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Three-dimensional quadrics in conformal geometric algebras and their versor transformations

Eckhard Hitzer

Abstract. This work explains how three dimensional quadrics can be defined by the outer products of conformal geometric algebra points in higher dimensions. These multivector expressions code all types of quadrics in arbitrary scale, location and orientation. Furthermore, a newly modified (compared to [1]) approach now allows not only the use of the standard intersection operations, but also of versor operators (scaling, rotation, translation). The new algebraic form of the theory will be explained in detail.

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1. Introduction

Three dimensional quadrics can be represented in extended conformal geometric algebras (CGA) in several ways. Most recently double CGA (DCGA)[4] and quadric CGA (QCGA)[1] have been proposed[1] as an extension to three dimensions of the outer product representation of conics, originally proposed by Perwass in Chapter 4.5 of [11]. Versors for rotation, translation and scaling have been proposed for conics in a modified version of Perwass' approach (a different definition of points being introduced) in [9]. Reverting to the original point definition of Perwass, [7] recently showed how the definition of versors for translation can be substantially simplified (avoiding quadratic terms in the translation parameters in versor exponents).

The present work shows, how the approach of [7] can also be extended to QCGA (with small modifications of the basis null vector definitions of QCGA) and leads to a consistent set of versors for rotation, translation and scaling. The original

[1]For further literature see the references in [1].

presentation of QCGA in [1] focused on ease of programming and computational efficiency, whereas we begin to present the elegant algebraic core structure of QCGA is in some detail, before advancing to the definition of versors for geometric transformations. To make the model fit for using versor transformations, the description of how round and flat conformal geometric algebra (CGA) objects are embedded also undergoes small modifications, as well as how computations with quadrics should be performed.

The paper is structured as follows. Section 2 introduces the basic definitions of QCGA and states a collection of useful algebraic identities, including the necessary modifications, needed later for the successful definition of geometric transformation versors. Section 3 explains how points are defined in QCGA, and how round and flat objects of CGA for three Euclidean dimensions, can be consistently embedded in QCGA, in a form compatible with the later definition of transformation versors. Section 4 shows the definition of quadrics by way of the outer product of nine contact points. Section 5 is on the computation of intersections of quadrics. Section 6 addresses the consistent definition of versors for rotations, translations and scaling, including the option of an alternative hybrid approach in combination with other extensions of CGA for the description of quadrics. The paper concludes with Section 7, followed by acknowledgments and references.

2. QCGA definition

This section introduces quadric conformal geometric algebra (QCGA) in slightly modified form, compared to [1]. We specify its basis vectors and show important blade computations. We keep the following notation: lower-case bold letters denote basis blades and multivectors (multivector $\mathbf{a}$). Italic lower-case letters refer to multivector components $(a_1, x, y^2, \cdots)$. For example, a_i is the i^{th} coordinate of the multivector $\mathbf{a}$. Constant scalars are denoted using lower-case default text font (constant radius r) or simply r. The superscripts star used in $\mathbf{x}^*$ represents the dualization of the multivector $\mathbf{x}$. Finally, subscript ε on $\mathbf{x}_\varepsilon$ refers to the Euclidean vector associated to the vector $\mathbf{x}$ of QCGA.

Note that when used in the geometric algebra inner product, contractions and the outer product have priority over the full geometric product. For instance, $\mathbf{a} \wedge \mathbf{b}\mathbf{I} = (\mathbf{a} \wedge \mathbf{b})\mathbf{I}$.

2.1. QCGA basis and metric

The algebraic equations in this section can be either computed by hand, expanding all blades in terms of basis vectors, or they can be computed with software, like The Clifford Toolbox for MATLAB[12]. The QCGA $Cl(9,6)$ is defined over a real 15-dimensional vector space $\mathbb{R}^{9,6}$. The base vectors of the space are basically divided into three groups: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (corresponding to the Euclidean vectors of $\mathbb{R}^3$), $\{\mathbf{e}_{o1}, \mathbf{e}_{o2}, \mathbf{e}_{o3}, \mathbf{e}_{o4}, \mathbf{e}_{o5}, \mathbf{e}_{o6}\}$, and $\{\mathbf{e}_{\infty 1}, \mathbf{e}_{\infty 2}, \mathbf{e}_{\infty 3}, \mathbf{e}_{\infty 4}, \mathbf{e}_{\infty 5}, \mathbf{e}_{\infty 6}\}$. The inner products between them are defined in Table 1.

For the efficient computation, a diagonal metric matrix may be useful. The algebra $Cl(9,6)$ generated by the Euclidean basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and six basis vectors $\{\mathbf{e}_{+1}, \mathbf{e}_{+2}, \mathbf{e}_{+3}, \mathbf{e}_{+4}, \mathbf{e}_{+5}, \mathbf{e}_{+6}\}$ squaring to $+1$ along with six other basis vectors

TABLE 1. Inner product between QCGA basis vectors.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{o1}$	$\mathbf{e}_{\infty1}$	$\mathbf{e}_{o2}$	$\mathbf{e}_{\infty2}$	$\mathbf{e}_{o3}$	$\mathbf{e}_{\infty3}$	$\mathbf{e}_{o4}$	$\mathbf{e}_{\infty4}$	$\mathbf{e}_{o5}$	$\mathbf{e}_{\infty5}$	$\mathbf{e}_{o6}$	$\mathbf{e}_{\infty6}$
$\mathbf{e}_1$	1	0	0	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_2$	0	1	0	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_3$	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{o1}$	·	·	·	0	−1	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{\infty1}$	·	·	·	−1	0	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{o2}$	·	·	·	·	·	0	−1	·	·	·	·	·	·	·	·
$\mathbf{e}_{\infty2}$	·	·	·	·	·	−1	0	·	·	·	·	·	·	·	·
$\mathbf{e}_{o3}$	·	·	·	·	·	·	·	0	−1	·	·	·	·	·	·
$\mathbf{e}_{\infty3}$	·	·	·	·	·	·	·	−1	0	·	·	·	·	·	·
$\mathbf{e}_{o4}$	·	·	·	·	·	·	·	·	·	0	−1	·	·	·	·
$\mathbf{e}_{\infty4}$	·	·	·	·	·	·	·	·	·	−1	0	·	·	·	·
$\mathbf{e}_{o5}$	·	·	·	·	·	·	·	·	·	·	·	0	−1	·	·
$\mathbf{e}_{\infty5}$	·	·	·	·	·	·	·	·	·	·	·	−1	0	·	·
$\mathbf{e}_{o6}$	·	·	·	·	·	·	·	·	·	·	·	·	·	0	−1
$\mathbf{e}_{\infty6}$	·	·	·	·	·	·	·	·	·	·	·	·	·	−1	0

$\{\mathbf{e}_{-1}, \mathbf{e}_{-2}, \mathbf{e}_{-1}, \mathbf{e}_{-4}, \mathbf{e}_{-5}, \mathbf{e}_{-6}\}$ squaring to -1 would correspond to a diagonal metric matrix. The transformation from the diagonal metric basis to that of Table 1 can be defined as follows[2]: for $1 \leq i, j \leq 6$,

$$\mathbf{e}_{\infty i} = \frac{1}{\sqrt{2}}(\mathbf{e}_{+i} + \mathbf{e}_{-i}), \qquad \mathbf{e}_{oi} = \frac{1}{\sqrt{2}}(\mathbf{e}_{-i} - \mathbf{e}_{+i}). \tag{1}$$

We further define for later use another pair of null vectors

$$\mathbf{e}_{\infty} = \tfrac{1}{3}(\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3}), \qquad \mathbf{e}_o = \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}. \tag{2}$$

Inner products lead to

$$\mathbf{e}_{\infty i} \cdot \mathbf{e}_{oi} = -1, \qquad \mathbf{e}_{\infty} \cdot \mathbf{e}_o = -1, \qquad \mathbf{e}_o^2 = \mathbf{e}_{\infty}^2 = 0, \tag{3}$$

$$\mathbf{e}_{\infty k} \cdot \mathbf{e}_o = -1 \quad (k = 1,2,3), \quad \mathbf{e}_{\infty l} \cdot \mathbf{e}_o = 0 \quad (l = 4,5,6), \quad \mathbf{e}_{\infty i} \cdot \mathbf{e}_{\infty} = 0, \tag{4}$$

