“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.”

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I. PRELIMINARY REMARKS.

(1) An abstract thus pertaining to the author’s original paper has been provided under the heading, **‘PREFACE’**, as is evident from Pages [3] – [4] of this submission.

(2) For further details, the reader should accordingly refer to the author’s concomitant submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - ADDITIONAL REMARKS pertaining to PART 1/10.”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 1/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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"A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions"

by Stephen C. Pearson (ATRACI),
Member of the
American Mathematical Society.

5th March 2001.

FINAL DRAFT pending further assessment.

PREFACE

As its title suggests, the overall aim of this paper is to provide a further insight into the properties of functions of a quaternion (hypercomplex) variable, bearing in mind that it is, in fact, a sequel to the author's first paper [5] and concomitant series of expository articles [6] – [8] based on this subject.

Hence, it seems appropriate that one should commence such a discourse by firstly defining the existence of the elementary quaternion hypercomplex functions as well as evaluating some of their algebraic properties (viz. Section I). With these ideas firmly established, the author then proceeds to investigate some additional analytical properties of quaternion functions in general (viz. Section II) and, finally, we conclude our formal analysis of this topic with an introductory treatment of the series expansions of quaternion hypercomplex functions (viz. Section III).

As a means of developing a totally rigorous approach to the material described above, the author has subsequently adopted Churchill et al. [1] as a suitable guideline for that purpose, in as much as the latter named authors have enunciated many notions, thus pertaining to complex variable analysis, which fortunately can be treated in an analogous manner with respect to quaternion numbers and their corresponding functions.

In summary, the author has made a number of pertinent comments in respect of both the above mentioned factors and other related matters – essentially, this was conducted in the spirit of an appraisal of all such results having been elucidated in the ensuing Sections I–III of this paper (viz. Section IV).

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TABLE OF CONTENTS

I. The Classification of Quaternion Hypercomplex Functions

The notion and subsequent properties of elementary functions, be they algebraic or transcendental by nature, have already been well established in terms of both real and complex variable analysis, as is evident from the mathematical texts compiled by Churchill et al. [1] and Salas and Einar Hille [2]. Invariably, one may even speculate on the feasibility of analogously extending such concepts into the realm of quaternion (hypercomplex) functions — a question which was initially posed in the author's first paper on this subject [5].

Hence, the remainder of this section is primarily devoted to defining the existence and deriving the algebraic properties of

(a) the polynomial function,
(b) the exponential function,
(c) the trigonometric functions,
(d) the hyperbolic functions,
(e) the logarithmic function,
(f) variables raised to the power of fractional indices; quaternion hypercomplex exponents,
(g) the inverse trigonometric functions,
(h) the inverse hyperbolic functions,

of the quaternion variable, $q = x + iy + j\hat{x} + k\hat{y}$ $(x, y, \hat{x}, \hat{y} \in \mathbb{R})$, in such a way that the logical connection between these functions and their more familiar analogues from complex variable analysis will become readily apparent to the reader.

— 2 —

1. Definition and Some Properties of Functional Arrays as a Prerequisite to the Study of Elementary Quaternion Hypercomplex Functions.

From complex variable analysis, we recall that elementary functions such as the logarithmic function are multi-valued in nature — an algebraic phenomenon for which Churchill et al. [1] have already provided an adequate account. Moreover, this same notion of multi-valued functions is further exemplified in the author's first paper [5] on quaternion functions by way of

(a) the quotient of any two quaternion numbers, q_1 and q_2, namely —

$$q_1/q_2 = \begin{cases} q_1(1/q_2) \\ (1/q_2)\,q_1 \end{cases} = \begin{cases} q_1 q_2^{-1} \\ q_2^{-1} q_1 \end{cases}, \ \forall q_2 \neq 0 \qquad (1-1);$$

(b) the first order derivative, with respect to 'q', of a quaternion function, $f(q)$, thus restricted to an arc, C, embedded in q-space,

$$\left[\frac{d}{dq}\right]_C (f(q)) = \begin{cases} \dfrac{\frac{d}{dt}\left[f(q(t))\right] \cdot \overline{\frac{d}{dt}\left[q(t)\right]}}{\left|\frac{d}{dt}\left[q(t)\right]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}\left[q(t)\right]} \cdot \frac{d}{dt}\left[f(q(t))\right]}{\left|\frac{d}{dt}\left[q(t)\right]\right|^2} \end{cases} \qquad (1-2),$$

provided that this limit (ie. derivative) exists.

Hence, in keeping with the notation employed in Eqs. (1-1) and (1-2) above, it seems only appropriate that every multi-valued quaternion hypercomplex function should likewise be represented as a functional array, whose existence we now formally define as follows :-

Definition DI-1.

A functional array, thus generated by a multi-valued function, F, is any predetermined collection of single-valued real, complex or quaternion hypercomplex functions, listed in tabular form.

Consequently, we denote any functional array of order, n, as

$$F = \{ f_{j^*} ; \forall j^* = 1, \ldots\ldots, n] = \begin{cases} f_1 & \Longleftrightarrow F \in \{ f_{j^*} \mid j^* = 1, \ldots\ldots\ldots, n \}, \\ \vdots & \\ f_{j^*} & \\ \vdots & \\ f_n & \end{cases}$$

where

(a) $n \in \mathbb{N}$, the set of natural numbers, denotes the <u>order</u> of the functional array,

(b) the single-valued function,

$f_{j^*} \in \mathbb{C} \cup \mathbb{H} \quad (\mathbb{R} \subset \mathbb{C})$,

(c) $f_1 \neq \ldots\ldots \neq f_{j^*} \neq \ldots\ldots\ldots \neq f_n$,

with respect to the specified domain of definition.

The reader will no doubt appreciate that the usage of the set-theoretic statement, $F \in \{ f_{j^*} \mid j^* = 1, \ldots\ldots\ldots, n \}$, is merely intended to illustrate the fact that one can arbitrarily equate the multi-valued function, F, with any one of the single-valued functions, $f_1, \ldots\ldots\ldots, f_n$, and hence Definition DI-1 remains logically consistent with the concept of a multi-valued function thus originating from complex variable analysis. Furthermore, in the spirit of maintaining this same logical consistency, we shall likewise clarify the notions of domain of definition and order, mentioned beforehand, as follows :-

Definition DI-2.

