“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 4/10.”

by Stephen C. Pearson MRACI
[T/A S. C. Pearson Technical Services],
Affiliate of the Royal Society of Chemistry &
Member of the London Mathematical Society,
Dorset. ENGLAND.

Email Address:- *scpearson1952@outlook.com* .

I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 4/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

*   *   *   *   *   *   *   *

****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****
****

such that the real variable function,

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right] \in [0,\pi],$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\}$, as required. <u>Q.E.D.</u>

Similarly, in view of Eq. (1-87), we shall likewise define the quaternion analogue of the multi-valued complex function, $\log(z)$, as follows :-

<u>Definition DI-15.</u>

Let there exist a <u>multi-valued logarithmic</u> quaternion hypercomplex function, $\log(q)$, whose domain,

$$dom(\log) \subseteq \mathbb{H} - \{0\}.$$

Subsequently, this function is denoted by the formula,

$$\log(q) = \mathrm{Log}(q) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] 2n\pi \quad (n \in \mathbb{Z}).$$

Churchill et al. [1] have established the following identities with respect to the multi-valued logarithmic function, $\log(z)$, namely -

$$\exp(\log(z)) = e^{\log(z)} = z \qquad (1-91);$$

****

$$\log(\exp(z)) = \log(e^{z}) = z + i2n\pi \quad (n \in \mathbb{Z}) \quad (1-92),$$

and thus we are motivated to derive quaternion analogues of these particular results by means of the next two theorems :-

****

****

****

Theorem TI-21.

Let there exist a multi-valued logarithmic quaternion hypercomplex function, $\log(q)$, whose domain,

$$dom(\log) \subseteq \mathbb{H} - \{0\}.$$

Subsequently, it may be proven that this function is characterised by the following property, i.e -

$$\exp(\log(q)) = q, \ \forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\}.$$

* * *

PROOF:-

From Definition DI-15, we recall that the multi-valued logarithmic function,

$$\log(q) = \mathrm{Log}(q) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi$$

$(n \in \mathbb{Z} \ \& \ q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\})$.

****

Finally, by virtue of Theorems TI-7 & TI-8 and Definition DI-14, we deduce that the function,

$$\exp(\log(q)) = \exp\left(\mathrm{Log}(q) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi\right)$$

$$= \exp(\mathrm{Log}(q)) \exp\left(\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi\right)$$

$= q \cdot 1 = q$, as required. Q.E.D.

Theorem TI-22.

Let there exist the multi-valued logarithmic quaternion hypercomplex function, $\log(q)$, having both an algebraic structure and associated properties as outlined in the preceding Definitions DI-14 & DI-15 and Theorems TI-20 & TI-21.

Henceforth, it may be shown that this function likewise possesses the additional property –

$$\log(\exp(q)) = q + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi, \ \forall n \in \mathbb{Z}, \text{ the set of integers.}$$

*　　　*　　　*

PROOF:-

From Theorem TI-21, we instantly recall that the exponential and logarithmic functions, 'exp' and 'log', are correlated by means of the formula –

$$\exp(\log(q)) = q$$

$$\therefore \exp(\log(x+iy+j\hat{x}+k\hat{y})) = x+iy+j\hat{x}+k\hat{y}$$

$$\Longrightarrow \exp(\log(X+iY+j\hat{X}+k\hat{Y})) = X+iY+j\hat{X}+k\hat{Y},$$

$\forall X, Y, \hat{X}, \hat{Y} \in \mathbb{R}$.

Subsequently, if we define $X, Y, \hat{X}$ & $\hat{Y}$ to be real variable functions of $x, y, \hat{x}$ & $\hat{y}$ such that

$$\exp(q) = X+iY+j\hat{X}+k\hat{Y},$$

it therefore follows that

$$\exp(\log(\exp(q))) = \exp(q).$$

Let us now define a function,

$$w = \log(\exp(q))$$

$$\Longrightarrow \exp(w) = \exp(q) \qquad \underline{(i)}.$$

However, from Theorem TI-8, we also perceive that the exponential function,

$$\exp(q) = \exp\left(q + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]2n\pi\right) \quad (n \in \mathbb{Z}) \qquad \underline{(ii)}.$$

Clearly, Eqs. (i) & (ii) become identical statements by setting

$$w = q + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi$$

$$\Longrightarrow \log(\exp(q)) = q + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi ,$$

$\forall n \in \mathbb{Z}$, the set of integers, as required. Q.E.D.

From complex variable analysis, we recall that, for any two non-zero complex numbers, z_1 and z_2, there exist the following logarithmic identities:-

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) \qquad (1-93);$$

$$\log(z_1/z_2) = \log(z_1) - \log(z_2) \qquad (1-94).$$

Hence, the purpose of our next theorem is to show that, subject to certain restrictions, quaternion analogues of Eqs. (1-93) & (1-94) likewise exist.

Theorem TI-23.

Let there exist two quaternion variables,

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 ,$$
$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2 , \quad \forall q_1, q_2 \in \mathbb{H} - \{0\}.$$

Subsequently, it may be shown that the logarithms of both the product, $q_1 q_2$, and also the quotient, q_1/q_2, are respectively written as

(a) $\log(q_1 q_2) = \log(q_1) + \log(q_2)$,

(b) $\log(q_1/q_2) = \log(q_1) - \log(q_2)$,

whenever $y_2 = \lambda y_1$; $\hat{x}_2 = \lambda \hat{x}_1$; $\hat{y}_2 = \lambda \hat{y}_1$, $\forall \lambda \in \mathbb{R}$.

* * *

PROOF:-

From Definition DI-15 and Theorem TI-20, we recall that the multi-valued logarithmic function,

$$\log(q) = \mathrm{Log}(q) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi$$

$$= \log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\Theta + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi$$

$$= \log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Theta + 2n\pi) \qquad \text{(i)},$$

such that the real variable function,

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}}\right] \in [0, \pi], \ \forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\},$$

and the integer, $n \in \mathbb{Z}$.

