N(b_N) - q_N(a_N)$$

$$= \sum_{n=1}^{N} q_n(b_n) - q_n(a_n),$$

where $L_{11}, L_{12}, L_{13}, L_{14}, \ldots\ldots\ldots, L_{N1}, L_{N2}, L_{N3}, L_{N4}$ are arbitrary real valued constants of integration. But since, as previously defined, the endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m), \ \forall m \in \{1, 2, 3, \ldots\ldots\ldots, N-1\}$,

(ii) $q_1(a_1) = q_N(b_N)$,

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it therefore follows that the quaternion sum,

$$\sum_{n=1}^{N} q_n(b_n) - q_n(a_n) = \sum_{n=1}^{N} q_n(b_n) - \sum_{n=1}^{N} q_n(a_n)$$

$$= q_1(b_1) + \ldots\ldots + q_{N-1}(b_{N-1}) + q_N(b_N) - (q_1(a_1) + q_2(a_2) + \ldots\ldots + q_N(a_N))$$

$$= q_1(b_1) + \ldots\ldots + q_{N-1}(b_{N-1}) + q_N(b_N) - q_1(a_1) - (q_2(a_2) + \ldots\ldots + q_N(a_N))$$

$$= q_1(b_1) + \ldots\ldots + q_{N-1}(b_{N-1}) - (q_2(a_2) + \ldots\ldots + q_N(a_N)) + q_N(b_N) - q_1(a_1)$$

$$= q_2(a_2) + \ldots\ldots + q_N(a_N) - (q_2(a_2) + \ldots\ldots + q_N(a_N)) + q_N(b_N) - q_1(a_1)$$

$$= 0,$$

and hence we conclude that the integral formula,

$\int_C dq = 0$, as required. <u>Q.E.D.</u>

Theorem TII-24.

Let there exist a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n(t), \ \forall t \in [a_n, b_n] \ \& \ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

insofar as the endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m)$, $\forall m \in \{1, 2, 3, \ldots\ldots\ldots, N-1\}$

(N is some finite positive integer)

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Subsequently, it may be proven that the integral formula,

$$\int_C q\, dq = 0,$$

is always valid in terms of the conditions specified above.

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PROOF:-

From the established theorems on differentiation, we deduce that the parametric first derivative of $f(q(t)) = (q(t))^2$, with respect to 't', is given by the formula,

$$\frac{d}{dt}[(q(t))^2] = q(t)\frac{d}{dt}[q(t)] + \frac{d}{dt}[q(t)]\,q(t),$$

when the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a, b],$$

denotes any arbitrary smooth arc, C. Hence, for any smooth arc, K_n, thus represented by the equation,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n(t), \ \forall t \in [a_n, b_n] \ \& \ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

the corresponding first derivative,

$$\begin{aligned} \frac{d}{dt}[(q_n(t))^2] &= q_n(t)\frac{d}{dt}[q_n(t)] + \frac{d}{dt}[q_n(t)]q_n(t) \\ &= q_n(t)\frac{d}{dt}[q_n(t)] + q_n(t)\frac{d}{dt}[q_n(t)] \\ &= 2q_n(t)\frac{d}{dt}[q_n(t)], \end{aligned}$$

bearing in mind the provisions of Theorem TI-10. However, since by definition the derivative,

$$\frac{d}{dt}\left(\int 2q_n(t)\frac{d}{dt}[q_n(t)]\,dt\right) = 2q_n(t)\frac{d}{dt}[q_n(t)],$$

then clearly the indefinite integral,

$$\int 2q_n(t)\frac{d}{dt}[q_n(t)]\,dt = (q_n(t))^2$$

$$\therefore \int q_n(t)\frac{d}{dt}[q_n(t)]\,dt = \tfrac{1}{2}(q_n(t))^2 \implies$$

$$\begin{aligned} \int_{a_n}^{b_n} q_n(t)\frac{d}{dt}[q_n(t)]\,dt &= \left[\tfrac{1}{2}(q_n(t))^2\right]_{a_n}^{b_n} \\ &= \tfrac{1}{2}\left((q_n(b_n))^2 - (q_n(a_n))^2\right). \end{aligned}$$

Furthermore, in view of the criteria laid down in Definition DII-19, we deduce that the definite integral,

$$\int_C q\,dq = \sum_{n=1}^{N}\int_{K_n} q\,dq$$

$$= \sum_{n=1}^{N}\int_{a_n}^{b_n} q_n(t)\frac{d}{dt}[q_n(t)]\,dt$$

$$= \sum_{n=1}^{N}\left[\tfrac{1}{2}(q_n(t))^2\right]_{a_n}^{b_n}$$

$$= \sum_{n=1}^{N}\tfrac{1}{2}\left((q_n(b_n))^2 - (q_n(a_n))^2\right)$$

$$= \tfrac{1}{2}\sum_{n=1}^{N}(q_n(b_n))^2 - (q_n(a_n))^2 .$$

But since, as previously stated, the endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m)$, $\forall m \in \{1,2,3,\ldots\ldots\ldots,N-1\}$,