The domain of definition, or more simply, domain, of any functional array is equal to that of its constituent single-valued functions, in other words -

$$\mathrm{dom}(\{f_j^*; \forall j^* = 1, \ldots\ldots\ldots, n]) = \mathrm{dom}(f_1) = \ldots\ldots = \mathrm{dom}(f_j^*) = \ldots\ldots = \mathrm{dom}(f_n).$$

Definition DI-3.

Let there exist a functional array of order, n. Hence, the total number of constituent single-valued functions listed therein may be written as -

$$\#\{f_j^*; \forall j^* = 1, \ldots\ldots\ldots, n] = n = \#(\{f_j^* | j^* = 1, \ldots\ldots\ldots, n\}),$$

which accordingly represents the order of any such array.

We surmise that Definition DI-2 is a perfectly natural one in the sense that the notion of a domain of definition, which was originally postulated in the case of single-valued functions, has now been extended to include multi-valued functions as well. Indeed, one can legitimately regard any given single-valued function as being a multi-valued function whose constituent single-valued functions have been made equal to one another over the same domain. We will summarise this same idea more concisely by means of the following definition :-

-5-

Definition DI - 4.

A functional array, $\{f_{j^*}; \forall j^* = 1, \ldots\ldots\ldots, n\}$, is said to be a totally degenerate functional array of order, n, provided that all of its 'n' constituent single-valued functions are equal to one another over the same domain. In the circumstances, we henceforth obtain a single-valued function,

$$f = \{f_{j^*}; \forall j^* = 1, \ldots\ldots\ldots, n\} = \begin{cases} f_1 & , \\ \vdots & \\ f_{j^*} & \\ \vdots & \\ f_n & \end{cases}$$

where $f_1 = \ldots\ldots\ldots = f_{j^*} = \ldots\ldots\ldots = f_n = f$.

The concepts expressed in Definitions DI-1 to 4 may be conveniently illustrated by means of the function diagrams provided in Figs. (1-1) and (1-2) overleaf, insofar as we recognise that the first function diagram is simply a generalisation of the second.

From the author's first paper [5] on quaternion functions, we recall that the composite quaternion hypercomplex function, $(f \circ g)(q)$, is denoted by the definitive formula,

$$(f \circ g)(q) = f(g(q)) \qquad (1-3),$$

which can also be written in the form of a <u>mapping (transformation)</u>, in other words —

$$\begin{tikzcd}
q \arrow[d, "g"'] \arrow[dr, "f \circ g"] & \\
g(q) \arrow[r, "f"'] & f(g(q))
\end{tikzcd} \qquad (1-4).$$

For simplicity's sake, it was originally assumed that the functions, $g(q)$ and $f(g(q))$, were both <u>single-valued</u>. However, we can easily modify this assumption to include the <u>compositions, $g \circ F$</u>, of any <u>multi-valued</u> function, $F \in \mathbb{C} \cup \mathbb{H}$, as is evident from our next two definitions which are enunciated as follows :-

<u>Definition DI-5.</u>

Let there exist a functional array, $F = \{f_j^*, \forall j^* = 1, \ldots\ldots, n]$. Henceforth, we shall define the existence of a <u>composite</u> functional array, $g(\{f_j^*; \forall j^* = 1, \ldots\ldots, n]) \in \mathbb{C} \cup \mathbb{H}$, such that

$$g(\{f_j^*; \forall j^* = 1, \ldots\ldots, n]) = \{g \circ f_j^*; \forall j^* = 1, \ldots\ldots, n] = \begin{cases} g \circ f_1, \\ \vdots \\ g \circ f_j^* \\ \vdots \\ g \circ f_n \end{cases}$$