Let there exist two quaternions,

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 \text{ and } q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2,$$

and, furthermore, by substituting the real and imaginary parts of q_1 and q_2 into Eq. (i) above, we respectively obtain

$$\log(q_1) = \log(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)$$
$$= \log\left(\sqrt{x_1^2 + y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) + \left[\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right](\Theta_1 + 2n_1\pi) \qquad \text{(ii)},$$

such that the real valued constant,

$$\Theta_1 = \cos^{-1}\left[\frac{x_1}{\sqrt{x_1^2 + y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right] \in [0, \pi],$$

and the integer, $n = n_1 \in \mathbb{Z}$

AND

$$\log(q_2) = \log(x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2)$$
$$= \log\left(\sqrt{x_2^2 + y_2^2 + \hat{x}_2^2 + \hat{y}_2^2}\right) + \left[\frac{iy_2 + j\hat{x}_2 + k\hat{y}_2}{\sqrt{y_2^2 + \hat{x}_2^2 + \hat{y}_2^2}}\right](\Theta_2 + 2n_2\pi) \qquad \text{(iii)},$$

such that the real valued constant,

$$\Theta_2 = \cos^{-1}\left[\frac{x_2}{\sqrt{x_2^2 + y_2^2 + \hat{x}_2^2 + \hat{y}_2^2}}\right] \in [0, \pi],$$

and the integer, $n = n_2 \in \mathbb{Z}$.

Since, as stated in the preamble to this proof, we initially set

$y_2 = \lambda y_1$; $\hat{x}_2 = \lambda \hat{x}_1$ and $\hat{y}_2 = \lambda \hat{y}_1$, $\forall \lambda \in \mathbb{R}$,

it therefore follows that Eq. (iii) can now be written as

$$\log(q_2) = \log(x_2 + i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1)$$

$$= \log\left(\sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2\hat{x}_1^2 + \lambda^2\hat{y}_1^2}\right) + \left[\frac{i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1}{\sqrt{\lambda^2 y_1^2 + \lambda^2\hat{x}_1^2 + \lambda^2\hat{y}_1^2}}\right](\Theta_2 + 2n_2\pi)$$

$$= \log\left(\sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2\hat{x}_1^2 + \lambda^2\hat{y}_1^2}\right) + \left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right](\Theta_2 + 2n_2\pi) \qquad \text{(iv)},$$

such that the real valued constant,

$$\Theta_2 = \cos^{-1}\left[\frac{x_2}{\sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2\hat{x}_1^2 + \lambda^2\hat{y}_1^2}}\right] \in [0, \pi],$$

and the integer, $n_2 \in \mathbb{Z}$.

(a) Let us define a quaternion constant, w, whose logarithm,

$$\log(w) = \log(q_1) + \log(q_2) \qquad (A-1),$$

$\therefore \; \exp(\log(w)) = \exp(\log(q_1) + \log(q_2))$.

From Theorem TI–21, however, we recall that

$$\exp(\log(q)) = q, \; \forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\},$$

and thus, by the 'dummy substitution of variables' process,

$$\exp(\log(w)) = w \implies$$

$$w = \exp(\log(q_1) + \log(q_2)) \qquad (A-2).$$

Moreover, let us construct two quaternion hypercomplex functions which are respectively denoted as –

$$\phi_1(q) = \log(q) \qquad (A-3);$$

$$\phi_2(q) = \log\left(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 \hat{x}^2 + \lambda^2 \hat{y}^2 + x_2^2 - x_1^2}\right) + \left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Psi + 2n_2\pi) \qquad (A-4),$$

such that the real variable function,

$$\Psi = \cos^{-1}\left[\frac{x + x_2 - x_1}{\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 \hat{x}^2 + \lambda^2 \hat{y}^2 + x_2^2 - x_1^2}}\right] \in [0, \pi],$$

and the integer, $n_2 \in \mathbb{Z}$.

****

In accordance with Theorem TI-7, it subsequently follows that

$$\exp(\phi_1(q) + \phi_2(q)) =$$

$$\exp\left[\begin{array}{l} \log(q) + \log(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 \hat{x}^2 + \lambda^2 \hat{y}^2 + z_2^2 - z_1^2}) \ + \\ \left(\dfrac{\lambda}{|\lambda|}\right)\left[\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Psi + 2n_2\pi) \end{array}\right]$$

****
****
****

$$= \exp\left[\begin{array}{l} \log(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}) \ + \ \left[\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Theta + 2n\pi) \ + \\ \log(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 \hat{x}^2 + \lambda^2 \hat{y}^2 + z_2^2 - x_1^2}) \ + \\ \left(\dfrac{\lambda}{|\lambda|}\right)\left[\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Psi + 2n_2\pi) \end{array}\right]$$

$$= \exp\left(\log(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}) \ + \ \left[\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Theta + 2n\pi)\right) \times$$

$$\exp\left[\begin{array}{l} \log(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 \hat{x}^2 + \lambda^2 \hat{y}^2 + z_2^2 - x_1^2}) \ + \\ \left(\dfrac{\lambda}{|\lambda|}\right)\left[\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Psi + 2n_2\pi) \end{array}\right] \qquad (A-5).$$

Now, by setting

$x = x_1$; $y = y_1$; $\hat{x} = \hat{x}_1$; $\hat{y} = \hat{y}_1$ and $n = n_1$

****

****

and hence substituting these values into Eqs. (A−3), (A−4) & (A−5), we further obtain

$$\begin{aligned}\phi_1(q_1) &= \phi_1(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)\\ &= \log(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)\\ &= \log(q_1) \qquad ;\end{aligned}$$

$$\phi_2(q_1) = \phi_2(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)$$

$$= \log\left(\sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2 \hat{x}_1^2 + \lambda^2 \hat{y}_1^2}\right) \;+$$

$$\left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right](\Theta_2 + 2n_2\pi)$$

****

$$= \log(q_2) \quad ;$$

$$\exp(\phi_1(q_1) + \phi_2(q_1)) = \exp(\phi_1(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1) + \phi_2(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1))$$

$$= \exp\left(\log\left(\sqrt{x_1^2 + y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) + \left[\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right](\Theta_1 + 2n_1\pi)\right) \;\times$$

$$\exp\left[\log\left(\sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2 \hat{x}_1^2 + \lambda^2 \hat{y}_1^2}\right) + \left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right](\Theta_2 + 2n_2\pi)\right]$$

$$= \exp(\log(q_1))\exp(\log(q_2))$$

****

****

$$\therefore \exp(\log(q_1) + \log(q_2)) = \exp(\log(q_1))\exp(\log(q_2)) \qquad (A-6),$$

since the real variable functions,

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right] = \cos^{-1}\left[\frac{x_1}{\sqrt{x_1^2+y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right] = \Theta_1$$