We further define the bivectors E_i, E, as

$$E_i = \mathbf{e}_{\infty i} \wedge \mathbf{e}_{oi} = \mathbf{e}_{+i}\mathbf{e}_{-i}, \qquad E = \mathbf{e}_{\infty} \wedge \mathbf{e}_o, \tag{5}$$

and obtain the following products

$$E_i^2 = 1, \qquad E_iE_j = E_jE_i, \tag{6}$$

$$\mathbf{e}_{oi}E_i = -E_i\mathbf{e}_{oi} = -\mathbf{e}_{oi}, \qquad \mathbf{e}_{\infty i}E_i = -E_i\mathbf{e}_{\infty i} = \mathbf{e}_{\infty i}, \tag{7}$$

$$\mathbf{e}_{oj}E_i \overset{i \neq i}{=} E_i\mathbf{e}_{oj}, \qquad \mathbf{e}_{\infty j}E_i \overset{i \neq i}{=} E_i\mathbf{e}_{\infty j}, \tag{8}$$

$$E^2 = 1, \quad \mathbf{e}_oE = -E\mathbf{e}_o = -\mathbf{e}_o, \quad \mathbf{e}_{\infty}E = -E\mathbf{e}_{\infty} = \mathbf{e}_{\infty}. \tag{9}$$

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[2]Traditionally, null basis vectors $\mathbf{e}_{\infty i} = \mathbf{e}_{+i} + \mathbf{e}_{-i}$, $\mathbf{e}_{oi} = \frac{1}{2}(\mathbf{e}_{-i} - \mathbf{e}_{+i})$, for $1 \leq i \leq 6$ are chosen, as in [1, 3, 8]. But in general any factor $\lambda_i \in \mathbb{R} \setminus \{0\}$, $1 \leq i \leq 6$, could be chosen to define $\mathbf{e}_{\infty i} = \frac{1}{\lambda_i\sqrt{2}}(\mathbf{e}_{+i} + \mathbf{e}_{-i})$, $\mathbf{e}_{oi} = \frac{\lambda_i}{\sqrt{2}}(\mathbf{e}_{-i} - \mathbf{e}_{+i})$, while preserving the scalar products of Table 1. This freedom to operate with continuous parametrized sets of horospheres has e.g. been used advantageously by El Mir et al for elegant algebraic view point change representation in [5].

For clarity, we also define the following blades:

$$\mathbf{I}_{\infty a} = \mathbf{e}_{\infty 1}\mathbf{e}_{\infty 2}\mathbf{e}_{\infty 3}, \quad \mathbf{I}_{\infty b} = \mathbf{e}_{\infty 4}\mathbf{e}_{\infty 5}\mathbf{e}_{\infty 6}, \quad \mathbf{I}_{\infty} = \mathbf{I}_{\infty a}\mathbf{I}_{\infty b}, \tag{10}$$

$$\mathbf{I}_{oa} = \mathbf{e}_{o1}\mathbf{e}_{o2}\mathbf{e}_{o3}, \quad \mathbf{I}_{ob} = \mathbf{e}_{o4}\mathbf{e}_{o5}\mathbf{e}_{o6}, \quad \mathbf{I}_{o} = \mathbf{I}_{oa}\mathbf{I}_{ob}, \tag{11}$$

$$\mathbf{I}_{\infty o} = \mathbf{I}_{\infty} \wedge \mathbf{I}_{o} = -E_1E_2E_3E_4E_5E_6, \tag{12}$$

$$\mathbf{I}^{\triangleright}_{\infty a} = (\mathbf{e}_{\infty 1} - \mathbf{e}_{\infty 2}) \wedge (\mathbf{e}_{\infty 2} - \mathbf{e}_{\infty 3}), \quad \mathbf{I}^{\triangleright}_{\infty} = \mathbf{I}^{\triangleright}_{\infty a}\mathbf{I}_{\infty b}, \tag{13}$$

$$\mathbf{I}^{\triangleright}_{oa} = (\mathbf{e}_{o1} - \mathbf{e}_{o2}) \wedge (\mathbf{e}_{o2} - \mathbf{e}_{o3}), \quad \mathbf{I}^{\triangleright}_{o} = \mathbf{I}^{\triangleright}_{oa}\mathbf{I}_{ob}, \quad \mathbf{I}^{\triangleright} = \mathbf{I}^{\triangleright}_{\infty} \wedge \mathbf{I}^{\triangleright}_{o}. \tag{14}$$

We note that[3]

$$\mathbf{I}^2_{\infty o} = 1, \quad \mathbf{I}_{o}\mathbf{I}_{\infty o} = \mathbf{I}_{\infty o}\mathbf{I}_{o} = -\mathbf{I}_{o}, \quad \mathbf{I}_{\infty}\mathbf{I}_{\infty o} = \mathbf{I}_{\infty o}\mathbf{I}_{\infty} = -\mathbf{I}_{\infty}, \tag{15}$$

$$\mathbf{I}_{\infty a} \wedge \mathbf{I}_{oa} = -E_1E_2E_3, \quad \mathbf{I}_{\infty b} \wedge \mathbf{I}_{ob} = -E_4E_5E_6, \tag{16}$$

$$\mathbf{I}^{\triangleright} = \mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa}\mathbf{I}_{\infty b} \wedge \mathbf{I}_{ob} = -\mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa}E_4E_5E_6, \tag{17}$$

$$(\mathbf{I}^{\triangleright})^2 = (\mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa})^2 = 9, \quad (\mathbf{I}^{\triangleright})^{-1} = \tfrac{1}{9}\mathbf{I}^{\triangleright}, \quad (\mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa})^{-1} = \tfrac{1}{9}\mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa}. \tag{18}$$

$$\mathbf{I}^{\triangleright}_{\infty} \cdot \mathbf{I}^{\triangleright}_{o} = \mathbf{I}^{\triangleright}_{o} \cdot \mathbf{I}^{\triangleright}_{\infty} = \mathbf{I}^{\triangleright}_{\infty} \rfloor \mathbf{I}^{\triangleright}_{o} = \mathbf{I}^{\triangleright}_{\infty} \lfloor \mathbf{I}^{\triangleright}_{o} = -3. \tag{19}$$

We have the following outer products

$$\begin{aligned}\mathbf{I}_{\infty a} &= \mathbf{e}_{\infty 1} \wedge \mathbf{I}^{\triangleright}_{\infty a} = \mathbf{e}_{\infty 2} \wedge \mathbf{I}^{\triangleright}_{\infty a} = \mathbf{e}_{\infty 3} \wedge \mathbf{I}^{\triangleright}_{\infty a} = \mathbf{e}_{\infty} \wedge \mathbf{I}^{\triangleright}_{\infty a} \\ &= \mathbf{e}_{\infty 1}\, \mathbf{I}^{\triangleright}_{\infty a} = \mathbf{e}_{\infty 2}\, \mathbf{I}^{\triangleright}_{\infty a} = \mathbf{e}_{\infty 3}\, \mathbf{I}^{\triangleright}_{\infty a} = \mathbf{e}_{\infty}\, \mathbf{I}^{\triangleright}_{\infty a},\end{aligned} \tag{20}$$

$$\begin{aligned}\mathbf{I}_{oa} &= \mathbf{e}_{o1} \wedge \mathbf{I}^{\triangleright}_{oa} = \mathbf{e}_{o2} \wedge \mathbf{I}^{\triangleright}_{oa} = \mathbf{e}_{o3} \wedge \mathbf{I}^{\triangleright}_{oa} = \tfrac{1}{3}\mathbf{e}_{o} \wedge \mathbf{I}^{\triangleright}_{oa} \\ &= \mathbf{e}_{o1}\, \mathbf{I}^{\triangleright}_{oa} = \mathbf{e}_{o2}\, \mathbf{I}^{\triangleright}_{oa} = \mathbf{e}_{o3}\, \mathbf{I}^{\triangleright}_{oa} = \tfrac{1}{3}\mathbf{e}_{o}\, \mathbf{I}^{\triangleright}_{oa},\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{I}_{\infty a} \wedge \mathbf{I}_{oa} &= E_1 \wedge \mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa} = E_2 \wedge \mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa} = E_3 \wedge \mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa} \\ &= \tfrac{1}{3}E \wedge \mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa} = \tfrac{1}{3}E\, \mathbf{I}^{\triangleright}_{\infty a} \wedge \mathbf{I}^{\triangleright}_{oa}.\end{aligned} \tag{22}$$