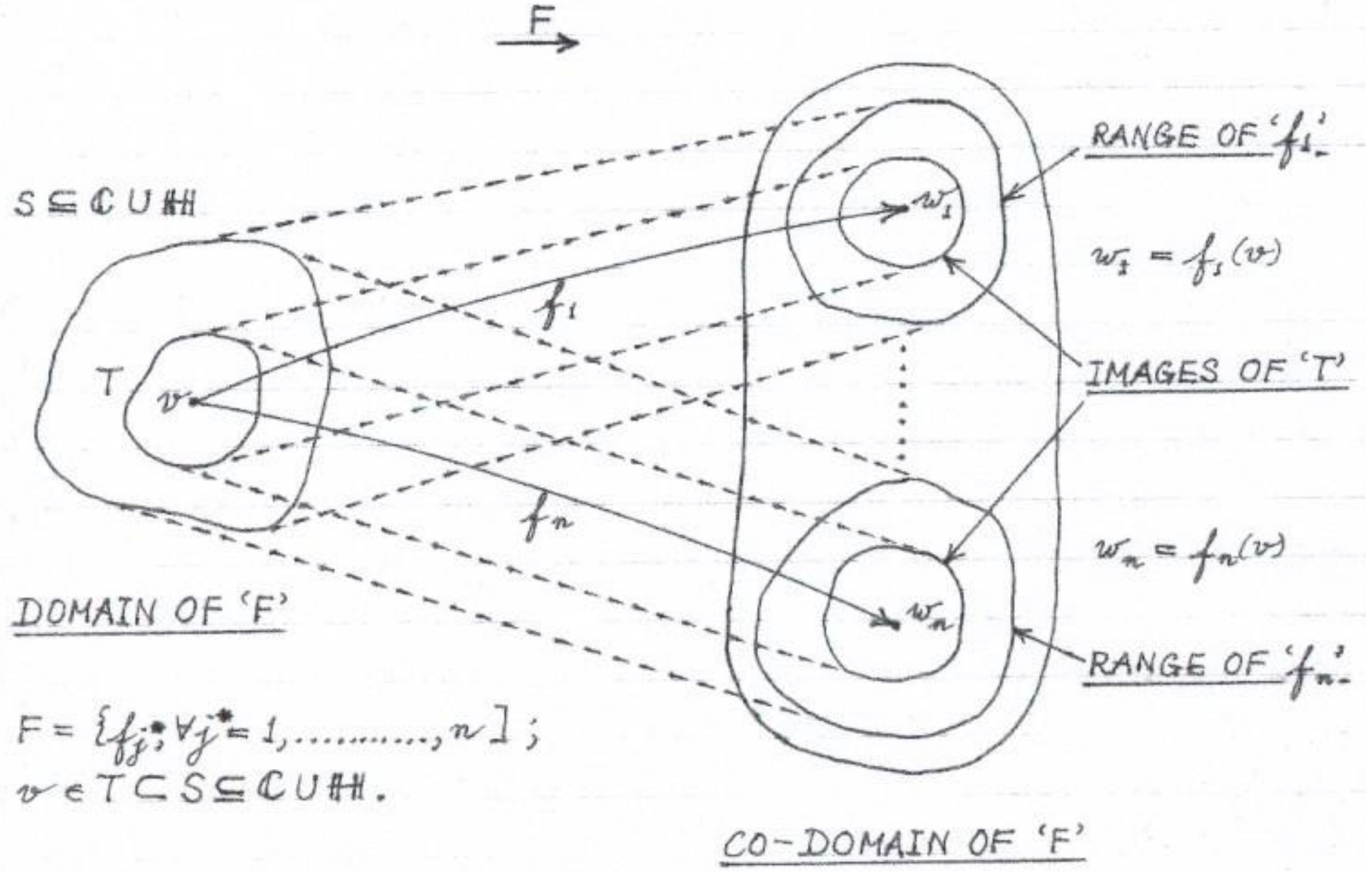


Fig. (1−1)

where the domain and co-domain of 'F' are both contained in $\mathbb{C} \cup \mathbb{H}$ $(\mathbb{R} \subset \mathbb{C})$.

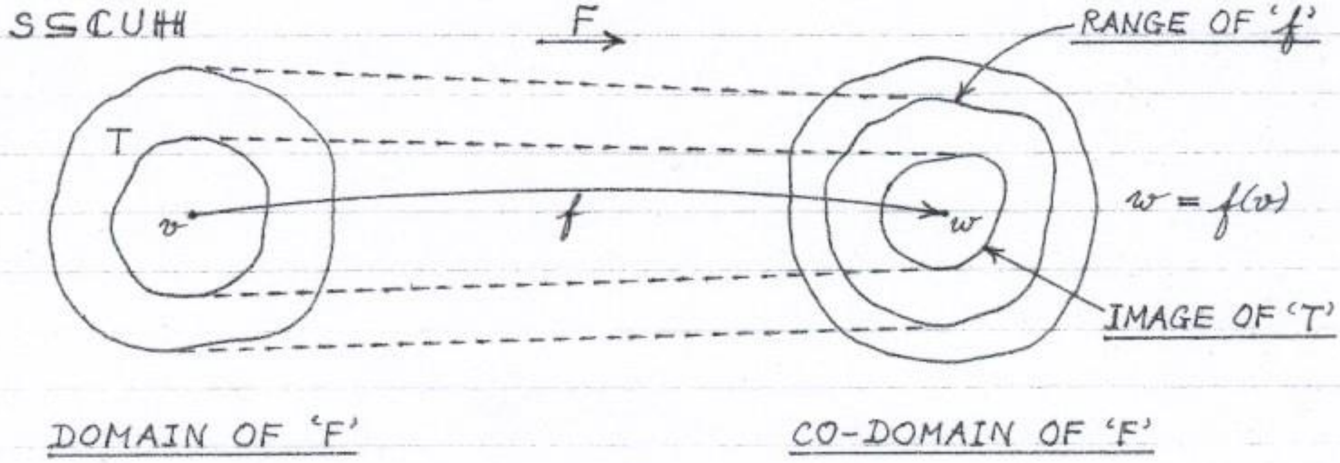
Definition DI−6.

Let there exist a composite functional array, $g(\{f_j^*, \forall j^* = 1, \ldots\ldots, n]) \in \mathbb{C} \cup \mathbb{H}$, such that

$$g(\{f_{j^*};\forall j^*=1,\ldots\ldots,n]) = \{g\circ f_{j^*};\forall j^*=1,\ldots\ldots,n] = \begin{cases} g\circ f_1 \\ \vdots \\ g\circ f_{j^*} \\ \vdots \\ g\circ f_n \end{cases}$$

Now, if each of the functions, $g\circ f_{j^*}$; $\forall j^*=1,\ldots\ldots,n$, is itself a multi-valued function such that we obtain the functional array,

$$g\circ f_{j^*} = \{(g\circ f_{j^*})_{k^*};\forall k^*=1,\ldots\ldots,m],$$



$F = \{f_{j^*};\forall j^*=1,\ldots\ldots,n] = f \Longrightarrow$
$f = f_1 = \ldots\ldots = f_{j^*} = \ldots\ldots = f_n;$
$v \in T \subset S \subseteq \mathbb{C}\cup\mathbb{H}.$