AND

$$\Psi = \cos^{-1}\left[\frac{x+x_2-x_1}{\sqrt{x^2+\lambda^2 y^2+\lambda^2\hat{x}^2+\lambda^2\hat{y}^2+x_2^2-x_1^2}}\right]$$

$$= \cos^{-1}\left[\frac{x_2}{\sqrt{x_2^2+\lambda^2 y_1^2+\lambda^2\hat{x}_1^2+\lambda^2\hat{y}_1^2}}\right] = \Theta_2,$$

after making the relevant algebraic substitutions into Eqs. (ii) & (iv). Once again, from Theorem TI-21, we deduce that

$$\exp(\log(q)) = q \Longrightarrow \begin{cases} \exp(\log(q_1)) = q_1 & , \\ \exp(\log(q_2)) = q_2 & \end{cases}$$

****
****
****
****

and thus Eq. (A-6) may be written as

$$\exp(\log(q_1) + \log(q_2)) = q_1 q_2 \qquad (A-7).$$

Finally, Eqs. (A-2) & (A-7) can be combined into a single statement, namely -

****

$w = q_1 q_2$,

insofar as direct substitution of this value for w into Eq. (A−1) likewise yields –

$\log(q_1 q_2) = \log(q_1) + \log(q_2)$, as required. <u>Q.E.D.</u>

(b) Let us define a quaternion constant, W, whose logarithm,

$$\log(W) = \log(q_1) - \log(q_2) \qquad (B-1),$$

$$\therefore \exp(\log(W)) = \exp(\log(q_1) - \log(q_2)).$$

From Theorem TI−21, however, we recall that

$$\exp(\log(q)) = q, \ \forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\},$$

and thus, by the 'dummy substitution of variables' process,

$$\exp(\log(W)) = W \implies$$

$$W = \exp(\log(q_1) - \log(q_2)) \qquad (B-2).$$

Moreover, we recall from part (a) of this proof that the quaternion functions,

$$\phi_1(q) = \log(q)$$

$$= \log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Theta + 2n\pi)$$

****

$$= \mathcal{U}_1(x,y,\hat{x},\hat{y}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\mathcal{V}_1(x,y,\hat{x},\hat{y}) \qquad (B-3);$$

$$\phi_2(q) = \log\left(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2\hat{x}^2 + \lambda^2\hat{y}^2 + x_2^2 - x_1^2}\right) +$$
$$\left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Psi + 2n_2\pi)$$

$$\therefore -\phi_2(q) = -\log\left(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2\hat{x}^2 + \lambda^2\hat{y}^2 + x_2^2 - x_1^2}\right) -$$
$$\left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right](\Psi + 2n_2\pi)$$

$$= \mathcal{U}_2(x,y,\hat{x},\hat{y}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\mathcal{V}_2(x,y,\hat{x},\hat{y})$$

$$= \phi_2^*(q) \qquad (B-4),$$

insofar as the real variable functions,

$$\mathcal{U}_1(x,y,\hat{x},\hat{y}) = \log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right);$$

$$\mathcal{V}_1(x,y,\hat{x},\hat{y}) = \Theta + 2n\pi;$$

$$\mathcal{U}_2(x,y,\hat{x},\hat{y}) = -\log\left(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2\hat{x}^2 + \lambda^2\hat{y}^2 + x_2^2 - x_1^2}\right);$$

$$\mathcal{V}_2(x,y,\hat{x},\hat{y}) = -\left(\frac{\lambda}{|\lambda|}\right)(\Psi + 2n_2\pi);$$

****

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}}\right] \in [0, \pi] \; ;$$

$$\Psi = \cos^{-1}\left[\frac{x + x_2 - x_1}{\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 \hat{x}^2 + \lambda^2 \hat{y}^2 + x_2^2 - x_1^2}}\right] \in [0, \pi] \, ,$$

and the integers, $n, n_2 \in \mathbb{Z}$.

In accordance with Theorem TI-7, it subsequently follows that the exponential function,

$$\exp(\phi_1(q) + \phi_2^*(q)) = \exp(\phi_1(q)) \exp(\phi_2^*(q))$$

$$\therefore \exp(\phi_1(q) - \phi_2(q)) = \exp(\phi_1(q)) \exp(-\phi_2(q)) \qquad (B-5).$$

Furthermore, we deduce from Theorem TI-6 that the formula,

$\exp(nq) = (\exp(q))^n$, where $n \in \mathbb{Z}$ and $q \in dom(\exp) \subseteq \mathbb{H}$,

likewise implies that for any quaternion variable, $Q^* \in dom(\exp) \subseteq \mathbb{H}$, the formula,

$$\exp(nQ^*) = (\exp(Q^*))^n \qquad (B-6),$$

is also valid. By setting $n = -1$ and $Q^* = \phi_2(q)$, we can rewrite Eq. (B-6) as

$$\exp(-\phi_2(q)) = (\exp(\phi_2(q)))^{-1}$$

and hence Eq. (B−5) can be rewritten as

$$\exp(\phi_1(q) - \phi_2(q)) = \exp(\phi_1(q))(\exp(\phi_2(q)))^{-1} \qquad (B-7).$$

From part (a) of this proof, we recall that for any quaternion constant, $q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 \in \mathbb{H} - \{0\}$,

$$\phi_1(q_1) = \log(q_1) \ \& \ \phi_2(q_1) = \log(q_2) \quad (q_2 = x_2 + i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1 \in \mathbb{H} - \{0\}),$$

and, after making the appropriate algebraic substitutions into Eq. (B−7), we thus obtain

$$\exp(\phi_1(q_1) - \phi_2(q_1)) = \exp(\phi_1(q_1))(\exp(\phi_2(q_1)))^{-1}$$

$$\therefore \exp(\log(q_1) - \log(q_2)) = \exp(\log(q_1))(\exp(\log(q_2)))^{-1} \qquad (B-8).$$