And we have the following inner products ($1 \le i \le 6$)

$$\mathbf{I}^{\triangleright}_{oa} = -3\mathbf{e}_{\infty} \cdot \mathbf{I}_{oa}, \quad \mathbf{I}^{\triangleright}_{o} = -3\mathbf{e}_{\infty} \cdot \mathbf{I}_{o}, \tag{23}$$

$$\mathbf{I}^{\triangleright}_{\infty a} = -\mathbf{e}_{o} \cdot \mathbf{I}_{\infty a}, \quad \mathbf{I}^{\triangleright}_{\infty} = -\mathbf{e}_{o} \cdot \mathbf{I}_{\infty}, \tag{24}$$

$$(\mathbf{e}_{oi} \cdot \mathbf{I}_{\infty}) \cdot \mathbf{I}_{o} = -\mathbf{e}_{oi}, \quad (\mathbf{e}_{\infty i} \cdot \mathbf{I}_{o}) \cdot \mathbf{I}_{\infty} = -\mathbf{e}_{\infty i}, \tag{25}$$

$$(\mathbf{e}_{o} \cdot \mathbf{I}_{\infty}) \cdot \mathbf{I}_{o} = -\mathbf{e}_{o}, \quad (\mathbf{e}_{\infty} \cdot \mathbf{I}_{o}) \cdot \mathbf{I}_{\infty} = -\mathbf{e}_{\infty} \tag{26}$$

$$\mathbf{e}_{\infty} \cdot \mathbf{I}_{\infty o} = -\tfrac{1}{3}\mathbf{I}_{\infty} \wedge \mathbf{I}^{\triangleright}_{o}, \qquad \mathbf{e}_{o} \cdot \mathbf{I}_{\infty o} = -\mathbf{I}^{\triangleright}_{\infty} \wedge \mathbf{I}_{o}, \tag{27}$$

$$\mathbf{e}_{\infty i} \cdot \mathbf{I}^{\triangleright}_{\infty a} = 0, \qquad \mathbf{e}_{\infty i} \cdot \mathbf{I}^{\triangleright}_{\infty} = 0, \qquad \mathbf{e}_{\infty} \cdot \mathbf{I}^{\triangleright}_{\infty a} = 0, \qquad \mathbf{e}_{\infty} \cdot \mathbf{I}^{\triangleright}_{\infty} = 0, \tag{28}$$

$$\mathbf{e}_{oi} \cdot \mathbf{I}^{\triangleright}_{oa} = 0, \qquad \mathbf{e}_{oi} \cdot \mathbf{I}^{\triangleright}_{o} = 0, \qquad \mathbf{e}_{o} \cdot \mathbf{I}^{\triangleright}_{oa} = 0, \qquad \mathbf{e}_{o} \cdot \mathbf{I}^{\triangleright}_{o} = 0, \tag{29}$$

$$\mathbf{e}_{\infty} \cdot \mathbf{I}^{\triangleright}_{oa} = 0, \qquad \mathbf{e}_{\infty} \cdot \mathbf{I}^{\triangleright}_{o} = 0, \qquad \mathbf{e}_{o} \cdot \mathbf{I}^{\triangleright}_{\infty a} = 0, \qquad \mathbf{e}_{o} \cdot \mathbf{I}^{\triangleright}_{\infty} = 0, \tag{30}$$

$$\mathbf{e}_{\infty} \cdot \mathbf{I}^{\triangleright} = 0, \qquad \mathbf{e}_{o} \cdot \mathbf{I}^{\triangleright} = 0, \qquad E \cdot \mathbf{I}^{\triangleright} = 0. \tag{31}$$

[3]Note, that the product symbols $\rfloor$ and $\lfloor$ express left- and right contraction, respectively.

As consequence we obtain ($1 \leq i \leq 3$)

$$\begin{aligned} \mathbf{I}_\infty &= \mathbf{e}_{\infty i} \wedge \mathbf{I}_\infty^\triangleright = -\mathbf{I}_\infty^\triangleright \wedge \mathbf{e}_{\infty i} = \mathbf{e}_{\infty i}\, \mathbf{I}_\infty^\triangleright = -\mathbf{I}_\infty^\triangleright\, \mathbf{e}_{\infty i} \\ &= \mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright = -\mathbf{I}_\infty^\triangleright \wedge \mathbf{e}_\infty = \mathbf{e}_\infty\, \mathbf{I}_\infty^\triangleright = -\mathbf{I}_\infty^\triangleright\, \mathbf{e}_\infty, \end{aligned} \tag{32}$$

$$\mathbf{I}_\infty \wedge \mathbf{I}_o^\triangleright = \mathbf{e}_{\infty i} \wedge \mathbf{I}^\triangleright = \mathbf{I}^\triangleright \wedge \mathbf{e}_{\infty i} = \mathbf{e}_\infty \wedge \mathbf{I}^\triangleright = \mathbf{I}^\triangleright \wedge \mathbf{e}_\infty = \mathbf{e}_\infty\, \mathbf{I}^\triangleright = \mathbf{I}^\triangleright\, \mathbf{e}_\infty, \tag{33}$$

$$\begin{aligned} \mathbf{I}_o &= \mathbf{e}_{oi} \wedge \mathbf{I}_o^\triangleright = -\mathbf{I}_o^\triangleright \wedge \mathbf{e}_{oi} = \mathbf{e}_{oi}\, \mathbf{I}_o^\triangleright = -\mathbf{I}_o^\triangleright\, \mathbf{e}_{oi} \\ &= \tfrac{1}{3}\mathbf{e}_o \wedge \mathbf{I}_o^\triangleright = -\tfrac{1}{3}\mathbf{I}_o^\triangleright \wedge \mathbf{e}_o = \tfrac{1}{3}\mathbf{e}_o \mathbf{I}_o^\triangleright = -\tfrac{1}{3}\mathbf{I}_o^\triangleright\, \mathbf{e}_o, \end{aligned} \tag{34}$$

$$-\mathbf{I}_\infty^\triangleright \wedge \mathbf{I}_o = \mathbf{e}_{oi} \wedge \mathbf{I}^\triangleright = \mathbf{I}^\triangleright \wedge \mathbf{e}_{oi} = \tfrac{1}{3}\mathbf{e}_o \wedge \mathbf{I}^\triangleright = \tfrac{1}{3}\mathbf{I}^\triangleright \wedge \mathbf{e}_o = \tfrac{1}{3}\mathbf{e}_o\, \mathbf{I}^\triangleright = \tfrac{1}{3}\mathbf{I}^\triangleright\, \mathbf{e}_o, \tag{35}$$

$$-3\,\mathbf{I}_{\infty o} = E\,\mathbf{I}^\triangleright = E \wedge \mathbf{I}^\triangleright = \mathbf{I}^\triangleright E, \qquad \mathbf{I}^\triangleright = -3E\,\mathbf{I}_{\infty o} = -3\mathbf{I}_{\infty o}\, E. \tag{36}$$