Fig. (1-2)

we therefore designate the composition,

$$g(\{f_{j^*}; \forall j^* = 1, \ldots\ldots, n]) = \begin{cases} g \circ f_1 \\ \vdots \\ g \circ f_{j^*} \\ \vdots \\ g \circ f_n \end{cases} = \begin{cases} \{(g \circ f_1)_{k^*}; \forall k^* = 1, \ldots\ldots, m] \\ \{(g \circ f_{j^*})_{k^*}; \forall k^* = 1, \ldots\ldots, m] \\ \{(g \circ f_n)_{k^*}; \forall k^* = 1, \ldots\ldots, m] \end{cases}$$

$$= \begin{cases} (g \circ f_1)_1 \;, \\ \vdots \\ (g \circ f_1)_m \\ \vdots \\ (g \circ f_n)_1 \\ \vdots \\ (g \circ f_n)_m \end{cases}$$

to be a <u>nested composite</u> functional array thus having an order given by the formula,

$$\# g(\{f_{j^*}; \forall j^* = 1, \ldots\ldots, n]) = mn .$$

Once again, we recognise that Definition DI-6 is simply a generalisation of Definition DI-5, which in turn is a generalisation of Eq. (1-3). Furthermore, the notions expressed in these same definitions can also be summarised via the following mappings, namely -

$$\begin{tikzcd}
 & f_1(v) \arrow[r, "g"] \arrow[dd, phantom, "\vdots"] & (g\circ f_1)(v) \arrow[dd, phantom, "\vdots"] \\
v \arrow[ur, "f_1"] \arrow[urr, "g\circ f_1"] \arrow[dr, "f_n"'] \arrow[drr, "g\circ f_n"'] & & \\
 & f_n(v) \arrow[r, "g"'] & (g\circ f_n)(v)
\end{tikzcd} \qquad (1-5),$$

$v \in \mathbb{C} \cup \mathbb{H}$

$$\begin{tikzcd}
 & f_1(v) \arrow[r, "g"] \arrow[dd, phantom, "\vdots"] & (g\circ f_1)(v) = \{(g\circ f_1)_{k^*}(v),\ \forall k^* = 1,\ldots\ldots,m\} \arrow[dd, phantom, "\vdots"] \\
v \arrow[ur, "f_1"] \arrow[urr, "g\circ f_1"] \arrow[dr, "f_n"'] \arrow[drr, "g\circ f_n"'] & & \\
 & f_n(v) \arrow[r, "g"'] & (g\circ f_n)(v) = \{(g\circ f_n)_{k^*}(v),\ \forall k^* = 1,\ldots\ldots,m\}
\end{tikzcd}$$

$v \in \mathbb{C} \cup \mathbb{H} \qquad (1-6),$

both of which are analogous extensions of the mapping designated as Eq. (1-4).

2. The Polynomial Function.

From complex variable analysis, we recall that any <u>polynomial function of degree n</u>,

$$p_n(z) = a_0 + a_1 z + \ldots\ldots\ldots\ldots + a_n z^n,$$

$$\forall a_0, a_1, \ldots\ldots\ldots\ldots, a_n, z \in \mathbb{C} \qquad (1-7),$$

is both single-valued and defined over the entire complex domain, $\mathbb{C}$. Hence, it seems only natural that should also regard the function,

$p_n(q) = a_0 + a_1 q + \dots\dots\dots\dots\dots + a_n q^n,$

$\forall a_0, a_1, \dots\dots\dots\dots, a_n, q \in \mathbb{H}$ (1-8),

as being a logical extension of Eq. (1-7) into the realm of single variable quaternion functions.

Granted then the existence of Eq. (1-8), let us firstly consider the case, $n = 1$, whence we may write

$p_1(q) = a_0 + a_1 q$, $\forall a_0, a_1, q \in \mathbb{H}$ (1-9) .

-10-

However, since it was already demonstrated in the author's first paper [5] on quaternion functions that, for any two quaternion numbers, a_1 and q, the quaternion product,

$a_1 q \neq q a_1$ (1-10),

it therefore follows that the polynomial function,

$p_1(q) = a_0 + a_1 q \neq a_0 + q a_1$ (1-11).

Similarly, let us consider the case, $n = 2$, such that Eq. (1-8) can now be written as

$p_2(q) = a_0 + a_1 q + a_2 q^2$, $\forall a_0, a_1, a_2, q \in \mathbb{H}$ (1-12).

Once again, we observe that the quaternion products,

$a_1 q \neq q a_1 \quad (1-13a),$

$a_2 q^2 \neq q^2 a_2 \quad (1-13b),$

and hence the polynomial function,

$$\begin{aligned} p_2(q) &= a_0 + a_1 q + a_2 q^2 \qquad (1-14). \\ &\neq a_0 + q a_1 + a_2 q^2 \\ &\neq a_0 + a_1 q + q^2 a_2 \\ &\neq a_0 + q a_1 + q^2 a_2 \end{aligned}$$

Moreover, since the algebraic entities, $a_0 + q a_1$, $a_0 + q a_1 + a_2 q^2$, $a_0 + a_1 q + q^2 a_2$, $a_0 + q a_1 + q^2 a_2$, are all polynomial expressions as indeed are $a_0 + a_1 q$ and $a_0 + a_1 q + a_2 q^2$ from Eqs. (1-9) and (1-12) respectively, we surmise that Eq. (1-8) alone cannot suffice as a definitive formula for the <u>polynomial quaternion hypercomplex function of degree n</u> and thus we are faced with the problem of finding just such a formula. To resolve this

problem, we shall utilise the concepts and notation of the preceding Part 1 of this section and subsequently construct the required formula along the following lines :-

(a) Consider the case, $n = 0$, such that there shall exist a <u>single-valued</u> quaternion function,

$$P_0(q) = a_0 \Longrightarrow \# P_0(q) = 1 \qquad (1-15),$$

where a_0 is any arbitrary quaternion constant.

(b) Consider the case, $n=1$. The non-commutative product, $a_1 q \neq q a_1$, induces us to define a multi-valued quaternion function,

$$P_1(q) = \begin{cases} a_0 + a_1 q \\ a_0 + q a_1 \end{cases} \Longrightarrow \# P_1(q) = 2 \qquad (1-16),$$

insofar as the single-valued polynomial expression, $a_0 + a_1 q \neq a_0 + q a_1$.

(c) Consider the case, $n=2$. The non-commutative quaternion products, $a_1 q \neq q a_1$ and $a_2 q^2 \neq q^2 a_2$, induce us to define a multi-valued quaternion function,

$$P_2(q) = \begin{cases} a_0 + a_1 q + a_2 q^2 \\ a_0 + q a_1 + a_2 q^2 \\ a_0 + a_1 q + q^2 a_2 \\ a_0 + q a_1 + q^2 a_2 \end{cases} \Longrightarrow \# P_2(q) = 4 \qquad (1-17),$$

insofar as the single-valued polynomial expression, $a_0 + a_1 q + a_2 q^2 \neq a_0 + q a_1 + a_2 q^2 \neq a_0 + a_1 q + q^2 a_2 \neq a_0 + q a_1 + q^2 a_2$.