(ii) $q_1(a_1) = q_N(b_N)$,

it therefore follows that the quaternion sum,

$$\sum_{n=1}^{N}(q_n(b_n))^2 - (q_n(a_n))^2 = \sum_{n=1}^{N}(q_n(b_n))^2 - \sum_{n=1}^{N}(q_n(a_n))^2$$

$$= (q_1(b_1))^2 + \ldots\ldots + (q_{N-1}(b_{N-1}))^2 + (q_N(b_N))^2 - \left[(q_1(a_1))^2 + (q_2(a_2))^2 + \ldots\ldots + (q_N(a_N))^2\right]$$

$$= (q_1(b_1))^2 + \ldots\ldots + (q_{N-1}(b_{N-1}))^2 + (q_N(b_N))^2 - (q_1(a_1))^2 - \left[(q_2(a_2))^2 + \ldots\ldots + (q_N(a_N))^2\right]$$

$$= (q_1(b_1))^2 + \ldots\ldots + (q_{N-1}(b_{N-1}))^2 - \left[(q_2(a_2))^2 + \ldots\ldots + (q_N(a_N))^2\right] + (q_N(b_N))^2 - (q_1(a_1))^2$$

$$= (q_2(a_2))^2 + \ldots\ldots + (q_N(a_N))^2 - \left[(q_2(a_2))^2 + \ldots\ldots + (q_N(a_N))^2\right] + (q_N(b_N))^2 - (q_1(a_1))^2$$

$= 0,$

and hence we conclude that the integral formula,

$\int_C q\,dq = \frac{1}{2} \times 0 = 0$, as required. Q.E.D.

Theorem TII-25.

Let there exist a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its component smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n(t) \ , \ \forall t \in [a_n, b_n] \ \& \ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

such that the corresponding endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m)$, $\forall m \in \{1, 2, 3, \ldots\ldots, N-1\}$,

(N is some finite positive integer)

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Hence, it may be shown that, if the quasi-complex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic at every point interior to and on the simple closed contour, C, then the definite integral,

$$\int_C f(q)\,dq = 0,$$

likewise exists, if and only if the contour, C, is sufficiently small so as to be entirely contained within the δ-neighbourhood of any fixed point, $q(t_0)$, located on the contour. This result shall otherwise be referred to as the

quaternion analogue of the Cauchy-Goursat Theorem.

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PROOF:-

From the preceding Theorem TII-13 and Definition DII-10, we recall that a quasi-complex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic at each point, $q = q_0 = q(t_0)$, on a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, if and only if the first derivative, $\frac{d}{dq}(f(q))$, exists not only at $q(t_0)$ but also at every point within some δ-neighbourhood of $q(t_0)$. From Definition DII-9, we also obtain

$$\left| \frac{f(q+\delta q) - f(q)}{\delta q} - \frac{d}{dq}(f(q)) \right| < \epsilon,$$

whenever $|\delta q| < \delta$,

thereby ensuring the existence of this particular derivative.

Now, by writing both the quaternion, q, and its corresponding increment, δq, respectively as

$$q = q(t_0),$$
$$\delta q = q_n(t) - q(t_0),$$

it is evident that the moduli,

$$\left| \frac{f(q+\delta q) - f(q)}{\delta q} - \frac{d}{dq}(f(q)) \right| = \left| \frac{f(q_n(t)) - f(q(t_0))}{q_n(t) - q(t_0)} - \frac{d}{dq}(f(q)) \bigg|_{q=q(t_0)} \right|,$$

$$|\delta q| = |q_n(t) - q(t_0)|,$$

whereupon the previously stated inequality,

$$\left| \frac{f(q_n(t)) - f(q(t_0))}{q_n(t) - q(t_0)} - \frac{d}{dq}(f(q)) \bigg|_{q=q(t_0)} \right| < \epsilon \implies$$

$$|q_n(t) - q(t_0)| < \delta,$$

provided that the contour, C, is sufficiently small, by virtue of Definition DII-10, so as to be entirely contained within the δ-neighbourhood of $q(t_0)$. Furthermore, by defining the existence of a quasi-complex parametric function,

$$\phi(q_n(t)) = \frac{f(q_n(t)) - f(q(t_0))}{q_n(t) - q(t_0)} - \frac{d}{dq}(f(q))\bigg|_{q=q(t_0)} \quad , \forall q_n(t) \neq q(t_0),$$

we deduce that

$$f(q_n(t)) - f(q(t_0)) - \left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)(q_n(t) - q(t_0))$$

$$= \phi(q_n(t))(q_n(t) - q(t_0))$$

$$\therefore f(q_n(t)) = \phi(q_n(t))(q_n(t) - q(t_0)) + f(q(t_0)) + \left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)(q_n(t) - q(t_0)).$$