We can summarize the important set of relations

$$\{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \wedge \mathbf{I}_\infty^\triangleright = \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}\, \mathbf{I}_\infty^\triangleright = \mathbf{I}_\infty^\triangleright\, \{1, -\mathbf{e}_o, -\mathbf{e}_\infty, E\}, \tag{37}$$

$$\{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \wedge \mathbf{I}_o^\triangleright = \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}\, \mathbf{I}_o^\triangleright = \mathbf{I}_o^\triangleright\, \{1, -\mathbf{e}_o, -\mathbf{e}_\infty, E\}, \tag{38}$$

$$\{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \wedge \mathbf{I}^\triangleright = \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}\, \mathbf{I}^\triangleright = \mathbf{I}^\triangleright\, \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}. \tag{39}$$

We define the pseudo-scalar $\mathbf{I}_\varepsilon$ in $\mathbb{R}^3$:

$$\mathbf{I}_\varepsilon = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \quad \mathbf{I}_\varepsilon^2 = -1, \quad \mathbf{I}_\varepsilon^{-1} = -\mathbf{I}_\varepsilon, \tag{40}$$

and the conformal pseudo-scalar $\mathbf{I}_\mathrm{C}$ by

$$\mathbf{I}_\mathrm{C} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\, \mathbf{e}_\infty \wedge \mathbf{e}_o = \mathbf{I}_\varepsilon E, \qquad \mathbf{I}_\mathrm{C}^2 = -1, \qquad \mathbf{I}_\mathrm{C}^{-1} = -\mathbf{I}_\mathrm{C}. \tag{41}$$

The full pseudo-scalar $\mathbf{I}$ and its inverse $\mathbf{I}^{-1}$ (used for dualization) are:

$$\mathbf{I} = \mathbf{I}_\varepsilon \mathbf{I}_{\infty o} = -\tfrac{1}{3}\mathbf{I}_\mathrm{C}\mathbf{I}^\triangleright = -\tfrac{1}{3}\mathbf{I}_\varepsilon E \mathbf{I}^\triangleright = -\mathbf{I}_\varepsilon E_1E_2E_3E_4E_5E_6, \tag{42}$$

$$\mathbf{I}^2 = -1, \quad \mathbf{I}^{-1} = -\mathbf{I}. \tag{43}$$

The dual of a multivector indicates division by the pseudo-scalar, e.g., $a^* = -a\mathbf{I}$, $a = a^*\mathbf{I}$. From eq. (1.19) in [8], we have the useful duality between outer and inner products of non-scalar blades A, B in geometric algebra:

$$(A \wedge B)^* = A \cdot B^*, \quad A \wedge (B^*) = (A \cdot B)^* \;\Leftrightarrow\; A \wedge (B\mathbf{I}) = (A \cdot B)\,\mathbf{I}, \tag{44}$$

which indicates that

$$A \wedge B = 0 \;\Leftrightarrow\; A \cdot B^* = 0, \qquad A \cdot B = 0 \;\Leftrightarrow\; A \wedge B^* = 0. \tag{45}$$

Useful duality relationships are

$$\begin{gathered} \mathbf{I}_{\infty o}^* = -\mathbf{I}_\varepsilon, \quad (\mathbf{I}_\infty \wedge \mathbf{I}_o^\triangleright)^* = -3\mathbf{I}_\varepsilon\, \mathbf{e}_\infty, \\ \big(\mathbf{I}_\varepsilon(\mathbf{e}_{oi} \cdot \mathbf{I}_\infty) \wedge \mathbf{I}_o\big)^* = -\mathbf{e}_{oi}, \quad \big(\mathbf{I}_\varepsilon \mathbf{I}_\infty \wedge (\mathbf{e}_{\infty i} \cdot \mathbf{I}_o)\big)^* = -\mathbf{e}_{\infty i}, \\ \big(\mathbf{I}_\varepsilon(\mathbf{e}_o \cdot \mathbf{I}_\infty) \wedge \mathbf{I}_o\big)^* = -\mathbf{e}_o, \quad \big(\mathbf{I}_\varepsilon \mathbf{I}_\infty \wedge (\mathbf{e}_\infty \cdot \mathbf{I}_o)\big)^* = -\mathbf{e}_\infty. \end{gathered} \tag{46}$$

3. QCGA objects

QCGA is an extension of CGA; thus the objects defined in CGA are also defined in QCGA. The following sections introduce the important definition of a general point in QCGA, and show next how all round and flat geometric objects (point pairs, flat points, circles, lines, spheres, planes) of CGA can straightforwardly be embedded in QCGA.

3.1. Point in QCGA

The point $\mathbf{x}$ of QCGA corresponding to the Euclidean point $\mathbf{x}_\varepsilon = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \in \mathbb{R}^3$, is defined as

$$\mathbf{x} = \mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty1} + y^2\mathbf{e}_{\infty2} + z^2\mathbf{e}_{\infty3}) + xy\mathbf{e}_{\infty4} + xz\mathbf{e}_{\infty5} + yz\mathbf{e}_{\infty6} + \mathbf{e}_o. \tag{47}$$

Note that the null vectors $\mathbf{e}_{o4}, \mathbf{e}_{o5}, \mathbf{e}_{o6}$ are not present in the definition of the point. This is merely to keep the convenient properties of the CGA points, namely, the inner product between two points is identical with the squared distance between them. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points, their inner product is

$$\begin{aligned}
&\mathbf{x}_1 \cdot \mathbf{x}_2 \\
&= (\mathbf{x}_{1\varepsilon} + \tfrac{1}{2}x_1^2\mathbf{e}_{\infty1} + \tfrac{1}{2}y_1^2\mathbf{e}_{\infty2} + \tfrac{1}{2}z_1^2\mathbf{e}_{\infty3} + x_1y_1\mathbf{e}_{\infty4} + x_1z_1\mathbf{e}_{\infty5} + y_1z_1\mathbf{e}_{\infty6} + \mathbf{e}_o) \\
&\quad \cdot (\mathbf{x}_{2\varepsilon} + \tfrac{1}{2}x_2^2\mathbf{e}_{\infty1} + \tfrac{1}{2}y_2^2\mathbf{e}_{\infty2} + \tfrac{1}{2}z_2^2\mathbf{e}_{\infty3} + x_2y_2\mathbf{e}_{\infty4} + x_2z_2\mathbf{e}_{\infty5} + y_2z_2\mathbf{e}_{\infty6} + \mathbf{e}_o).
\end{aligned} \tag{48}$$

from which together with Table 1, it follows that

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_{1\varepsilon} \cdot \mathbf{x}_{2\varepsilon} - \tfrac{1}{2}(x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2) = -\tfrac{1}{2}(\mathbf{x}_{1\varepsilon} - \mathbf{x}_{2\varepsilon})^2. \tag{49}$$

We see that the inner product is equivalent to the minus half of the squared Euclidean distance between $\mathbf{x}_1$ and $\mathbf{x}_2$.

In the remainder of the paper the following result will be useful, because it relates a point in QCGA to the representation it would have in CGA $\mathbb{R}^{4,1}$ with vector basis $\{\mathbf{e}_o, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_\infty\}$.