(d) In view of the algebraic structures exhibited by Eqs. (1-15), (1-16) and (1-17), let us now consider the general case of $n \in \{0, 1, 2, \ldots\ldots, \infty\} = \mathbb{N} \cup \{0\}$. Consequently, the non-commutative quaternion products,

$a_1 q \neq q a_1$, $a_2 q^2 \neq q^2 a_2$, ..., $a_n q^n \neq q^n a_n$, induce us to define a multi-valued quaternion function,

$$P_n(q) = \begin{cases} a_0 + a_1 q + \ldots\ldots\ldots\ldots + a_n q^n \Longrightarrow \# P_n(q) = N_n \quad (1\text{-}18), \\ \vdots \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_0 + q a_1 + \ldots\ldots\ldots\ldots + q^n a_n \end{cases}$$

insofar as

(i) the value of the positive integer, N_n, has yet to be determined;

(ii) the single-valued polynomial expression, $a_0 + a_1 q + \ldots\ldots\ldots + a_n q^n \neq \ldots$
$\ldots\ldots\ldots \neq a_0 + q a_1 + \ldots\ldots\ldots + q^n a_n$.

The ideas conveyed in the preceding arguments may be conveniently summarized by means of our next definition, namely —

Definition DI-7.

Let there exist an 'nth' degree homogeneous polynomial quaternion hypercomplex function, $P_n(q)$, having a domain,

$dom\,(P_n) \subseteq \mathbb{H} \quad (n = 0, 1, 2, \ldots\ldots\ldots, \infty)$.

Subsequently, we define $P_n(q)$ to be a multi-valued function such that

$$P_n(q) = \{ P_{nj}(q), \forall j = 1, \ldots\ldots\ldots\ldots, N_n \}$$

$$= \begin{cases} a_0 + a_1 q + \ldots\ldots\ldots\ldots + a_n q^n \\ \vdots \qquad \vdots \qquad\qquad\qquad \vdots \\ a_0 + q a_1 + \ldots\ldots\ldots\ldots + q^n a_n \end{cases} = \begin{cases} P_{n1}(q) \\ \vdots \\ P_{nN_n}(q) \end{cases},$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(P_n) \subseteq \mathbb{H}$,

where the coefficients, $a_0, a_1, \ldots\ldots\ldots, a_n$, are deemed to be arbitrary <u>quaternion constants</u>

<u>AND</u>

$\# P_n(q) = \# \{P_{nj}(q), \forall j = 1, \ldots\ldots\ldots, N_n\} = N_n$,

being a positive integer whose value has yet to be determined.

We will finalise our discussion of the <u>polynomial</u> quaternion hyper-complex function with the following remarks :-

<u>(a)</u> By making each of the constant coefficients, $a_0, a_1, \ldots\ldots\ldots\ldots, a_n \in \mathbb{R}$, Eq. (1-18) automatically reduces to a <u>single-valued</u> polynomial function,

$$P_n(q) = a_0 + a_1 q + \ldots\ldots\ldots\ldots + a_n q^n \qquad (1-19),$$

since the quaternion products,

$$a_r q^r = q^r a_r \,, \; \forall r \in \{0, 1, \ldots\ldots\ldots\ldots, n\} \qquad (1-20).$$

<u>(b)</u> Similarly, it follows that the restrictions,

<u>(i)</u> $q = x + iy$ $\qquad (1-21a)$,
$a_r = \alpha_r + i\beta_r$

(ii) $q = x + j\hat{x}$ (1–21b),
$a_r = \alpha_r + j\beta_r$

(iii) $q = x + k\hat{y}$ (1–21c),
$a_r = \alpha_r + k\beta_r$

$\forall r \in \{0,1,\ldots\ldots,n\}$ & $\forall \alpha_r, \beta_r \in \mathbb{R}$, respectively yield single-valued polynomial functions of the form,

$$P_n(q) = a_0 + a_1 q + \ldots\ldots\ldots + a_n q^n \quad (1\text{–}22),$$

bearing in mind the commutativity of the quaternion products, $a_r q^r$, $\forall r \in \{0,1,\ldots\ldots,n\}$ in each case.

(c) Finally, the reader will undoubtedly recall from Definition DI–7 that the adjective homogeneous was used to describe the 'nth' degree polynomial quaternion function previously designated as Eq. (1–18). We intend to clarify this choice of terminology by means of the following simple illustration. Consider the case of the first degree polynomial function,

$$P_1(q) = \begin{cases} a_0 + a_1 q \\ a_0 + q a_1 \end{cases} \Longleftrightarrow P_1(q) \in \{a_0 + a_1 q, a_0 + q a_1\} \quad (1\text{–}23),$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(P_1) \subseteq \mathbb{H}$.

Moreover, let us define a dual-valued quaternion function,

$$\phi(q) = \begin{cases} a_1 q \\ q a_1 \end{cases} \Longleftrightarrow \phi(q) \in \{a_1 q, q a_1\} \qquad (1-24),$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(P_1) \subseteq \mathbb{H}$.