Subsequently, the multiplication of both sides of this equation by the derivative, $\frac{d}{dt}[q_n(t)]$, and then integrating this product with respect to 't' between the limits, 'a_n' and 'b_n', likewise yields

$$f(q_n(t))\frac{d}{dt}[q_n(t)] = \phi(q_n(t))(q_n(t) - q(t_0))\frac{d}{dt}[q_n(t)] + f(q(t_0))\frac{d}{dt}[q_n(t)] + \left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q_n(t)\frac{d}{dt}[q_n(t)] - \left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q(t_0)\frac{d}{dt}[q_n(t)]$$

$$\therefore \int_{a_n}^{b_n} f(q_n(t))\frac{d}{dt}[q_n(t)]\,dt = \int_{a_n}^{b_n} \phi(q_n(t))(q_n(t)-q(t_0))\frac{d}{dt}[q_n(t)]\,dt \;+$$

$$\int_{a_n}^{b_n} f(q(t_0))\frac{d}{dt}[q_n(t)]\,dt \;+$$

$$\int_{a_n}^{b_n}\left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q_n(t)\frac{d}{dt}[q_n(t)]\,dt \;-$$

$$\int_{a_n}^{b_n}\left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q(t_0)\frac{d}{dt}[q_n(t)]\,dt$$

$$= \int_{a_n}^{b_n} \phi(q_n(t))(q_n(t)-q(t_0))\frac{d}{dt}[q_n(t)]\,dt + f(q(t_0))\int_{a_n}^{b_n}\frac{d}{dt}[q_n(t)]\,dt \;+$$

$$\left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)\int_{a_n}^{b_n} q_n(t)\frac{d}{dt}[q_n(t)]\,dt - \left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q(t_0)\int_{a_n}^{b_n}\frac{d}{dt}[q_n(t)]\,dt$$

$$\therefore \int_{K_n} f(q)\,dq = \int_{a_n}^{b_n} \phi(q_n(t))(q_n(t)-q(t_0))\frac{d}{dt}[q_n(t)]\,dt + f(q(t_0))\int_{K_n} dq \;+$$

$$\left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)\int_{K_n} q\,dq - \left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q(t_0)\int_{K_n} dq,$$

whereupon we deduce that the integral sum,

$$\sum_{n=1}^{N}\int_{K_n} f(q)\,dq = \sum_{n=1}^{N}\int_{a_n}^{b_n} \phi(q_n(t))(q_n(t)-q(t_0))\frac{d}{dt}[q_n(t)]\,dt \;+$$

$$\sum_{n=1}^{N} f(q(t_0))\int_{K_n} dq + \sum_{n=1}^{N}\left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)\int_{K_n} q\,dq \;-$$

$$\sum_{n=1}^{N}\left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q(t_0)\int_{K_n}dq$$

$$= \sum_{n=1}^{N}\int_{a_n}^{b_n}\phi(q_n(t))(q_n(t)-q(t_0))\frac{d}{dt}[q_n(t)]\,dt \;+$$

$$f(q(t_0))\sum_{n=1}^{N}\int_{K_n}dq \;+\; \left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)\sum_{n=1}^{N}\int_{K_n}q\,dq \;-$$

$$\left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q(t_0)\sum_{n=1}^{N}\int_{K_n}dq$$

$$\therefore \int_C f(q)\,dq = \sum_{n=1}^{N}\int_{a_n}^{b_n}\phi(q_n(t))(q_n(t)-q(t_0))\frac{d}{dt}[q_n(t)]\,dt \;+$$

$$f(q(t_0))\int_C dq \;+\; \left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)\int_C q\,dq \;-$$

$$\left(\frac{d}{dq}(f(q))\Big|_{q=q(t_0)}\right)q(t_0)\int_C dq,$$

by virtue of Definition DII-19. However, from Theorems TII-23 and TII-24, we instantly recall that, for a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, the constituent smooth arcs, K_n, are represented by the formulae —

$$q_n(t) = x_n(t) + i y_n(t) + j\hat{x}_n(t) + k\hat{y}_m(t),\ \forall t \in [a_n, b_n], \implies$$

$$\int_C dq = 0 \quad \left(\frac{y_n(t)}{\lambda_1} = \frac{\hat{x}_n(t)}{\lambda_2} = \frac{\hat{y}_m(t)}{\lambda_3} = \frac{\xi_n(t)}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right)$$

AND

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n(t), \ \forall t \in [a_n, b_n] \ \& \ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

$$\Longrightarrow \int_C q \, dq = 0,$$

and hence the integral sum, $\int_C f(q)\, dq$, is reduced to

$$\int_C f(q)\, dq = \sum_{n=1}^{N} \int_{a_n}^{b_n} \phi(q_n(t))(q_n(t) - q(t_0)) \frac{d}{dt}[q_n(t)]\, dt.$$

Bearing in mind the provisions of Theorem TII-18, we similarly perceive that the modulus,

$$\left|\int_C f(q)\, dq\right| = \left|\sum_{n=1}^{N} \int_{a_n}^{b_n} \phi(q_n(t))(q_n(t) - q(t_0)) \frac{d}{dt}[q_n(t)]\, dt\right|$$

$$\leq \sum_{n=1}^{N} \left|\int_{a_n}^{b_n} \phi(q_n(t))(q_n(t) - q(t_0)) \frac{d}{dt}[q_n(t)]\, dt\right|$$

$$\leq \sum_{n=1}^{N} \int_{a_n}^{b_n} \left|\phi(q_n(t))(q_n(t) - q(t_0)) \frac{d}{dt}[q_n(t)]\right| dt.$$