Once again, from Theorem TI−21, we deduce that

$$\exp(\log(q)) = q \Longrightarrow \begin{cases} \exp(\log(q_1)) = q_1 \ , \\ \exp(\log(q_2)) = q_2 \end{cases}$$

whereupon Eq. (B−8) may be written as

$$\exp(\log(q_1) - \log(q_2)) = q_1 q_2^{-1} \qquad (B-9).$$

Clearly, Eqs. (B−2) & (B−9) can be combined into a single statement, namely —

$$W = q_1 q_2^{-1} ,$$

insofar as direct substitution of this value for W into Eq.(B−1) likewise yields –

$$\log(q_1 q_2^{-1}) = \log(q_1) - \log(q_2) \qquad (B-10).$$

Since the quaternion constants,

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 = x_1 + \left[\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right]\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2} \;;$$

$$q_2 = x_2 + i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1 = x_2 + \left[\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right]\lambda\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2} \;,$$

then, by virtue of Theorem TI−10, we can set

$$q_1 = U_1 + QV_1 \;\&\; q_2 = U_2 + QV_2 \Longrightarrow q_2^{-1} = \bar{q}_2/|q_2|^2 = \frac{\overline{U_2 + QV_2}}{|U_2 + QV_2|^2} \;,$$

where the real valued constants,

$$U_1 = x_1 \;;\; V_1 = \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2} \;;\; U_2 = x_2 \;;\; V_2 = \lambda\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2} \;,$$

and the quaternion constant,

$$Q = \frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}} \Longrightarrow Q^2 = -1 ,$$

such that we obtain the quotient value,

$$q_1/q_2 = \frac{U_1 + QV_1}{U_2 + QV_2} = \frac{(U_1 + QV_1)\overline{(U_2 + QV_2)}}{|U_2 + QV_2|^2} = \frac{\overline{(U_2 + QV_2)}(U_1 + QV_1)}{|U_2 + QV_2|^2}$$

$$= q_1 q_2^{-1} = q_2^{-1} q_1 \qquad (B-11).$$

Finally, in view of this particular result, Eq. (B−10) may now be written as

$\log(q_1/q_2) = \log(q_1) - \log(q_2)$, as required. Q.E.D.

We conclude our discussion of the logarithmic quaternion hypercomplex function with the following remarks:-

(a) The results of the preceding Theorems TI−20 & TI−22 are analogous with Eqs. (1−88) & (1−92) insofar as the complex number, i, is replaced by the quaternion variable,

$$\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}},$$

and the modulus, $r = |z| = \sqrt{x^2 + y^2}$, is replaced by the modulus, $r = |q| = \sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}$.

(b) The algebraic property enunciated in Theorem TI−21 is wholly analogous with Eq. (1−91).

(c) The algebraic properties enunciated in Theorem TI-23, however, are comparable to but not wholly analogous with Eqs. (1-93) & (1-94) in view of the restrictions placed on the quaternions, q_1 and q_2, via the formulae,

$$\log(q_1 q_2) = \log(q_1) + \log(q_2) \qquad (1-95);$$

$$\log(q_1/q_2) = \log(q_1) - \log(q_2) \qquad (1-96).$$

7. Variables raised to the Power of Fractional Indices; Quaternion Hypercomplex Exponents.

From complex variable analysis, we recall that any non-zero complex number, z, raised to the power of a fractional index, $1/n$ ($n \in \mathbb{N}$), accordingly defines an irrational 'nth root' function, $z^{1/n}$, which takes on 'n' distinct values. Indeed, Churchill et al. [1] provide us with the following definitive formula, namely –

$$z^{1/n} = \exp\left(\frac{\log(z)}{n}\right) = \sqrt[n]{r}\,\exp\left[\frac{i(\Theta + 2k\pi)}{n}\right]$$

$$(r = |z| > 0;\ -\pi < \Theta < \pi;\ k \in \{0, 1, 2, \ldots\ldots, n-1\}) \qquad (1-97).$$

Furthermore, it should be noted that this particular multi-valued function also satisfies the property –

$$(z^{1/n})^n = z \quad (z \neq 0) \qquad (1-98).$$

Hence, in view of Eqs. (1-97) & (1-98), we shall derive the quaternion analogue of $z^{1/n}$ by means of the next definition and theorem :–

Definition DI−16.

Let there exist a quaternion variable,

$$q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\}.$$

The nth root of 'q' is subsequently denoted as $q^{1/n}$ and thus satisfies the property –

$$(q^{1/n})^n = q, \ \forall n \in \mathbb{N}, \text{ the set of natural numbers.}$$

Theorem TI−24.

Let there exist a quaternion variable,

$$q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\}.$$

In the circumstances, it may be shown that the irrational 'nth root' quaternion hypercomplex function,

$$q^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\Theta + 2K\pi}{n}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\sin\left(\frac{\Theta + 2K\pi}{n}\right)\right],$$

where the real variable functions,

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}}\right] \in [0, \pi] \ \& \ r = \sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2},$$

and the integers,

$K \in \{0, 1, \ldots\ldots\ldots, (n-1)\}$ & $n \in \mathbb{N}$, the set of natural numbers.

* * *

PROOF:-

From Definition DI-16, we initially recall that the nth root of a quaternion number is defined by the equation,

$(q^{1/n})^n = q$, $\forall n \in \mathbb{N}$, the set of natural numbers.

Furthermore, by virtue of the preceding Definitions DI-14 & DI-15 and Theorems TI-20 & TI-21, we deduce that

$$\begin{aligned}(q^{1/n})^n &= q = \exp(\log(q)) \\ &= e^{\left[\log(\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right](2Z\pi+\Theta)\right]},\end{aligned}$$

$\forall Z \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots, \pm\infty\}$,

such that the real variable function,

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right] \in [0, \pi].$$

Now let the <u>nth root</u> quaternion hypercomplex function,

$$q^{1/n} = e^{\left(U + \left[\frac{i\dot{y} + j\hat{x} + k\hat{y}}{\sqrt{\dot{y}^2 + \hat{x}^2 + \hat{y}^2}}\right]V\right)},$$

such that the undefined variables, $U, V \in \mathbb{R}$.