Clearly, this function yields the formulae,

(i) $\phi(q) = a_1 q \Longrightarrow a_0 + \phi(q) = a_0 + a_1 q \qquad (1-25a)$,

(ii) $\phi(q) = q a_1 \Longrightarrow a_0 + \phi(q) = a_0 + q a_1 \qquad (1-25b)$,

such that we obtain the set-theoretic statement,

$$a_0 + \phi(q) \in \{a_0 + a_1 q, a_0 + q a_1\} \Longleftrightarrow$$

$$a_0 + \phi(q) = \begin{cases} a_0 + a_1 q = P_1(q) \\ a_0 + q a_1 \end{cases} \qquad (1-26),$$

by virtue of Definition DI-1. Hence, it is a matter of routine algebra to show that the square of the function, $P_1(q)$, namely –

$$\begin{aligned} (P_1(q))^2 &= (a_0 + \phi(q))^2 \\ &= (a_0 + \phi(q))(a_0 + \phi(q)) \\ &= a_0(a_0 + \phi(q)) + \phi(q)(a_0 + \phi(q)) \\ &= a_0^2 + a_0\phi(q) + \phi(q)a_0 + (\phi(q))^2 \qquad (1-27), \end{aligned}$$

and thus

$$\phi(q) = a_1 q \Longrightarrow (P_1(q))^2 = a_0^2 + a_0 a_1 q + a_1 q a_0 + a_1 q a_1 q \qquad (1-28a),$$

$$\phi(q) = q a_1 \Longrightarrow (P_1(q))^2 = a_0^2 + a_0 q a_1 + q a_1 a_0 + q a_1 q a_1 \qquad (1-28b).$$

Once again, we obtain the set-theoretic statement,

$$(P_1(q))^2 \in \left\{ \begin{matrix} a_0^2 + a_0 a_1 q + a_1 q a_0 + a_1 q a_1 q, \\ a_0^2 + a_0 q a_1 + q a_1 a_0 + q a_1 q a_1 \end{matrix} \right\} \Longleftrightarrow$$

$$(P_1(q))^2 = \begin{cases} a_0^2 + a_0 a_1 q + a_1 q a_0 + a_1 q a_1 q \\ a_0^2 + a_0 q a_1 + q a_1 a_0 + q a_1 q a_1 \end{cases} \qquad (1-29),$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(P_1) \subseteq \mathbb{H}$, by virtue of Definition DI-1. A visual inspection of Eq. (1-29) would have shown that its algebraic structure does not conform to that of Eq. (1-18) and so $(P_1(q))^2$ cannot be a homogeneous polynomial function in terms of Definition DI-7.

For reasons of brevity, however, we shall not pursue this matter any further for the time being, but rather we will discuss some of its ramifications in Section IV of this paper.

3. The Exponential Function.

From complex variable analysis, we recall that the exponential function $exp(z)$, is defined by the formula,

$$exp(z) = exp(x + iy) = e^x e^{iy}$$

$$= e^x \cos y + i e^x \sin y, \ \forall z = x + iy \in \mathbb{C} \qquad (1-30),$$

which is single-valued throughout the entire complex domain, $\mathbb{C}$. Furthermore, according to Churchill et al. [1], this particular transcendental function also plays a crucial rôle in the formulation of other elementary complex valued functions such as the trigonometric and hyperbolic functions and so on and hence its overall importance in this regard cannot be emphasised strongly enough.

However, since the existence of the complex exponential function is wholly dependant upon Euler's formula, namely –

$$e^{i\theta} = \cos\theta + i\sin\theta\ , \ \forall\,\theta \in \mathbb{R} \qquad (1-31)\ ,$$

we immediately recognise that the algebraic structure of an exponential quaternion hypercomplex function will likewise depend on our derivation of a suitable quaternion analogue of Eq. (1-31). To derive such an analogue, we must firstly construct a set of quaternion numbers, Q, which when squared always yield a product of −1, and to this end we enunciate the following theorems :–

Theorem TI–1.

Let there exist a quaternion number, Q, such that

$$Q^2 = -1\ .$$

Hence, it may be shown that this particular quaternion may be

expressed as

$$Q = iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2},$$

$\forall \hat{X} \in [-\sqrt{1-Y^2}, \sqrt{1-Y^2}]$ & $\forall Y \in [-1, 1]$ such that

$1 - Y^2 - \hat{X}^2 \geq 0$.

* * *

PROOF :-

Let the quaternion,

$$Q = X + iY + j\hat{X} + k\hat{Y},$$

be so defined such that we obtain

$$Q^2 = (X + iY + j\hat{X} + k\hat{Y})^2 = -1.$$

Expansion of the above equation in accordance with the established rules of quaternion multiplication henceforth yields the product,

$$X^2 - Y^2 - \hat{X}^2 - \hat{Y}^2 + i\,2XY + j\,2X\hat{X} + k\,2X\hat{Y} = -1,$$

whence we deduce that the corresponding real and imaginary parts thereof are given by

$$X^2 - Y^2 - \hat{X}^2 - \hat{Y}^2 = -1,$$

$$2XY = 2X\hat{X} = 2X\hat{Y} = 0.$$

Since we require that $X, Y, \hat{X}, \hat{Y} \in \mathbb{R}$, then the only possible solutions to the above stated set of equations are respectively

$X = 0$ and $-Y^2 - \hat{X}^2 - \hat{Y}^2 = -1 \Longrightarrow Y^2 + \hat{X}^2 + \hat{Y}^2 = 1$

$\therefore \hat{Y} = \pm\sqrt{1 - Y^2 - \hat{X}^2}$.

We further observe that the stipulation $\hat{Y} \in \mathbb{R}$ naturally implies that

$1 - Y^2 - \hat{X}^2 \geqq 0$

$\therefore -Y^2 - \hat{X}^2 \geqq -1$

$\therefore Y^2 + \hat{X}^2 \leqslant 1$

$\therefore \hat{X}^2 \leqslant 1 - Y^2$

$\therefore -\sqrt{1-Y^2} \leqslant \hat{X} \leqslant \sqrt{1-Y^2} \Longrightarrow \hat{X} \in [-\sqrt{1-Y^2}, \sqrt{1-Y^2}]$,

provided that

$1 - Y^2 \geqq 0$

$\therefore -Y^2 \geqq -1$

$\therefore Y^2 \leqslant 1$

$\therefore -1 \leqslant Y \leqslant 1 \Longrightarrow Y \in [-1, 1]$.

Finally, after making the appropriate algebraic substitutions, we perceive that

$$
\begin{aligned}
Q &= X + iY + j\hat{X} + k\hat{Y} \\
&= 0 + iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2} \\
&= iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2},
\end{aligned}
$$

$\forall Y \in [-1, 1]$ and $\forall \hat{X} \in [-\sqrt{1-Y^2}, \sqrt{1-Y^2}]$ such that $1 - Y^2 - \hat{X}^2 \geq 0$, as required. Q.E.D.