But since, as previously stated, the moduli,

$$|\phi(q_n(t))| = \left| \frac{f(q_n(t)) - f(q(t_0))}{q_n(t) - q(t_0)} - \frac{d}{dq}(f(q))\Big|_{q=q(t_0)} \right| < \epsilon,$$

$$|q_n(t) - q(t_0)| < \delta,$$

such that the modulus product,

$$\left|\phi(q_n(t))(q_n(t) - q(t_0))\tfrac{d}{dt}[q_n(t)]\right| = \left|\phi(q_n(t))\right|\left|q_n(t) - q(t_0)\right|\left|\tfrac{d}{dt}[q_n(t)]\right|$$

$$< \epsilon\delta\left|\tfrac{d}{dt}[q_n(t)]\right| \qquad (a_n \leq t \leq b_n),$$

it therefore follows from real variable analysis that the definite integral,

$$\int_{a_n}^{b_n}\left|\phi(q_n(t))(q_n(t) - q(t_0))\tfrac{d}{dt}[q_n(t)]\right| dt < \int_{a_n}^{b_n}\epsilon\delta\left|\tfrac{d}{dt}[q_n(t)]\right| dt$$

$$= \epsilon\delta\int_{a_n}^{b_n}\left|\tfrac{d}{dt}[q_n(t)]\right| dt$$

$$= \epsilon\delta\, L(a_n, b_n),$$

where the definite integral,

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$$L(a_n, b_n) = \int_{a_n}^{b_n}\left|\tfrac{d}{dt}[q_n(t)]\right| dt > 0,$$

and thus the modular inequality,

$$\left|\int_C f(q)\,dq\right| \leq \sum_{n=1}^{N}\int_{a_n}^{b_n}\left|\phi(q_n(t))(q_n(t) - q(t_0))\tfrac{d}{dt}[q_n(t)]\right| dt$$

$$< \sum_{n=1}^{N}\epsilon\delta\, L(a_n, b_n) = \epsilon\delta\sum_{n=1}^{N} L(a_n, b_n) = \gamma > 0.$$

Clearly, the simultaneous existence of the two inequalities,

$$\left|\int_C f(q)\,dq\right| < \gamma \,,$$

$$\left|q_n(t) - q(t_0)\right| < \delta \,,$$

is the very condition which we require for the existence of the limit,

$$\lim_{q_n(t) \to q(t_0)} \left[\int_C f(q)\,dq\right] = 0 .$$

But since the definite integral,

$$\int_C f(q)\,dq = \sum_{n=1}^{N} \int_{K_n} f(q)\,dq = \sum_{n=1}^{N} \int_{a_n}^{b_n} f(q_n(t)) \frac{d}{dt}[q_n(t)]\,dt = Q_0 \,,$$

where Q_0 is some arbitrary quaternion constant, we therefore conclude that the limit,

$$\lim_{q_n(t) \to q(t_0)} \left[Q_0\right] = 0 \Longrightarrow Q_0 = 0 \,,$$

and hence the definite integral,

$$\int_C f(q)\,dq = 0 ,$$ as required. Q.E.D.

Definition DII-20.

Let there exist a quasi-complex quaternion number,

$$q = x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi \qquad (\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}) \,,$$

such that the vector subspace,

$$\Pi = \left\langle 1, \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\right\rangle = \left\{ x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi \;\middle|\; x, \xi \in \mathbb{R} \right\}$$

$$\subset \mathbb{H} = \langle 1, i, j, k \rangle .$$

Bearing in mind the algebraic properties,

(a) $$\left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]^2 = -1 \;;$$

(b) $$\left| x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi \right| = \sqrt{x^2+\xi^2} \;,$$

we postulate that every such quasi-complex quaternion number, q, may be represented as a point on some Cartesian coordinate system, which shall otherwise be referred to as the quaternion analogue of the Argand diagram, insofar as its horizontal x-axis thus represents the set of all real numbers,

$$\mathbb{R} = \{ x \mid x \in \mathbb{R} \} = \langle 1 \rangle ,$$

and, similarly, its vertical ξ-axis represents the set of all 'pure' quaternions,

$$\left\{ \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi \;\middle|\; \xi \in \mathbb{R} \right\} = \left\langle \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right] \right\rangle .$$

Definition DII-21.

Let there exist a quasi-complex quaternion number,

$$q = x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi \qquad (\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}),$$

on the vector subspace,

$$\Pi = \left\langle 1, \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\right\rangle = \left\{x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi \,\middle|\, x, \xi \in \mathbb{R}\right\}$$

$$\subset \mathbb{H}.$$

A simply connected domain, $D \subseteq \Pi$, is a domain such that every simple closed contour located therein encloses only those points belonging to D. By direct contrast, however, a domain that is not simply connected is said to be a multiply connected domain.

Definition DII-22.