Subsequently, in the light of Theorem TI-6, we perceive, after making the appropriate algebraic substitutions, that

$$\left(e^{\left(U + \left[\frac{i\dot{y} + j\hat{x} + k\hat{y}}{\sqrt{\dot{y}^2 + \hat{x}^2 + \hat{y}^2}}\right]V\right)}\right)^n = e^{n\left(U + \left[\frac{i\dot{y} + j\hat{x} + k\hat{y}}{\sqrt{\dot{y}^2 + \hat{x}^2 + \hat{y}^2}}\right]V\right)} =$$

$$e^{\left(nU + \left[\frac{i\dot{y} + j\hat{x} + k\hat{y}}{\sqrt{\dot{y}^2 + \hat{x}^2 + \hat{y}^2}}\right]nV\right)} = e^{\left[\log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right) + \left[\frac{i\dot{y} + j\hat{x} + k\hat{y}}{\sqrt{\dot{y}^2 + \hat{x}^2 + \hat{y}^2}}\right](2Z\pi + \Theta)\right]}$$

-112-

$$\Longrightarrow nU = \log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right) \ \& \ nV = 2Z\pi + \Theta$$

$$\therefore U = \tfrac{1}{n}\log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right) = \tfrac{1}{n}\log(r) = \log(\sqrt[n]{r}),$$

and similarly,

$$V = \frac{2Z\pi + \Theta}{n}.$$

Henceforth, we may write the <u>nth root</u> quaternion function, $q^{1/n}$, as

$$\begin{aligned} q^{1/n} &= e^{\left(U + \left[\frac{i\dot{y} + j\hat{x} + k\hat{y}}{\sqrt{\dot{y}^2 + \hat{x}^2 + \hat{y}^2}}\right]V\right)} \\ &= e^{U} e^{\left[\frac{i\dot{y} + j\hat{x} + k\hat{y}}{\sqrt{\dot{y}^2 + \hat{x}^2 + \hat{y}^2}}\right]V} \\ &= e^{\log(\sqrt[n]{r})} e^{\left[\frac{i\dot{y} + j\hat{x} + k\hat{y}}{\sqrt{\dot{y}^2 + \hat{x}^2 + \hat{y}^2}}\right]\left(\frac{2Z\pi + \Theta}{n}\right)} \end{aligned}$$

$$= \sqrt[n]{r}\left[\cos\left(\frac{2Z\pi + \Theta}{n}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\sin\left(\frac{2Z\pi + \Theta}{n}\right)\right],$$

by virtue of Theorem TI−3, and, whilst we had originally postulated that the integer, $Z \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, \pm\infty\}$, it is also evident from real variable analysis that the periodicity of the trigonometric functions, $\cos[(2Z\pi + \Theta)/n]$ and $\sin[(2Z\pi + \Theta)/n]$, will always ensure that the <u>nth root</u> function, $q^{1/n}$, can only take on 'n' distinct values whenever $Z = 0, 1, \ldots\ldots\ldots\ldots, n-1$.

In the circumstances, we now let the integer, $K \in \{0, 1, \ldots\ldots\ldots\ldots, (n-1)\}$, act as a dummy substitute for the integer, $Z \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, \pm\infty\}$, and hence we conclude that the <u>nth root</u> function,

$$q^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\Theta + 2K\pi}{n}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\sin\left(\frac{\Theta + 2K\pi}{n}\right)\right],$$

where the real variable functions,

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}}\right] \in [0, \pi] \ \& \ r = \sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2} > 0,$$

and the integer,

$K \in \{0, 1, \ldots\ldots\ldots\ldots, (n-1)\}$, $\forall n \in \mathbb{N}$, the set of natural numbers,

as required. <u>Q.E.D.</u>

Churchill et al. [1] have defined the multi-valued complex exponent functions, z^c and c^z, in terms of the following formulae –

$$z^c = \exp(c\log(z)) \qquad (z \neq 0) \qquad (1-99);$$

$$c^z = \exp(z\log(c)) \qquad (c \neq 0) \qquad (1-100),$$

where c is some arbitrary complex constant. Needless to say, the purpose of our next definition is to enunciate the quaternion analogues of Eqs. (1-99) & (1-100):–

Definition DI–17.

Let there exist two quaternions, 'q' and 'c', where 'c' is some arbitrary constant. We accordingly define the existence of two quaternion hypercomplex exponent functions, q^c and c^q, such that

(a) $$q^c = \begin{cases} \exp(c\log(q)) & (q \in \mathbb{H} - \{0\} \ \& \ c \in \mathbb{H}) \\ \exp(\log(q)c) & \end{cases}$$

AND

(b) $$c^q = \begin{cases} \exp(q\log(c)) & (q \in \mathbb{H} \ \& \ c \in \mathbb{H} - \{0\}). \\ \exp(\log(c)q) & \end{cases}$$

We conclude our discussion of variables raised to the power of fractional indices and quaternion hypercomplex exponents with the following remarks:-

(a) The algebraic property enunciated in Theorem TI-24 is analogous with Eq.(1-97) insofar as the complex number, i, is replaced by the quaternion variable,

$$\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}},$$

and the modulus, $r = |z| = \sqrt{x^2 + y^2}$, is replaced by the modulus, $r = |q| = \sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}$.

(b) The algebraic properties enunciated in Definition DI-17, however, are comparable to but not wholly analogous with Eqs.(1-99) & (1-100) since the quaternion products, $c\log(q)$, $\log(q)c$, $q\log(c)$ and $\log(c)q$, are generally non-commutative, in other words –

$$c\log(q) \neq \log(q)c \quad \& \quad q\log(c) \neq \log(c)q.$$

8. The Inverse Trigonometric Functions.

From complex variable analysis, we recall that the inverse trigonometric functions, $\sin^{-1}(z)$, $\cos^{-1}(z)$ and $\tan^{-1}(z)$, are defined in the following manner:-

(a) the inverse sine function,

$$w = \sin^{-1}(z) \Longrightarrow z = \sin(w) \qquad (1-101);$$

(b) the inverse cosine function,

$$w = \cos^{-1}(z) \Longrightarrow z = \cos(w) \qquad (1\text{-}102);$$

(c) the inverse tangent function,

$$w = \tan^{-1}(z) \Longrightarrow z = \tan(w) \qquad (1\text{-}103).$$

Moreover, Churchill et al. [1] provide us with three supplementary formulae for $\sin^{-1}(z)$, $\cos^{-1}(z)$ and $\tan^{-1}(z)$, namely –

$$\sin^{-1}(z) = -i \log\left[iz + (1-z^2)^{1/2}\right] \qquad (1\text{-}104);$$

$$\cos^{-1}(z) = -i \log\left[z + i(1-z^2)^{1/2}\right] \qquad (1\text{-}105);$$

$$\tan^{-1}(z) = \tfrac{1}{2} i \log\left(\frac{i+z}{i-z}\right) \qquad (1\text{-}106).$$

Hence, the purpose of our next definition and theorem is to derive the quaternion analogues of Eqs. (1-101) $\longrightarrow$ (1-106).