Theorem TI-2.

Let there exist a quaternion,

$$Q = i\left(\frac{y}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right) + j\left(\frac{\hat{x}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right) + k\left(\frac{\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right).$$

Hence, it may be shown that

$$Q^2 = \left[i\left(\frac{y}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right) + j\left(\frac{\hat{x}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right) + k\left(\frac{\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right)\right]^2 = -1,$$

$\forall y, \hat{x}, \hat{y} \in \mathbb{R}$.

* * *

PROOF :-

We firstly observe that

$$\left(\frac{y}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)^2 + \left(\frac{\hat{x}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)^2 + \left(\frac{\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)^2 = 1 ,$$

$\forall y, \hat{x}, \hat{y} \in \mathbb{R}$, and hence it logically follows that

$$\frac{\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} = \pm\sqrt{1-\left(\frac{y}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)^2-\left(\frac{\hat{x}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)^2} .$$

In the circumstances, it is now evident that this particular quaternion has precisely the same algebraic form as the quaternion,

$$Q = iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2} ,$$

$\forall Y \in [-1,1]$ & $\forall \hat{X} \in [-\sqrt{1-Y^2}, \sqrt{1-Y^2}]$ such that $1-Y^2-\hat{X}^2 \geq 0$,

-20-

in as much as we may write

$$Y = \frac{y}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} \quad \text{and} \quad \hat{X} = \frac{\hat{x}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} ,$$

whereupon we conclude in accordance with Theorem TI-1 that

$$Q^2 = \left[i\left(\frac{y}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) + j\left(\frac{\hat{x}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) + k\left(\frac{\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\right]^2$$

$$= -1 ,$$

$\forall y, \hat{x}, \hat{y} \in \mathbb{R}$, as required. <u>Q.E.D.</u>

Since it may be shown that any complex number,

$$z = x + iy = re^{i\theta} \qquad (1-32),$$

obeys the following indicial laws, namely –

$$z^{m+n} = z^m z^n \qquad (1-33a),$$

$$z^{m-n} = z^m / z^n \qquad (1-33b),$$

$$(z^m)^n = z^{mn} \qquad (1-33c),$$

$\forall m, n \in \mathbb{Z}$, the set of integers, insofar as the <u>modulus</u> of z,

$$|z| = r \qquad (1-34),$$

and also the <u>exponential function, i.e. Euler's formula,</u>

$$e^{i\theta} = \cos\theta + i\sin\theta, \ \forall \theta \in \mathbb{R} \qquad (1-35),$$

we therefore postulate that any quaternion,

-21-

$$\begin{aligned} q &= x + iy + j\hat{x} + k\hat{y} \\ &= x + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\sqrt{y^2 + \hat{x}^2 + \hat{y}^2} \qquad (1-36), \end{aligned}$$

will likewise possess a set of properties which are akin to Eqs. (1–33a) to (1–33c). Such an assumption was made solely on the basis that the previously established property via Theorem TI–2,

$$\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]^2 = -1 \qquad (1-37),$$

is entirely analogous to the definitive formula,

$$i^2 = -1 \qquad (1-38),$$

thus pertaining to the imaginary number, $i = (0,1) \in \mathbb{C}$.

A concise summary of these important points is provided by way of our next definition :—

Definition DI-8.

Let there exist a quaternion number

$$q = x + iy + j\hat{x} + k\hat{y} = x + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\sqrt{y^2 + \hat{x}^2 + \hat{y}^2},$$

whence we note that the property,

$$\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]^2 = -1,$$

is entirely analogous to the definitive formula,

$$i^2 = -1 \quad (i = (0,1) \in \mathbb{C}),$$

in the light of the preceding Theorem TI−2. Subsequently, we postulate that the following indicial laws, namely –

(a) $q^{m+n} = q^m q^n$,

(b) $q^{m-n} = q^m / q^n$,

(c) $(q^m)^n = q^{mn}$,

are valid, $\forall m, n \in \mathbb{Z}$, the set of all integers.

Indeed, Churchill et al. [1] have clearly demonstrated that the validity of Eqs. (1−33a) to (1−33c) not only depends upon the existence of Euler's formula but also on the result,

$$(e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta),\ \forall n \in \mathbb{Z} \qquad (1-39),$$

commonly referred to as De Moivre's theorem. This being the case, can we now construct suitable quaternion analogues of Eqs. (1−35) and (1−39)? The answer is, quite simply, yes, and our justification for this assertion will accordingly be revealed by way of the following theorem :–

Theorem TI−3.

Let there exist a quaternion number, $\exp[(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta]$, such that

$$\exp[(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta] = e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}$$

$$= \cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta,$$

$\forall \theta \in \mathbb{R}$, $\forall Y \in [-1,1]$ & $\forall \hat{X} \in [-\sqrt{1-Y^2}, \sqrt{1-Y^2}]$, which we shall accordingly designate to be the <u>quaternion analogue of Euler's Formula</u> thus arising from complex variable analysis.

Henceforth, it may be proven that the formula,

$$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})n\theta} = \left[e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^n =$$

$$\cos(n\theta) + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta),$$

is also valid, $\forall n \in \{0, \pm1, \pm2, \ldots\ldots\ldots\ldots, \pm\infty\}$. This particular result shall likewise be referred to as the <u>quaternion analogue of De Moivre's Theorem</u>.