$$\begin{aligned}
\mathbf{x} \wedge \mathbf{I}_\infty^\triangleright &= \Big(\mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty1} + y^2\mathbf{e}_{\infty2} + z^2\mathbf{e}_{\infty3}) + \mathbf{e}_o\Big) \wedge \mathbf{I}_\infty^\triangleright \\
&= (\mathbf{x}_\varepsilon + \mathbf{e}_o) \wedge \mathbf{I}_\infty^\triangleright + \tfrac{1}{2}(x^2\mathbf{e}_{\infty1} + y^2\mathbf{e}_{\infty2} + z^2\mathbf{e}_{\infty3}) \wedge \mathbf{I}_\infty^\triangleright \\
&= (\mathbf{x}_\varepsilon + \mathbf{e}_o) \wedge \mathbf{I}_\infty^\triangleright + \tfrac{1}{2}(x^2 + y^2 + z^2)\,\mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright \\
&= (\mathbf{x}_\varepsilon + \mathbf{e}_o) \wedge \mathbf{I}_\infty^\triangleright + \tfrac{1}{2}\mathbf{x}_\varepsilon^2\,\mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright \\
&= (\mathbf{x}_\varepsilon + \tfrac{1}{2}\mathbf{x}_\varepsilon^2\mathbf{e}_\infty + \mathbf{e}_o) \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_\mathrm{C} \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_\mathrm{C}\,\mathbf{I}_\infty^\triangleright,
\end{aligned} \tag{50}$$

where we have dropped in the first line the cross terms $xy\mathbf{e}_{\infty4} + xz\mathbf{e}_{\infty5} + yz\mathbf{e}_{\infty6}$, because wedging with $\mathbf{I}_{\infty a}$, a factor in $\mathbf{I}_\infty^\triangleright$, eliminates them. Therefore, if a point in QCGA appears wedged with $\mathbf{I}_\infty^\triangleright$, we can replace it by the form

$$\mathbf{x}_\mathrm{C} = \mathbf{x}_\varepsilon + \tfrac{1}{2}\mathbf{x}_\varepsilon^2\mathbf{e}_\infty + \mathbf{e}_o \overset{(19)}{=} -\tfrac{1}{3}(\mathbf{x} \wedge \mathbf{I}_\infty^\triangleright) \rfloor \mathbf{I}_o^\triangleright. \tag{51}$$

it would have in CGA. This in turn means, that we can embed in QCGA the known CGA representations of round and flat objects, by taking the outer products of between one and five points with $\mathbf{I}_\infty^\triangleright$, as shown in the following.

3.2. Round and flat objects in QCGA

With round objects, we mean points, point pairs, circles and spheres with uniform curvature. Similar to CGA, these can be defined by the outer product of one to four points with $\mathbf{I}_\infty^\triangleright$. Their center $\mathbf{c}_\mathrm{C}$, radius r and Euclidean carrier blade D can be easily extracted. Alternatively, they can be directly constructed from their center $\mathbf{c}_\mathrm{C}$, radius r and Euclidean carrier D.

Wedging any round object with the point at infinity $\mathbf{e}_\infty$, gives the corresponding flat object multivector. From it the orthogonal distance to the origin $\mathbf{c}_{\varepsilon\perp}$ and the Euclidean carrier D can easily be extracted.

We now briefly review the CGA description of round and flat objects embedded in QCGA. The round objects are point, point pair, circle and sphere,

$$P = \mathbf{x} \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_\mathrm{C}\,\mathbf{I}_\infty^\triangleright, \tag{52}$$

$$Pp = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1\mathrm{C}} \wedge \mathbf{x}_{2\mathrm{C}}\,\mathbf{I}_\infty^\triangleright, \tag{53}$$

$$Circle = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1\mathrm{C}} \wedge \mathbf{x}_{2\mathrm{C}} \wedge \mathbf{x}_{3\mathrm{C}}\,\mathbf{I}_\infty^\triangleright, \tag{54}$$

$$Sphere = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1\mathrm{C}} \wedge \mathbf{x}_{2\mathrm{C}} \wedge \mathbf{x}_{3\mathrm{C}} \wedge \mathbf{x}_{4\mathrm{C}}\,\mathbf{I}_\infty^\triangleright. \tag{55}$$

The corresponding flat objects are flat point, line, plane and the whole three-dimensional space[4],

$$Flatp = -P \wedge \mathbf{e}_\infty = \mathbf{x} \wedge \mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_\mathrm{C} \wedge \mathbf{e}_\infty\,\mathbf{I}_\infty^\triangleright, \tag{56}$$

$$Line = -Pp \wedge \mathbf{e}_\infty = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1\mathrm{C}} \wedge \mathbf{x}_{2\mathrm{C}} \wedge \mathbf{e}_\infty\,\mathbf{I}_\infty^\triangleright, \tag{57}$$

$$Plane = -Circle \wedge \mathbf{e}_\infty = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1\mathrm{C}} \wedge \mathbf{x}_{2\mathrm{C}} \wedge \mathbf{x}_{3\mathrm{C}} \wedge \mathbf{e}_\infty\,\mathbf{I}_\infty^\triangleright, \tag{58}$$

$$\begin{aligned} Space &= -Sphere \wedge \mathbf{e}_\infty = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 \wedge \mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright \\ &= \mathbf{x}_{1\mathrm{C}} \wedge \mathbf{x}_{2\mathrm{C}} \wedge \mathbf{x}_{3\mathrm{C}} \wedge \mathbf{x}_{4\mathrm{C}} \wedge \mathbf{e}_\infty\,\mathbf{I}_\infty^\triangleright. \end{aligned} \tag{59}$$

The above embeddings by means of the outer product with $\mathbf{I}_\infty^\triangleright$, allow to make use of standard CGA results found in [8]. All embedded round entities of point, point pair, circle and sphere have one common multivector form[5]

$$S = \Big(D \wedge \mathbf{c}_\varepsilon + [\tfrac{1}{2}(\mathbf{c}_\varepsilon^2 + r^2)D - \mathbf{c}_\varepsilon \mathbf{c}_\varepsilon \rfloor D]\mathbf{e}_\infty + D\mathbf{e}_o + D\lfloor \mathbf{c}_\varepsilon E\Big)\mathbf{I}_\infty^\triangleright = S_\mathrm{C}\,\mathbf{I}_\infty^\triangleright, \tag{60}$$

$$S_\mathrm{C} = -\tfrac{1}{3} S \lfloor \mathbf{I}_o^\triangleright.$$

The Euclidean carriers D are for each object

$$D = \begin{cases} 1, & \text{point } \mathbf{x} \\ \mathbf{d}_\varepsilon, & \text{point pair } Pp \\ \mathbf{i}_c, & \text{circle } Circle \\ \mathbf{I}_\varepsilon, & \text{sphere } Sphere, \end{cases} \tag{61}$$

where the unit point pair connection direction vector is $\mathbf{d}_\varepsilon = (\mathbf{x}_{1\varepsilon} - \mathbf{x}_{2\varepsilon})/2r$ and the Euclidean circle plane bivector $\mathbf{i}_c$. The radius r of a round object and its center $\mathbf{c}_\mathrm{C}$

[4] The leading minus sign comes from $\mathbf{I}_\infty^\triangleright \wedge \mathbf{e}_\infty = -\mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright$ of (37).

[5] Note, that the left- and right contraction $\rfloor$ and $\lfloor$, respectively, are needed essentially.

are generally determined by

$$r^2 = \frac{S_\mathrm{C}\widetilde{S_\mathrm{C}}}{(S_\mathrm{C}\wedge\mathbf{e}_\infty)(S_\mathrm{C}\wedge\mathbf{e}_\infty)^\sim}, \qquad \mathbf{c}_\mathrm{C} = S_\mathrm{C}\,\mathbf{e}_\infty\,S_\mathrm{C}. \tag{62}$$

where $\widetilde{S_\mathrm{C}}$ indicates the reverse of S_C.

All embedded flat entities of flat point, line, plane and space have one common multivector form

$$F = -S\wedge\mathbf{e}_\infty = (D\wedge\mathbf{c}_\varepsilon\mathbf{e}_\infty - DE)\,\mathbf{I}_\infty^\triangleright = (D\mathbf{c}_{\varepsilon\perp}\mathbf{e}_\infty - DE)\,\mathbf{I}_\infty^\triangleright = F_\mathrm{C}\,\mathbf{I}_\infty^\triangleright, \tag{63}$$
$$F_\mathrm{C} = -S_\mathrm{C}\wedge\mathbf{e}_\infty = \tfrac{1}{3}F\,\lfloor\,\mathbf{I}_o^\triangleright,$$

where the orthogonal Euclidean distance of the flat object from the origin is

$$\mathbf{c}_{\varepsilon\perp} = \begin{cases} \mathbf{x}_\varepsilon, & \text{finite-infinite point pair } Flatp \\ \mathbf{c}_{\varepsilon\perp}, & \text{line } Line \\ \mathbf{c}_{\varepsilon\perp}, & \text{plane } Plane \\ 0, & \text{3D space } Space. \end{cases} \tag{64}$$

The Euclidean carrier blade D, and the orthogonal Euclidean distance vector of F from the origin, can both be directly determined from the flat object multivector as

$$D = F_\mathrm{C}\lfloor E, \qquad \mathbf{c}_{\varepsilon\perp} = D^{-1}(F_\mathrm{C}\wedge\mathbf{e}_o)\lfloor E. \tag{65}$$

4. Quadric surfaces

This section describes how QCGA handles quadric surface. All embedded CGA objects in QCGA defined in Section 3 are thus part of a more general framework.