Definition DI-18.

Let there exist three inverse trigonometric quaternion hypercomplex functions, $\sin^{-1}(q)$, $\cos^{-1}(q)$ and $\tan^{-1}(q)$, whereupon $q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H}$. In the circumstances, we postulate that

(a) the inverse sine function,

****

$$w = \sin^{-1}(q) \Longrightarrow q = \sin(w),$$

(b) the <u>inverse cosine</u> function,

$$w = \cos^{-1}(q) \Longrightarrow q = \cos(w),$$

****
****
****
****
****
****

(c) the <u>inverse tangent</u> function,

$$w = \tan^{-1}(q) \Longrightarrow q = \tan(w).$$

<u>Theorem TI-25.</u>

Let there exist the <u>inverse trigonometric</u> functions, $\sin^{-1}(q)$, $\cos^{-1}(q)$ and $\tan^{-1}(q)$, as previously defined. Subsequently, we may prove that the following formulae, namely –

(a) $\sin^{-1}(q) = -Q \log\left[Qq + (1-q^2)^{1/2}\right],$

(b) $\cos^{-1}(q) = -Q \log\left[q + Q(1-q^2)^{1/2}\right],$

(c) $\tan^{-1}(q) = \frac{1}{2} Q \log\left(\dfrac{Q+q}{Q-q}\right),$

are likewise valid, insofar as the auxiliary function,

****

$$Q = \frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}} .$$

* * *

PROOF :-

From Definitions DI-10, DI-11 and DI-18, we initially recall that

(i) the inverse sine function,

$$w = \sin^{-1}(q) \Longrightarrow q = \sin(w) = \frac{\exp(Q^*w) - \exp(-Q^*w)}{2Q^*} ,$$

(ii) the inverse cosine function,

$$w = \cos^{-1}(q) \Longrightarrow q = \cos(w) = \frac{\exp(Q^*w) + \exp(-Q^*w)}{2} ,$$

(iii) the inverse tangent function,

$$w = \tan^{-1}(q) \Longrightarrow q = \tan(w) = \sin(w)/\cos(w)$$

$$= \frac{\dfrac{\exp(Q^*w) - \exp(-Q^*w)}{2Q^*}}{\dfrac{\exp(Q^*w) + \exp(-Q^*w)}{2}} ,$$

such that the auxiliary function,

$$Q^* = \frac{iv_1 + ju_2 + kv_2}{\sqrt{v_1^2 + u_2^2 + v_2^2}}\,, \quad \text{whenever } w = u_1 + iv_1 + ju_2 + kv_2 .$$

We will now examine each of these inverse trigonometric functions separately.

(a) Let us define a variable, W_1, such that

(i) $W_1 = w = \sin^{-1}(q) \Longrightarrow q = \sin(w) = \sin(W_1)$;

(ii) $w = W_1 = u_1 + iv_1 + ju_2 + kv_2 = U_1 + \left(\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right) V_1 \Longrightarrow$

$$u_1 = U_1,\ v_1 = \frac{yV_1}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}},\ u_2 = \frac{\hat{x}V_1}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}} \text{ and } v_2 = \frac{\hat{y}V_1}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}};$$

(iii) the variables, $U_1, V_1 \in \mathbb{R}$.

Hence, it follows that the auxiliary function,

$$Q^* = \frac{\dfrac{iyV_1 + j\hat{x}V_1 + k\hat{y}V_1}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}}{\sqrt{\dfrac{y^2V_1^2}{y^2 + \hat{x}^2 + \hat{y}^2} + \dfrac{\hat{x}^2V_1^2}{y^2 + \hat{x}^2 + \hat{y}^2} + \dfrac{\hat{y}^2V_1^2}{y^2 + \hat{x}^2 + \hat{y}^2}}}$$

$$= \frac{V_1}{|V_1|}\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right)$$

****

$$= I^*Q,$$

where $Q = \dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}$ and $I^* = V_1/|V_1| = \pm 1 \Longrightarrow I^{*2} = 1.$

Subsequently, we may write

$$q = \sin(W_1) = \frac{\exp(I^*QW_1) - \exp(-I^*QW_1)}{2I^*Q}$$

$$= -\tfrac{1}{2}I^*Q\left(\exp(I^*QW_1) - \exp(-I^*QW_1)\right)$$

$$\therefore \exp(I^*QW_1) - \exp(-I^*QW_1) = 2I^*Qq = 2qI^*Q,$$

by virtue of Theorem TI-10, such that

$$(\exp(I^*QW_1))^2 - \exp(-I^*QW_1)\exp(I^*QW_1) = 2I^*Qq\exp(I^*QW_1)$$

$$\therefore (\exp(I^*QW_1))^2 - \exp(-I^*QW_1 + I^*QW_1) = 2I^*Qq\exp(I^*QW_1)$$

$$\therefore (\exp(I^*QW_1))^2 - 2I^*Qq\exp(I^*QW_1) - \exp(0) = 0$$

$$\therefore (\exp(I^*QW_1))^2 - 2I^*Qq\exp(I^*QW_1) - 1 = 0$$

$$\therefore (\exp(I^*QW_1))^2 - 2I^*Qq\exp(I^*QW_1) = 1$$

****
****
****
****

$$\therefore (\exp(I^*QW_1))^2 - 2I^*Qq\exp(I^*QW_1) + I^{*2}Q^2q^2 = 1 + I^{*2}Q^2q^2$$