Let there exist a vector subspace,

$$\Pi = \left\langle 1, \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\right\rangle = \left\{x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi \,\middle|\, x, \xi \in \mathbb{R}\right\}$$

$$\subset \mathbb{H}.$$

A simple closed contour, $C = \bigcup_{n=1}^{N} K_n \subset \Pi$, is said to be

(a) positively oriented, if and only if the consecutive endpoints of each constituent smooth arc, K_n, are described in a counterclockwise manner with respect to some fixed point, q_0, interior to C;

(b) negatively oriented, if and only if the consecutive endpoints of each constituent smooth arc, K_n, are described in a clockwise manner with respect to some fixed point, q_0, interior to C.

Definition DII-23.

Let there exist a contour, $C = \bigcup_{n=1}^{N} K_n$, whose constituent smooth arcs, K_n, are denoted by the equations,

$$q_n(t) = x_n(t) + iy_n(t) + j\hat{x}_n(t) + k\hat{y}_n(t), \quad \forall t \in [a_n, b_n].$$

Consequently, we postulate the existence of the definite integral,

$$\int_C f(q)\,dq = \sum_{n=1}^{N} \int_{a_n}^{b_n} f(q_n(t)) \frac{d}{dt}[q_n(t)]\,dt$$

$$= \int_{\alpha}^{\beta} f(q)\,dq,$$

insofar as

(a) the quaternion constants, $\alpha = q(a) = q(a_1)$ and $\beta = q(b) = q(b_N)$, represent the endpoints of the contour, C;

(b) the functions, $f(q_n(t))\frac{d}{dt}[q_n(t)]$, are integrable with respect to the real parameter 't';

(c) the value of the definite integral,

$$\int_C f(q)\,dq = \int_\alpha^\beta f(q)\,dq,$$

is <u>independent</u> of any contour, C, joining the two endpoints, α and β.

<u>Theorem TII-26.</u>

Let there exist a parametric quaternion function,

$$F(t) = U_1(t) + i\,V_1(t) + j\,U_2(t) + k\,V_2(t),$$

which is integrable over the domain, $[a,b] = \{t \mid a \leq t \leq b\}$. Subsequently, it may be proven that the definite integral,

$$\int_a^b F(t)\,dt = \int_{-b}^{-a} F(-t)\,dt.$$

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<u>PROOF</u>:-

Let the parametric quaternion function,

$$F(t) = U_1(t) + i\,V_1(t) + j\,U_2(t) + k\,V_2(t),$$

imply the existence of another parametric function,

$$\begin{aligned}\Phi(t) &= F(-t)\\ &= U_1(-t) + i\,V_1(-t) + j\,U_2(-t) + k\,V_2(-t),\end{aligned}$$

by virtue of the 'dummy substitution of variables' process. Since, as previously stated in the preamble to this proof, the function, $F(t)$, is integrable over the domain, $[a, b] = \{t \mid a \leq t \leq b\}$, it therefore follows from the established theorems on integration that the definite integral,

$$\int_a^b F(t)\,dt = \int_a^b U_1(t)\,dt + i\int_a^b V_1(t)\,dt + j\int_a^b U_2(t)\,dt + k\int_a^b V_2(t)\,dt.$$

Furthermore, by letting the function, $\Phi(t) = F(-t)$, be integrable over the domain, $[-b, -a] = \{t \mid -b \leq t \leq -a\}$, we similarly deduce that the definite integral,

$$\int_{-b}^{-a}\Phi(t)\,dt = \int_{-b}^{-a}F(-t)\,dt$$

$$= \int_{-b}^{-a}U_1(-t)\,dt + i\int_{-b}^{-a}V_1(-t)\,dt + j\int_{-b}^{-a}U_2(-t)\,dt + k\int_{-b}^{-a}V_2(-t)\,dt.$$

From the calculus of single variable real functions, we recall that the definite integral,

$$\int_a^b g(x)\,dx = G(b) - G(a) \Longrightarrow \int_{X_1}^{X_2} g(X)\,dX = G(X_2) - G(X_1),$$

by virtue of the 'dummy substitution of variables' process. Now, let

$$X = -x \Longrightarrow X_2 = -x_2\,;\ X_1 = -x_1\,;\ dX = -dx\,,$$

whereupon we deduce that the definite integral,

$$\int_{X_1}^{X_2} g(X)\,dX = \int_{x_1}^{x_2} g(-x) - dx \qquad \text{(note the 'change of variable' transformation)}$$

$$= -\int_{x_1}^{x_2} g(-x)\,dx$$

$$\therefore -\int_{x_1}^{x_2} g(-x)\,dx = G(X_2) - G(X_1)$$
$$= G(-x_2) - G(-x_1)$$

$$\therefore \int_{x_1}^{x_2} g(-x)\,dx = G(-x_1) - G(-x_2).$$

By setting, $x_2 = -a$ and $x_1 = -b$, we further obtain

$$\int_{-b}^{-a} g(-x)\,dx = G(-(-b)) - G(-(-a))$$
$$= G(b) - G(a)$$
$$= \int_a^b g(x)\,dx\,,$$

which likewise implies the existence of the definite integrals,

$$\int_{-b}^{-a} U_1(-t)\,dt = \int_a^b U_1(t)\,dt \ ; \ \int_{-b}^{-a} V_1(-t)\,dt = \int_a^b V_1(t)\,dt \ ;$$

$$\int_{-b}^{-a} U_2(-t)\,dt = \int_a^b U_2(t)\,dt \ ; \ \int_{-b}^{-a} V_2(-t)\,dt = \int_a^b V_2(t)\,dt .$$