*　　　　*　　　　*

<u>PROOF</u>:-

Let there exist a function,

$$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta} = \cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta,$$

$\forall \theta \in \mathbb{R}$. Now, for any $\theta_1, \theta_2 \in \mathbb{R}$, it therefore follows that

$$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta_1} = \cos\theta_1 + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_1;$$

$$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta_2} = \cos\theta_2 + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_2$$

and hence the product function,

$$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta_1} e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta_2}$$

$$= \left\{ \begin{array}{l} \left[\cos\theta_1 + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_1\right] \times \\ \left[\cos\theta_2 + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_2\right] \end{array} \right\}$$

$$\begin{aligned} &= \cos\theta_1\left[\cos\theta_2 + (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_2\right] + \\ &\quad (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_1\left[\cos\theta_2 + (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_2\right] \end{aligned}$$

$$\begin{aligned} &= \cos\theta_1\cos\theta_2 + (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\cos\theta_1\sin\theta_2 + \\ &\quad (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_1\cos\theta_2 + \\ &\quad (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})^2\sin\theta_1\sin\theta_2 \end{aligned}$$

$$\begin{aligned} &= \cos\theta_1\cos\theta_2 + (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\cos\theta_1\sin\theta_2 + \\ &\quad (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{aligned}$$

$$\begin{aligned} &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \\ &\quad (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \end{aligned}$$

$$= \cos(\theta_1+\theta_2) + (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(\theta_1+\theta_2)$$

$$= e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})(\theta_1+\theta_2)}.$$

By utilising this result, we can now show that the formula,

$$e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})n\theta} = \left[e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^n,$$

$\forall n \in \{0, \pm1, \pm2, \ldots\ldots\ldots\ldots, \pm\infty\}$,

is also valid with the aid of the principle of mathematical induction.

(i) By definition, we obtain

$$1 = e^{0} = e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\cdot 0}$$

$$= e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})(\theta - \theta)}$$

$$= e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\, e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})(-\theta)}$$

and hence multiplication of both sides by $\left[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^{-1}$

yields

$$\left[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^{-1} = e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})(-\theta)}.$$

(ii) The result,

$$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta} = \left[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^{1},$$

is clearly self-explanatory, since any quaternion raised to the power of 1 is its own exponential.

(iii) We likewise deduce that

$$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})2\theta} = e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})(\theta + \theta)}$$

$$= e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\, e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}$$

$$= \left[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^{2}.$$

(iv) Finally, if the formula,

$$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})n\theta} = [e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^n,$$

is valid for all integer values of n, we then perceive that

$$e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})(n+1)\theta} = e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})(n\theta+\theta)}$$

$$= e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})n\theta} e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}$$

$$= [e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^n [e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^1$$

$$= [e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^{(n+1)},$$

bearing in mind the provisions of Definition DI−8, and since we have

shown that this formula holds for the special cases of $n=-1$, $n=1$, $n=2$, we therefore conclude that it is true, $\forall n \in \{0, \pm 1, \pm 2, \ldots\ldots \ldots\ldots, \pm\infty\}$, as required.

It remains for us to prove that the formula,

$$[e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^n = \cos(n\theta) + (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta),$$

is likewise valid, $\forall n \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots, \pm\infty\}$.

(i) Firstly, consider the case, $n=-1$. Hence, we obtain

$$[e^{(iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^{-1} = [\cos\theta + (iY+j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta]^{-1}$$

$$= \frac{\overline{\cos\theta + (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta}}{|\cos\theta + (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta|}$$

$$= \frac{\cos\theta - (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \cos\theta - (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta$$

$$= \cos(-\theta) + (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin(-\theta),$$

since from elementary trigonometry we recall the formulae,

$\cos^2\theta + \sin^2\theta = 1$,
$\cos\theta = \cos(-\theta)$,
$-\sin\theta = \sin(-\theta)$, as required.

(ii) Consider the case, $n = 1$. Hence, we obtain

$$\left[e^{(iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\theta}\right]^1 = \left[\cos\theta + (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta\right]^1$$

$$= \cos\theta + (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta, \text{ as required.}$$

(iii) Consider the case, $n = 2$. Hence, we obtain

$$\left[e^{(iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\theta}\right]^2 = \left[\cos\theta + (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta\right]^2$$

$$= (\cos\theta + (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta)(\cos\theta + (iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2})\sin\theta)$$

$$= \cos\theta\,(\cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta) +$$
$$(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta\,(\cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta)$$

$$= \cos^2\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta\cos\theta +$$
$$(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta\cos\theta +$$
$$(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})^2\sin^2\theta$$

$$= \cos^2\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\,2\sin\theta\cos\theta - \sin^2\theta$$

$$= \cos^2\theta - \sin^2\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\,2\sin\theta\cos\theta$$

$$= \cos(2\theta) + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(2\theta),$$

since from elementary trigonometry we recall the formulae,

$$\cos^2\theta - \sin^2\theta = \cos(2\theta)$$
$2\sin\theta\cos\theta = \sin(2\theta)$, as required.

(iv) Finally, if the formula,

$$[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^n = \cos(n\theta) + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta),$$

is valid for all integer values of n, then we perceive that

$$[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^{n+1} =$$

$$[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^n\,[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}]^1$$

$$= \left\{ \begin{array}{l} \left[\cos(n\theta) + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta)\right] \times \\ \left[\cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta\right] \end{array} \right\}$$

$$= \cos(n\theta)\left[\cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta\right] +$$
$$(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta)\left[\cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin\theta\right]$$

$$= \cos(n\theta)\cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\cos(n\theta)\sin\theta +$$
$$(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta)\cos\theta +$$
$$(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})^2\sin(n\theta)\sin\theta$$

$$= \cos(n\theta)\cos\theta + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\left[\sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta\right]$$
$$- \sin(n\theta)\sin\theta$$

$$= \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta +$$
$$(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\left[\sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta\right]$$

$$= \cos(n\theta + \theta) + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta + \theta)$$

$$= \cos[(n+1)\theta] + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin[(n+1)\theta]$$

and since we have also shown that this formula holds for the special cases of $n = -1, n = 1, n = 2$, we therefore conclude that it is true, $\forall n \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, \pm\infty\}$, as required.

In summary, we have consecutively proven that

(a) $$e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})n\theta} = \left[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^n$$

AND

(b) $$\left[e^{(iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^n = \cos(n\theta) + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta),$$

whence it naturally follows that

To be continued via the author's next submission, namely -

> ***"A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 2/10."***

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