****

$$\therefore\ (\exp(I^*QW_1) - I^*Qq)^2 = 1 - q^2 \qquad (\underline{\underline{N.B.}}\ I^{*2}Q^2 = -1)$$

$$\therefore\ \exp(I^*QW_1) - I^*Qq = (1-q^2)^{1/2}$$

$$\therefore\ \exp(I^*QW_1) = I^*Qq + (1-q^2)^{1/2} \Longrightarrow$$

$$I^*QW_1 = \log[I^*Qq + (1-q^2)^{1/2}]$$

$$\therefore\ -I^{*2}Q^2W_1 = -I^*Q\log[I^*Qq + (1-q^2)^{1/2}]$$

$$\therefore\ W_1 = -I^*Q\log[I^*Qq + (1-q^2)^{1/2}].$$

However, since we had originally postulated that the inverse function,

$$W_1 = w = \sin^{-1}(q) \Longrightarrow q = \sin(w) = \sin(W_1),$$

we therefore obtain the result,

$$\sin^{-1}(q) = -I^*Q\log[I^*Qq + (1-q^2)^{1/2}]. \qquad (A-1)$$

Let us now define another variable, W_2, such that

(iv) $W_2 = w = \sin^{-1}(q) \Longrightarrow q = \sin(w) = \sin(W_2)$;

(v) $w = W_2 = u_1 + iv_1 + ju_2 + kv_2 = U_2 + \left(\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right)V_2 \Longrightarrow$

$$u_1 = U_2,\ v_1 = \frac{yV_2}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}},\ u_2 = \frac{\hat{x}V_2}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} \text{ and } v_2 = \frac{\hat{y}V_2}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}};$$

(vi) the variables, $U_2 \in \mathbb{R}$ and $V_2 \in [0,\infty)$.

****

Hence, it follows that the auxiliary function,

$$Q^* = \frac{\dfrac{iyV_2 + j\hat{x}V_2 + k\hat{y}V_2}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}}{\sqrt{\dfrac{y^2V_2^2}{y^2+\hat{x}^2+\hat{y}^2} + \dfrac{\hat{x}^2V_2^2}{y^2+\hat{x}^2+\hat{y}^2} + \dfrac{\hat{y}^2V_2^2}{y^2+\hat{x}^2+\hat{y}^2}}}$$

$$= \frac{V_2}{|V_2|}\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)$$

$$= Q,$$

where $Q = \dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$ and $V_2/|V_2| = 1$,

since, as previously stated, $V_2 = [0,\infty) \Longrightarrow V_2 = |V_2|$.

Subsequently, we may write

$$q = \sin(W_2) = \frac{\exp(QW_2) - \exp(-QW_2)}{2Q}$$

$$= -\tfrac{1}{2}Q(\exp(QW_2) - \exp(-QW_2))$$

$\therefore \exp(QW_2) - \exp(-QW_2) = 2Qq = 2qQ,$

by virtue of Theorem TI-10, such that

$$(\exp(QW_2))^2 - \exp(-QW_2)\exp(QW_2) = 2Qq\exp(QW_2)$$

****

****

$$\therefore (\exp(QW_2))^2 - \exp(-QW_2 + QW_2) = 2Qq \exp(QW_2)$$

$$\therefore (\exp(QW_2))^2 - 2Qq \exp(QW_2) - \exp(0) = 0$$

****
****
****

−121−

$$\therefore (\exp(QW_2))^2 - 2Qq \exp(QW_2) - 1 = 0$$

$$\therefore (\exp(QW_2))^2 - 2Qq \exp(QW_2) = 1$$

$$\therefore (\exp(QW_2))^2 - 2Qq \exp(QW_2) + Q^2 q^2 = 1 + Q^2 q^2$$

$$\therefore (\exp(QW_2) - Qq)^2 = 1 - q^2$$

$$\therefore \exp(QW_2) - Qq = (1-q^2)^{1/2}$$

$$\therefore \exp(QW_2) = Qq + (1-q^2)^{1/2} \Longrightarrow$$

$$QW_2 = \log[Qq + (1-q^2)^{1/2}]$$

$$\therefore -Q^2 W_2 = -Q \log[Qq + (1-q^2)^{1/2}]$$

$$\therefore W_2 = -Q \log[Qq + (1-q^2)^{1/2}].$$

However, since we had originally postulated that the inverse function,

$$W_2 = w = \sin^{-1}(q) \Longrightarrow q = \sin(w) = \sin(W_2),$$

we therefore obtain the result,

****

$$\sin^{-1}(q) = -Q\log[Qq + (1-q^2)^{1/2}]. \qquad (A-2)$$

We now wish to prove that

$$\sin^{-1}(q) = -I^*Q\log[I^*Qq + (1-q^2)^{1/2}] = -Q\log[Qq + (1-q^2)^{1/2}],$$

where $I^* = \pm 1$. To do this, let us consider the separate cases where $I^* = 1$ and $I^* = 1$. Firstly, by putting $I^* = 1$, we observe that

$$-I^*Q\log[I^*Qq + (1-q^2)^{1/2}] = -Q\log[Qq + (1-q^2)^{1/2}].$$

-122-

Secondly, by putting $I^* = -1$, we likewise note that

$$-I^*Q\log[I^*Qq + (1-q^2)^{1/2}] = Q\log[-Qq + (1-q^2)^{1/2}]$$

$$= -Q(-\log[-Qq + (1-q^2)^{1/2}])$$

$$= -Q\log\left[\frac{1}{-Qq + (1-q^2)^{1/2}}\right] = -Q\log\left[\frac{1}{(1-q^2)^{1/2} - Qq}\right],$$

by virtue of Theorem TI-6, since it is evident that the exponential function,

$$\exp(-\log[-Qq + (1-q^2)^{1/2}]) = (\exp(\log[-Qq + (1-q^2)^{1/2}]))^{-1}$$

$$= [-Qq + (1-q^2)^{1/2}]^{-1}$$

$$= \frac{1}{-Qq + (1-q^2)^{1/2}}$$

$$\Longrightarrow -\log[-Qq + (1-q^2)^{1/2}] = \log\left[\frac{1}{-Qq + (1-q^2)^{1/2}}\right].$$