Finally, we conclude, after making the appropriate algebraic substitutions, that the definite integral,

$$\int_{-b}^{-a} F(-t)\,dt = \int_{-b}^{-a} U_1(-t)\,dt + i\int_{-b}^{-a} V_1(-t)\,dt + j\int_{-b}^{-a} U_2(-t)\,dt + k\int_{-b}^{-a} V_2(-t)\,dt$$

$$= \int_a^b U_1(t)\,dt + i\int_a^b V_1(t)\,dt + j\int_a^b U_2(t)\,dt + k\int_a^b V_2(t)\,dt$$

$$= \int_a^b F(t)\,dt,$$ as required. Q.E.D.

Theorem TII-27.

Let there exist a contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + iy_n(t) + j\hat{x}_n(t) + k\hat{y}_n(t), \quad \forall t \in [a_n, b_n],$$

whereupon we make the additional stipulation that the endpoints thereof are subject to the condition,

$$q_{m+1}(a_{m+1}) = q_m(b_m), \quad \forall m \in \{1,2,3,\ldots\ldots,N-1\}$$ (N is some finite positive integer).

Subsequently, by constructing another contour, $-C = \bigcup_{n=1}^{N} -K_n$, where

$q_{m+1}(-b_{m+1}) = q_m(-a_m), \quad \forall m \in \{1,2,3,\ldots\ldots,N-1\},$

we can prove that the definite integral,

$$\int_{-C} f(q)\,dq = -\int_{C} f(q)\,dq,$$

provided that the functions, $f(q_n(t))\frac{d}{dt}[q_n(t)]$ and $f(q_n(-t))\frac{d}{dt}[q_n(-t)]$, are likewise integrable with respect to the real parameter 't'.

* * *

PROOF:-

Let us define a set of parametric quaternion functions,

$$F_n(t) = f(q_n(t))\frac{d}{dt}[q_n(t)] = f(q_n(t))\left(\frac{d}{dt}\circ q_n\right)(t),$$

which are both integrable over their respective domains, $[a_n, b_n] = \{t \mid a_n \le t \le b_n\}$, and furthermore imply the existence of another set of parametric functions,

$$F_n(-t) = f(q_n(-t))\left(\frac{d}{dt}\circ q_n\right)(-t),$$

by virtue of the 'dummy substitution of variables' process. Now, if we let each of these functions be integrable over the domain, $[-b_n, -a_n] = \{t \mid -b_n \le t \le -a_n\}$, it therefore follows from Theorem TII-26 that the definite integral,

$$\int_{a_n}^{b_n} F_n(t)\,dt = \int_{-b_n}^{-a_n} F_n(-t)\,dt$$

$$\therefore \int_{a_n}^{b_n} f(q_n(t))\left(\frac{d}{dt}\circ q_n\right)(t)\,dt = \int_{-b_n}^{-a_n} f(q_n(-t))\left(\frac{d}{dt}\circ q_n\right)(-t)\,dt$$

$$\therefore \int_{a_n}^{b_n} f(q_n(t))\frac{d}{dt}[q_n(t)]\,dt = \int_{-b_n}^{-a_n} f(q_n(-t))\frac{d}{dt}[q_n(-t)]\,dt$$

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$$\therefore \int_{K_n} f(q)\,dq = -\int_{-b_n}^{-a_n} f(q_n(-t))\left(-\frac{d}{dt}[q_n(-t)]\right)dt$$

$$\therefore \int_{K_n} f(q)\,dq = -\int_{-K_n} f(q)\,dq,$$

in accordance with Definition DII-18 and hence the definite integral,

$$\int_{-K_n} f(q)\,dq = -\int_{K_n} f(q)\,dq.$$

Finally, the summation of both sides of this equation from $n = 1$ to N yields

$$\sum_{n=1}^{N} \int_{-K_n} f(q)\,dq = \sum_{n=1}^{N}\left(-\int_{K_n} f(q)\,dq\right)$$

$$\therefore \sum_{n=1}^{N} \int_{-K_n} f(q)\,dq = -\sum_{n=1}^{N} \int_{K_n} f(q)\,dq$$

$$\therefore \int_{-C} f(q)\,dq = -\int_{C} f(q)\,dq,$$

bearing in mind the provisions of Definition DII-19, as required. Q.E.D.

Theorem TII-28.