A quadric in $\mathbb{R}^3$ is formulated as

$$F(x,y,z) = \mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + \mathrm{d}xy + \mathrm{e}xz + \mathrm{f}yz + \mathrm{g}x + \mathrm{h}y + \mathrm{i}z + \mathrm{j} = 0. \tag{66}$$

A quadric surface is constructed by wedging nine points together as follows

$$\mathbf{q} = \mathbf{x}_1\wedge\mathbf{x}_2\wedge\cdots\wedge\mathbf{x}_9. \tag{67}$$

The multivector $\mathbf{q}$ corresponds to the primal form of a quadric in QCGA, with grade 9 and 12 components. Again three of these components have the same coefficient and can be combined together in a form defined by only ten coefficients a, b, ..., j. If we further wedge the 9-blade $\mathbf{q}$ with the simple 5-vector $\mathbf{I}_0^\triangleright$, we obtain a 14-blade and its dual vector $(\mathbf{q}\wedge\mathbf{I}_0^\triangleright)^*$

$$\begin{aligned}(\mathbf{q}\wedge\mathbf{I}_0^\triangleright)^* &= \mathbf{I}_\varepsilon\big((2\mathrm{a}\mathbf{e}_{o1} + 2\mathrm{b}\mathbf{e}_{o2} + 2\mathrm{c}\mathbf{e}_{o3} + \mathrm{d}\mathbf{e}_{o4} + \mathrm{e}\mathbf{e}_{o5} + \mathrm{f}\mathbf{e}_{o6})\cdot\mathbf{I}_\infty\big)\wedge\mathbf{I}_o \\ &\qquad + (\mathrm{g}\mathbf{e}_1 + \mathrm{h}\mathbf{e}_2 + \mathrm{i}\mathbf{e}_3)\mathbf{I}_\varepsilon\,\mathbf{I}_{\infty o} + \mathrm{j}\,\mathbf{I}_\varepsilon\,\mathbf{I}_\infty\wedge(\mathbf{e}_\infty\cdot\mathbf{I}_o) \\ &= \big(-(2\mathrm{a}\mathbf{e}_{o1} + 2\mathrm{b}\mathbf{e}_{o2} + 2\mathrm{c}\mathbf{e}_{o3} + \mathrm{d}\mathbf{e}_{o4} + \mathrm{e}\mathbf{e}_{o5} + \mathrm{f}\mathbf{e}_{o6}) \\ &\qquad + \mathrm{g}\mathbf{e}_1 + \mathrm{h}\mathbf{e}_2 + \mathrm{i}\mathbf{e}_3 - \mathrm{j}\mathbf{e}_\infty\big)\,\mathbf{I} \\ &= (\mathbf{q}\wedge\mathbf{I}_0^\triangleright)^*\,\mathbf{I},\end{aligned} \tag{68}$$

where in the second equality we used the duality relationships of (46). The expression for the dual vector $(\mathbf{q}\wedge\mathbf{I}_0^{\triangleright})^*$ is therefore

$$\begin{aligned}(\mathbf{q}\wedge\mathbf{I}_0^{\triangleright})^* = -(2\mathrm{a}\mathbf{e}_{o1}+2\mathrm{b}\mathbf{e}_{o2}+2\mathrm{c}\mathbf{e}_{o3}+\mathrm{d}\mathbf{e}_{o4}+\mathrm{e}\mathbf{e}_{o5}+\mathrm{f}\mathbf{e}_{o6})\\ +\mathrm{g}\mathbf{e}_1+\mathrm{h}\mathbf{e}_2+\mathrm{i}\mathbf{e}_3-\mathrm{j}\mathbf{e}_\infty.\end{aligned} \tag{69}$$

Proposition 4.1. *A point* $\mathbf{x}$ *lies on the quadric surface* $\mathbf{q}$ *if and only* $\mathbf{x}\wedge\mathbf{q}\wedge\mathbf{I}_o^{\triangleright}=0$.

Proof.

$$\begin{aligned}\mathbf{x}\wedge(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright}) &= \mathbf{x}\wedge((\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\mathbf{I}) = \mathbf{x}\cdot(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\,\mathbf{I}\\ &= \mathbf{x}\cdot\big(-(2\mathrm{a}\mathbf{e}_{o1}+2\mathrm{b}\mathbf{e}_{o2}+2\mathrm{c}\mathbf{e}_{o3}+\mathrm{d}\mathbf{e}_{o4}+\mathrm{e}\mathbf{e}_{o5}+\mathrm{f}\mathbf{e}_{o6})\\ &\qquad +\mathrm{g}\mathbf{e}_1+\mathrm{h}\mathbf{e}_2+\mathrm{i}\mathbf{e}_3-\mathrm{j}\mathbf{e}_\infty\big)\,\mathbf{I}\\ &= (\mathrm{a}x^2+\mathrm{b}y^2+\mathrm{c}z^2+\mathrm{d}xy+\mathrm{e}xz+\mathrm{f}yz+\mathrm{g}x+\mathrm{h}y+\mathrm{i}z+\mathrm{j})\,\mathbf{I}.\end{aligned} \tag{70}$$

This corresponds to the formula representing a general quadric. □

The dualization of the primal quadric wedged with $\mathbf{I}_o^{\triangleright}$ leads to the 1-vector dual form $(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*$ of (69).

Corollary 4.2. *A point* $\mathbf{x}$ *lies on the quadric defined by* $\mathbf{q}$ *if and only if* $\mathbf{x}\cdot(\mathbf{q}\wedge\mathbf{I}_0^{\triangleright})^*=0$.

The ten coefficients $\{a,\ldots,j\}$ of the quadric equation (66) can be easily extracted from the quadric 9-blade $\mathbf{q}$ of (67) by computing the following scalar products with vector $(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*$ as

$$\begin{aligned}&a=\tfrac{1}{2}(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty1},\quad b=\tfrac{1}{2}(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty2},\quad c=\tfrac{1}{2}(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty3},\\ &d=(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty4},\quad e=(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty5},\quad f=(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_{\infty6},\\ &g=(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_1,\quad h=(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_2,\quad i=(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_3,\\ &j=(\mathbf{q}\wedge\mathbf{I}_o^{\triangleright})^*\cdot\mathbf{e}_o.\end{aligned} \tag{71}$$

5. Intersections

Any number of linearly independent round or flat embedded CGA objects in QCGA and any number of quadrics $\{\mathbf{A},\mathbf{B},\ldots,\mathbf{Z}\}$, after wedging with the simple 5-vector $\mathbf{I}_o^{\triangleright}$, can be intersected by computing the dual of the outer product of their duals

$$(\mathrm{intersect}\wedge\mathbf{I}_o^{\triangleright})^* = (\mathbf{A}\wedge\mathbf{I}_o^{\triangleright})^*\wedge(\mathbf{B}\wedge\mathbf{I}_o^{\triangleright})^*\wedge\ldots\wedge(\mathbf{Z}\wedge\mathbf{I}_o^{\triangleright})^*. \tag{72}$$

A criterion for a general point $\mathbf{x}$ to be on the intersection is

$$\mathbf{x}\cdot(\mathrm{intersect}\wedge\mathbf{I}_o^{\triangleright})^*=0,\qquad \mathrm{intersect}=-\tfrac{1}{3}\big((\mathrm{intersect}\wedge\mathbf{I}_o^{\triangleright})^*\mathbf{I}\big)\rfloor\mathbf{I}_\infty^{\triangleright}. \tag{73}$$

For cases that one object is wholly included in another object (like a line in a plane), the proper meet operation has to be defined, taking into account the subspace spanned by the join of the two objects [10].