Furthermore, let the dual-valued function,

$$(1-q^2)^{1/2} = X + QY \quad (X, Y \in \mathbb{R}) \qquad (A-3).$$

Subsequently, the square of this function,

$$1 - q^2 = (X + QY)^2$$

$$\therefore 1 - q^2 = X^2 + 2XYQ - Y^2$$

$$\therefore 1 - (x + Q\sqrt{y^2 + \hat{x}^2 + \hat{y}^2})^2 = X^2 + 2XYQ - Y^2$$

$$\therefore 1 - (x^2 + 2x\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\,Q - (y^2 + \hat{x}^2 + \hat{y}^2)) = X^2 - Y^2 + 2XYQ$$

$$\therefore 1 - x^2 - 2x\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\,Q + y^2 + \hat{x}^2 + \hat{y}^2 = X^2 - Y^2 + 2XYQ$$

$$\therefore 1 - x^2 + y^2 + \hat{x}^2 + \hat{y}^2 - 2x\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\,Q = X^2 - Y^2 + 2XYQ.$$

By equating the corresponding real and imaginary parts, we thus obtain the pair of simultaneous equations -

$$X^2 - Y^2 = 1 - x^2 + y^2 + \hat{x}^2 + \hat{y}^2 \quad \text{AND}$$

$$XY = -x\sqrt{y^2+\hat{x}^2+\hat{y}^2}\,.$$

The solution of these equations for X and Y leads us to conclude that Eq. (A–3) is a valid statement insofar as the function, $(1-q^2)^{1/2}$, will form a <u>commutative</u> product with Qq, in other words –

$$Qq\,(1-q^2)^{1/2} = (1-q^2)^{1/2}Qq. \quad \left[\underline{\underline{\text{N.B.}}}\ \begin{aligned} Qq &= Q(x+Q\sqrt{y^2+\hat{x}^2+\hat{y}^2}) \\ &= -\sqrt{y^2+\hat{x}^2+\hat{y}^2}+Qx \end{aligned}\right]$$

In the circumstances, we further deduce that

$$-I^*Q\log\left[I^*Qq+(1-q^2)^{1/2}\right] = -Q\log\left[\frac{1}{(1-q^2)^{1/2}-Qq}\right]$$

$$= -Q\log\left[\frac{(1-q^2)^{1/2}+Qq}{((1-q^2)^{1/2}-Qq)((1-q^2)^{1/2}+Qq)}\right]$$

$$= -Q\log\left[\frac{Qq+(1-q^2)^{1/2}}{1-q^2+(1-q^2)^{1/2}Qq-Qq(1-q^2)^{1/2}-Q^2q^2}\right]$$

$$= -Q\log\left[\frac{Qq+(1-q^2)^{1/2}}{1-q^2+(1-q^2)^{1/2}Qq-(1-q^2)^{1/2}Qq+q^2}\right]$$

$$= -Q\log\left[\frac{Qq+(1-q^2)^{1/2}}{1}\right]$$

$$= -Q\log\left[Qq+(1-q^2)^{1/2}\right].$$

Clearly, we have proven that the function,

$$-I^*Q\log[I^*Qq + (1-q^2)^{1/2}] = -Q\log[Qq + (1-q^2)^{1/2}],$$

whenever $I^* = \pm 1$, and, in the light of Eqs.(A-1) and (A-2), it naturally follows that the <u>inverse sine</u> function,

$$\sin^{-1}(q) = -I^*Q\log[I^*Qq + (1-q^2)^{1/2}] = -Q\log[Qq + (1-q^2)^{1/2}],$$

as required. <u>Q.E.D.</u>

<u>(b)</u> From Theorem TI-15, we recall that the trigonometric identity,

$$\cos(q) = \sin(\tfrac{\pi}{2} - q),$$

is valid for any $q \in \mathbb{H}$. Hence, it analogously follows that, for any $w \in \mathbb{H}$, the cosine function,

$$\cos(w) = \sin(\tfrac{\pi}{2} - w).$$

Moreover, we likewise note from Definition DI-18 that the inverse cosine function,

$$w = \cos^{-1}(q) \Longrightarrow q = \cos(w),$$

$$\therefore q = \sin(\tfrac{\pi}{2} - w)$$

$$\therefore \tfrac{\pi}{2} - w = \sin^{-1}(q)$$

$$\therefore w - \tfrac{\pi}{2} = -\sin^{-1}(q)$$

$$\therefore w = \tfrac{\pi}{2} - \sin^{-1}(q)$$

$$= \frac{\pi}{2} - (-Q \log[Qq + (1-q^2)^{1/2}])$$

$$= \frac{\pi}{2} + Q \log[Qq + (1-q^2)^{1/2}]$$

$$= -Q^2\frac{\pi}{2} + Q \log[Qq + (1-q^2)^{1/2}]$$

$$= -Q(Q\frac{\pi}{2} - \log[Qq + (1-q^2)^{1/2}])$$

$$= -Q(Q(2n+\tfrac{1}{2})\pi - \log[Qq + (1-q^2)^{1/2}] - Q2n\pi)$$

$$= -Q(\log(Q) - \log[Qq + (1-q^2)^{1/2}] - Q2n\pi)$$

$$= -Q(\log(Q) - (\log[Qq + (1-q^2)^{1/2}] + Q2n\pi))$$

$$= -Q(\log(Q) - \log[Qq + (1-q^2)^{1/2}])$$

$$= -Q \log\left[\frac{Q}{Qq + (1-q^2)^{1/2}}\right],$$

by virtue of Theorem TI-23, since the dual-valued function,

$$Qq + (1-q^2)^{1/2} = X - \sqrt{y^2 + \hat{x}^2 + \hat{y}^2} + Q(x+Y),$$

as previously stated in the proof of part (a) of this theorem [viz. Eq.(A-3)], and hence the logarithmic function,

$$\log[Qq + (1-q^2)^{1/2}] = \log[X - \sqrt{y^2 + \hat{x}^2 + \hat{y}^2} + Q(x+Y)]$$

$$= \log\left[\sqrt{\left(X - \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\right)^2 + (x+Y)^2}\right] +$$

$$\frac{Q(x+Y)}{|x+Y|}\left(\cos^{-1}\left[\frac{X - \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}{\sqrt{\left(X - \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\right)^2 + (x+Y)^2}}\right] + 2N\pi\right) \quad (N \in \mathbb{Z}),$$

as a direct consequence of Theorem TI-20 and Definition DI-15. In view of

To be continued via the author's next submission, namely -

"__A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 5/10__."

* * * * * * * *

* * * * * * * *

[THIS PAGE HAS BEEN INTENTIONALLY LEFT BLANK.]