Let there exist some arbitrary positively oriented simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent smooth arcs, K_n, is accordingly represented by the equations,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n(t), \ \forall t \in [a_n, b_n] \ \& \ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

whereupon the corresponding endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m)$, $\forall m \in \{1, 2, 3, \ldots\ldots, N-1\}$ (N is some finite positive integer),

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Furthermore, let there exist a set of non-intersecting positively oriented simple closed contours, Γ_μ, $\forall \mu \in \{1, 2, 3, \ldots\ldots\ldots, M\}$, which are located entirely within C. Now, let R be the closed region consisting of all points located within and on C, except for those which are interior to each Γ_μ. Similarly, let the entire oriented boundary of R, namely -

$$B = C \cup -\Gamma_1 \cup -\Gamma_2 \cup \ldots\ldots\ldots \cup -\Gamma_M.$$

Subsequently, it may be shown that, if the quasi-complex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

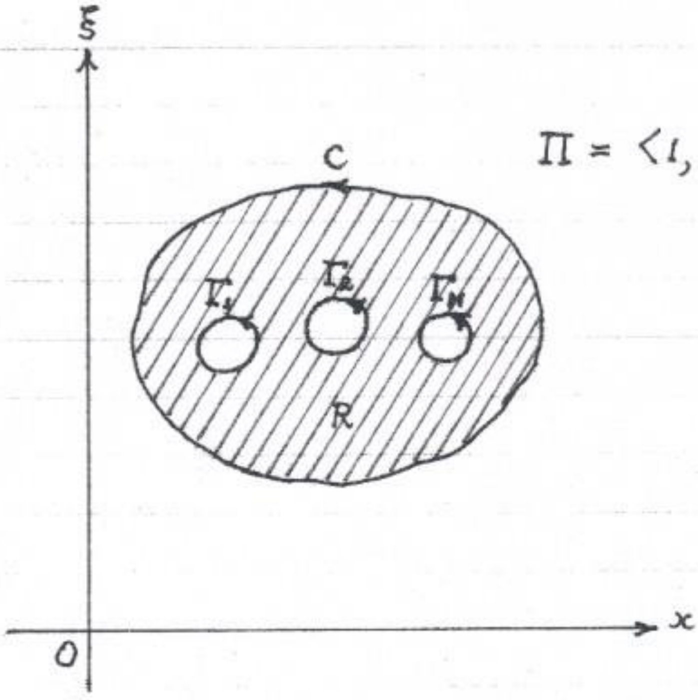
is analytic in R, then the definite integral,

$$\int_B f(q)\,dq = 0,$$

likewise exists in terms of the conditions specified above.

* * *

PROOF :-

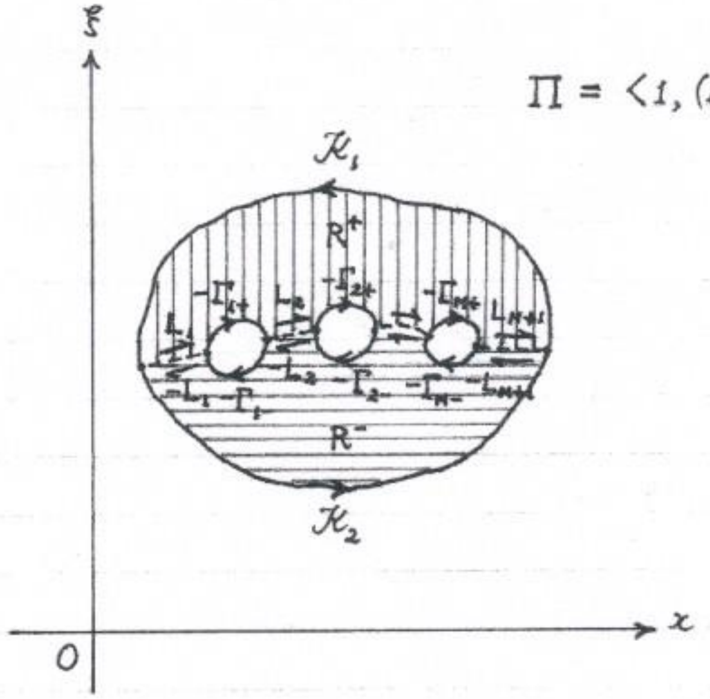


$$\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle$$

Fig. 1.

From Fig. 1 above, we observe that the region, R, thus bounded by the closed contour, C, is a <u>multiply connected domain</u>, bearing in mind the provisions of Definition DII–21. Consequently, in order to make the results of the preceding theorems on integration directly applicable to our geometric construction, we shall now modify Fig. 1 so as to facilitate the proof of this theorem by means of the following diagram depicted below:–

$$\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle$$



<u>Fig. 2.</u>

Hence, from Fig. 2 above, we immediately perceive that

(a) the <u>positively oriented</u> simple closed contour,

$C = \mathcal{K}_1 \cup \mathcal{K}_2$,

(b) the region bounded by C, namely –

$$R = R^+ \cup R^-,$$

(c) the existence of positively oriented simple closed contours, Γ_μ, $\forall \mu \in \{1, 2, 3, \ldots \ldots \ldots, M\}$, likewise implies the existence of negatively oriented simple closed contours,

$$-\Gamma_\mu = -\Gamma_{\mu+} \cup -\Gamma_{\mu-},$$

such that we let the corresponding definite integrals,

$$\int_{-\Gamma_\mu} f(q)\,dq = \int_{-\Gamma_{\mu+}} f(q)\,dq + \int_{-\Gamma_{\mu-}} f(q)\,dq,$$

in view of the criteria laid down in Definition DII-19.