6. Versors for rotation, translation and scaling

For the successful implementation of rotations together with a comparatively simple form for the translation versors, we found it essential to define the null vector pairs $\{\mathbf{e}_{\infty i}, \mathbf{e}_{oi}\}$, $i = 1, \ldots, 6$, in the *symmetric* fashion of (1)

$$\mathbf{e}_{\infty i} = \tfrac{1}{\sqrt{2}}(\mathbf{e}_{+i} + \mathbf{e}_{-i}), \qquad \mathbf{e}_{oi} = \tfrac{1}{\sqrt{2}}(\mathbf{e}_{-i} - \mathbf{e}_{+i}).$$

Only with this definition[6] for $\{\mathbf{e}_{\infty i}, \mathbf{e}_{oi}\}$, $i = 4,5,6$, were we able to keep the number of versor factors to a minimum for achieving translations, as seen below in equations (87) to (92), our approach thus has the advantage of completely eliminating the quadratic terms in the translation vector coordinates from the exponents of the factors in the translation operators, compared to [9], where these terms are already needed to achieve translations by versors in two dimensions.

6.1. Versors for rotation of three dimensional quadrics

Rotations around the z-axis are generated by the following five bivectors

$$\begin{gathered} e_{12}, \quad B_{z2} = \tfrac{1}{2}\mathbf{e}_{o4}\mathbf{e}_{\infty 12}^{\triangleright}, \quad B_{z3} = \mathbf{e}_{\infty 4}\mathbf{e}_{o12}^{\triangleright}, \\ B_{z4} = -\mathbf{e}_{o5}\mathbf{e}_{\infty 6}, \quad B_{z5} = -\mathbf{e}_{\infty 5}\mathbf{e}_{o6}, \quad B_{zi}^2 = 0 \quad (i = 2,3,4,5), \end{gathered} \tag{74}$$

in the rotor form of $R_z = R_{z1}(R_{z2} \wedge R_{z3})(R_{z4} \wedge R_{z5})$, where

$$\begin{gathered} R_{z1} = e^{\frac{\varphi}{2}e_{12}}, \qquad R_{zi} = \cos\varphi + \sin\varphi B_{zi} \quad (i = 2,3), \\ R_{zi} = \cos\tfrac{\varphi}{2} + \sin\tfrac{\varphi}{2} B_{zi} \quad (i = 4,5). \end{gathered} \tag{75}$$

Applying the rotor R_z to a conformal point $\mathbf{x}$ leads to

$$\mathbf{x}' = \widetilde{R_z}\mathbf{x}R_z, \qquad \mathbf{x}'_\varepsilon = x'\mathbf{e}_1 + y'\mathbf{e}_2 + z'\mathbf{e}_3, \tag{76}$$

$$x' = \cos\varphi x - \sin\varphi y, \qquad y' = \sin\varphi x + \cos\varphi y, \qquad z' = z. \tag{77}$$

an anticlockwise (mathematically positive) rotation in the e_{12}-plane, as seen from the positive z-direction.

Rotations around the x-axis are generated by the following five bivectors

$$\begin{gathered} e_{23}, \quad B_{x2} = \tfrac{1}{2}\mathbf{e}_{o6}(\mathbf{e}_{\infty 2} - \mathbf{e}_{\infty 3}), \quad B_{x3} = \mathbf{e}_{\infty 6}(\mathbf{e}_{o2} - \mathbf{e}_{o3}), \\ B_{x4} = -\mathbf{e}_{o4}\mathbf{e}_{\infty 5}, \quad B_{x5} = -\mathbf{e}_{\infty 4}\mathbf{e}_{o5}, \quad B_{xi}^2 = 0 \quad (i = 2,3,4,5), \end{gathered} \tag{78}$$

in the rotor form of $R_x = R_{x1}(R_{x2} \wedge R_{x3})(R_{x4} \wedge R_{x5})$, where

$$\begin{gathered} R_{x1} = e^{\frac{\varphi}{2}e_{23}}, \qquad R_{xi} = \cos\varphi + \sin\varphi B_{xi} \quad (i = 2,3), \\ R_{xi} = \cos\tfrac{\varphi}{2} + \sin\tfrac{\varphi}{2} B_{xi} \quad (i = 4,5). \end{gathered} \tag{79}$$

Applying the rotor R_x to a conformal point $\mathbf{x}$ leads to

$$\mathbf{x}' = \widetilde{R_x}\mathbf{x}R_x, \qquad \mathbf{x}'_\varepsilon = x'\mathbf{e}_1 + y'\mathbf{e}_2 + z'\mathbf{e}_3, \tag{80}$$

$$x' = x, \qquad y' = \cos\varphi y - \sin\varphi z, \qquad z' = \sin\varphi y + \cos\varphi z. \tag{81}$$

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[6]For $\{\mathbf{e}_{\infty k}, \mathbf{e}_{ok}\}$, $k = 1,2,3$, we could also have chosen $\mathbf{e}_{\infty i} = \mathbf{e}_{+i} + \mathbf{e}_{-i}$, $\mathbf{e}_{oi} = \frac{1}{2}(\mathbf{e}_{-i} - \mathbf{e}_{+i})$ as in [8], without altering our form of the transformation versors given below. But for aesthetic reasons, we decided in (1) to simply set all six coefficients symmetrically to $1/\sqrt{2}$.

an anticlockwise (mathematically positive) rotation in the e_{23}-plane, as seen from the positive x-direction.

Rotations around the y-axis are generated by the following five bivectors

$$e_{31}, \quad B_{y2} = \tfrac{1}{2}\mathbf{e}_{o5}(\mathbf{e}_{\infty 3} - \mathbf{e}_{\infty 1}), \quad B_{y3} = \mathbf{e}_{\infty 5}(\mathbf{e}_{o3} - \mathbf{e}_{o1}),$$
$$B_{y4} = -\mathbf{e}_{o6}\mathbf{e}_{\infty 4}, \quad B_{y5} = -\mathbf{e}_{\infty 6}\mathbf{e}_{o4}, \quad B_{yi}^2 = 0 \quad (i = 2,3,4,5), \tag{82}$$

in the rotor form of $R_y = R_{y1}(R_{y2} \wedge R_{y3})(R_{y4} \wedge R_{y5})$, where

$$R_{y1} = e^{\frac{\varphi}{2}e_{31}}, \qquad R_{yi} = \cos\varphi + \sin\varphi B_{yi} \quad (i = 2,3),$$
$$R_{yi} = \cos\tfrac{\varphi}{2} + \sin\tfrac{\varphi}{2} B_{yi} \quad (i = 4,5). \tag{83}$$

Applying the rotor R_y to a conformal point $\mathbf{x}$ leads to

$$\mathbf{x}' = \widetilde{R}_y \mathbf{x} R_y, \qquad \mathbf{x}'_\varepsilon = x'\mathbf{e}_1 + y'\mathbf{e}_2 + z'\mathbf{e}_3, \tag{84}$$
$$x' = \sin\varphi z + \cos\varphi x, \qquad y' = y, \qquad z' = \cos\varphi z - \sin\varphi x. \tag{85}$$

an anticlockwise (mathematically positive) rotation in the e_{31}-plane, as seen from the positive y-direction.