Furthermore, as is evident from our modified geometric construction, the simple closed contours, $-\Gamma_\mu$, may also be connected to each other as well as to the simple closed contour, C, by means of a finite set of rectilinear segments, $\{L_1, L_2, \ldots\ldots\ldots, L_{M+1}\}$, which in turn implies the existence of another set of rectilinear segments oriented in the opposite direction, namely – $\{-L_1, -L_2, \ldots\ldots\ldots\ldots, -L_{M+1}\}$.

Subsequently, if a quasi-complex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is <u>analytic</u> in R, then this function is also analytic in the sub-regions, R^+ and R^-. Moreover, we observe from Fig. 2 that

(i) the sub-region, R^+, is bounded by the <u>positively oriented</u> simple closed contour,

$$C_1 = \mathcal{K}_1 \cup L_1 \cup -\Gamma_{1+} \cup \ldots\ldots\ldots\ldots \cup -\Gamma_{M+} \cup L_{M+1},$$

(ii) the sub-region, R^-, is bounded by the <u>positively oriented</u> simple closed contour,

$$C_2 = \mathcal{K}_2 \cup -L_{M+1} \cup -\Gamma_{M-} \cup \ldots\ldots\ldots\ldots \cup -\Gamma_{1-} \cup -L_1,$$

whence, by virtue of the preceding Theorem TII-25, the definite integrals,

$$\int_{C_1} f(q)\,dq = 0 \quad \text{and} \quad \int_{C_2} f(q)\,dq = 0,$$

likewise exist, in view of the arguments outlined above, and therefore imply the existence of the integral sum,

$$\int_{C_1} f(q)\,dq + \int_{C_2} f(q)\,dq = 0.$$

However, since the definite integrals,

$$\int_{C_1} f(q)\,dq = \int_{\mathcal{K}_1} f(q)\,dq + \int_{L_1} f(q)\,dq + \int_{-\Gamma_{1+}} f(q)\,dq + \ldots\ldots\ldots\ldots +$$

$$\int_{-\Gamma_{M+}} f(q)\,dq + \int_{L_{M+1}} f(q)\,dq$$

$$= \int_{\mathcal{K}_1} f(q)\,dq + \sum_{n=1}^{M+1} \int_{L_n} f(q)\,dq + \sum_{\mu=1}^{M} \int_{-\Gamma_{\mu+}} f(q)\,dq$$

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AND

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$$\int_{C_2} f(q)\,dq = \int_{K_2} f(q)\,dq + \int_{-L_{M+1}} f(q)\,dq + \int_{-\Gamma_{M-}} f(q)\,dq + \dots\dots\dots\dots +$$

$$\int_{-\Gamma_{1-}} f(q)\,dq + \int_{-L_1} f(q)\,dq$$

$$= \int_{K_2} f(q)\,dq + \sum_{n=1}^{M+1} \int_{-L_n} f(q)\,dq + \sum_{\mu=1}^{M} \int_{-\Gamma_{\mu-}} f(q)\,dq \,,$$

by virtue of Definition DII-19, the above integral sum can accordingly be written as –

$$\int_{K_1} f(q)\,dq + \sum_{n=1}^{M+1} \int_{L_n} f(q)\,dq + \sum_{\mu=1}^{M} \int_{-\Gamma_{\mu+}} f(q)\,dq +$$

$$\int_{K_2} f(q)\,dq + \sum_{n=1}^{M+1} \int_{-L_n} f(q)\,dq + \sum_{\mu=1}^{M} \int_{-\Gamma_{\mu-}} f(q)\,dq = 0$$

$$\therefore \int_{K_1} f(q)\,dq + \int_{K_2} f(q)\,dq + \sum_{n=1}^{M+1} \int_{L_n} f(q)\,dq + \sum_{n=1}^{M+1} \int_{-L_n} f(q)\,dq +$$

$$\sum_{\mu=1}^{M} \int_{-\Gamma_{\mu+}} f(q)\,dq + \sum_{\mu=1}^{M} \int_{-\Gamma_{\mu-}} f(q)\,dq = 0$$

$$\therefore \int_{C} f(q)\,dq + \sum_{n=1}^{M+1} \int_{L_n} f(q)\,dq + \sum_{n=1}^{M+1} -\left(\int_{L_n} f(q)\,dq\right) +$$

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$$\sum_{\mu=1}^{M}\left(\int_{-\Gamma_{\mu+}} f(q)\,dq + \int_{-\Gamma_{\mu-}} f(q)\,dq\right) = 0$$

$$\therefore \int_{C} f(q)\,dq + \sum_{n=1}^{M+1}\int_{L_n} f(q)\,dq - \sum_{n=1}^{M+1}\int_{L_n} f(q)\,dq + \sum_{\mu=1}^{M}\int_{-\Gamma_{\mu}} f(q)\,dq = 0$$

$$\therefore \int_{C} f(q)\,dq + \sum_{\mu=1}^{M}\int_{-\Gamma_{\mu}} f(q)\,dq = 0$$

To be continued via the author's next submission, namely -

> "***A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 9/10.***"

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