We note the useful invariance relationships, that

$$\widetilde{R}\mathbf{a}R = \mathbf{a} \quad \forall \mathbf{a} \in \{1, \mathbf{e}_o, \mathbf{e}_\infty, E, \mathbf{I}_\varepsilon, \mathbf{I}_C, \mathbf{I}_o^\triangleright, \mathbf{I}_o, \mathbf{I}_\infty^\triangleright, \mathbf{I}_\infty, \mathbf{I}_{\infty o}, \mathbf{I}^\triangleright, \mathbf{I}\}. \tag{86}$$

Because embedded flat and round CGA objects and all quadrics are constructed from outer products of between one and nine points and $\mathbf{I}_\infty^\triangleright$, these objects are naturally covariant under rotations. Another consequence of this invariance is, that equations like (72), involving outer products with $\mathbf{I}_o^\triangleright$, are also covariant under rotations. Rotation covariance will certainly be of great value, when QCGA is employed, e.g., for the construction of feature multivectors. We further note, that in general the vectors $\mathbf{e}_{\infty i}$, $i = 4,5,6$, are not rotation invariant, which is natural for outer product factors used to yield axis aligned quadrics, which being axis aligned, can not be expected to be rotation covariant.

General rotations can be achieved by a sequence of Euler angle rotations around the z-, the x'-, and again the z''-axis, combining the above axis rotations. In doing so, the factors $R_{z1}R_{x'1}R_{z''1}$ will commute with all other factors, and form a conventional three dimensional Euclidean geometric algebra rotor. This description of rotations in QCGA with versors is possible, but it may not yet be the most elegant one. However, the aim of the current work is first of all to constructively demonstrate, that the description of rotations in QCGA by applying suitable versors is at all indeed possible.

6.2. Versors for translation of three dimensional quadrics

Translation by distance $a \in \mathbb{R}$ in the direction of $\mathbf{e}_1$ is achieved by the versor $T_x = T_{x1}T_{x2}T_{x3}$, with

$$T_{x1} = e^{\frac{1}{2}a\mathbf{e}_1\mathbf{e}_{\infty 1}} = 1 + \tfrac{1}{2}a\mathbf{e}_1\mathbf{e}_{\infty 1},$$
$$T_{x2} = e^{\frac{1}{2}a\mathbf{e}_2\mathbf{e}_{\infty 4}} = 1 + \tfrac{1}{2}a\mathbf{e}_2\mathbf{e}_{\infty 4}, \qquad T_{x3} = e^{\frac{1}{2}a\mathbf{e}_3\mathbf{e}_{\infty 5}} = 1 + \tfrac{1}{2}a\mathbf{e}_3\mathbf{e}_{\infty 5}, \tag{87}$$

which leads to

$$\mathbf{x}' = \widetilde{T}_x \mathbf{x} T_x, \qquad \mathbf{x}'_\varepsilon = (x+a)\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3. \tag{88}$$

Note, that the products of any two null bivector generator blades from $\{\mathbf{e}_1\mathbf{e}_{\infty 1}, \mathbf{e}_2\mathbf{e}_{\infty 4}, \mathbf{e}_3\mathbf{e}_{\infty 5}\}$, commute and form null quadvector blades (squaring to zero), and the product of all three null bivectors forms a null 6-blade. Therefore, the versor factors $\{T_{x1}, T_{x2}, T_{x3}\}$ commute pairwise. Analogous relationships apply to the two sets of three null bivectors each, generating the translations in the y- and z-directions, as can be seen below.

Translation by distance $b \in \mathbb{R}$ in the direction of $\mathbf{e}_2$ is similarly achieved by the versor $T_y = T_{y1}T_{y2}T_{y3}$, with

$$\begin{aligned} &T_{y1} = e^{\frac{1}{2}b\mathbf{e}_2\mathbf{e}_{\infty 2}} = 1 + \tfrac{1}{2}b\mathbf{e}_2\mathbf{e}_{\infty 2}, \\ &T_{y2} = e^{\frac{1}{2}b\mathbf{e}_3\mathbf{e}_{\infty 6}} = 1 + \tfrac{1}{2}b\mathbf{e}_3\mathbf{e}_{\infty 6}, \qquad T_{y3} = e^{\frac{1}{2}b\mathbf{e}_1\mathbf{e}_{\infty 4}} = 1 + \tfrac{1}{2}b\mathbf{e}_1\mathbf{e}_{\infty 4}, \end{aligned} \tag{89}$$

which leads to

$$\mathbf{x}' = \widetilde{T}_y \mathbf{x} T_y, \qquad \mathbf{x}'_\varepsilon = x\mathbf{e}_1 + (y+b)\mathbf{e}_2 + z\mathbf{e}_3. \tag{90}$$

Note, that the versor factors $\{T_{y1}, T_{y2}, T_{y3}\}$ commute pairwise.

Translation by distance $c \in \mathbb{R}$ in the direction of $\mathbf{e}_3$ is similarly achieved by the versor $T_z = T_{z1}T_{z2}T_{z3}$, with

$$\begin{aligned} &T_{z1} = e^{\frac{1}{2}b\mathbf{e}_3\mathbf{e}_{\infty 3}} = 1 + \tfrac{1}{2}b\mathbf{e}_3\mathbf{e}_{\infty 3}, \\ &T_{z2} = e^{\frac{1}{2}b\mathbf{e}_1\mathbf{e}_{\infty 5}} = 1 + \tfrac{1}{2}b\mathbf{e}_1\mathbf{e}_{\infty 5}, \qquad T_{z3} = e^{\frac{1}{2}b\mathbf{e}_2\mathbf{e}_{\infty 6}} = 1 + \tfrac{1}{2}b\mathbf{e}_2\mathbf{e}_{\infty 6}, \end{aligned} \tag{91}$$

which leads to

$$\mathbf{x}' = \widetilde{T}_z \mathbf{x} T_z, \qquad \mathbf{x}'_\varepsilon = x\mathbf{e}_1 + y\mathbf{e}_2 + (z+c)\mathbf{e}_3. \tag{92}$$

Note, that the versor factors $\{T_{z1}, T_{z2}, T_{z3}\}$ commute pairwise.

We point out, that the versors $\{T_x, T_y, T_z\}$ do not commute pairwise.

In total we have for the application to quadric conformal points

$$\mathbf{x}' = \widetilde{T}\mathbf{x}T, \quad T = T_xT_yT_z, \quad \mathbf{x}'_\varepsilon = (x+a)\mathbf{e}_1 + (y+b)\mathbf{e}_2 + (z+c)\mathbf{e}_3. \tag{93}$$

It is interesting to remark, that in spite of the above noted lack of pairwise commutation of $\{T_x, T_y, T_z\}$, we do have applied to any point $\mathbf{x}$ that

$$\mathbf{x}' = \widetilde{T}\mathbf{x}T = \widetilde{T_xT_yT_z}\mathbf{x}T_xT_yT_z = \widetilde{T_yT_xT_z}\mathbf{x}T_yT_xT_z, \quad \text{etc.} \tag{94}$$

That is applied to a point $\mathbf{x}$, the versor factors $\{T_x, T_y, T_z\}$ in $T = T_xT_yT_z$ can be freely commuted, without changing the result. We further note, that we have only bivector terms *linear* in displacement distances a, b, c, in each of the nine elementary versors $T_{x1}, \ldots, T_{z3}$. We expect this to be advantageous in application to optimization problems.

We note the useful invariance relationships, that

$$\widetilde{T}\mathbf{a}T = \mathbf{a} \quad \forall \mathbf{a} \in \{1, \mathbf{e}_{\infty 1}, \mathbf{e}_{\infty 2}, \mathbf{e}_{\infty 3}, \mathbf{e}_{\infty 4}, \mathbf{e}_{\infty 5}, \mathbf{e}_{\infty 6}, \mathbf{e}_\infty, \mathbf{I}_\infty^\triangleright, \mathbf{I}_\infty, \mathbf{I}\}. \tag{95}$$

By construction therefore all flat and round CGA objects and all quadrics are translation covariant, this includes, because of $T\mathbf{e}_{\infty i}\widetilde{T} = \mathbf{e}_{\infty i}$, $i = 4,5,6$, axis aligned quadrics as well, since translation has no effect on axis alignment.

6.3. Versors for scaling of three dimensional quadrics

Scaling by positive scalar $\alpha \in \mathbb{R}$ is achieved with the help of the scaling operator or scaling versor (scalor) $S = S_1S_2S_3S_4S_5S_6$, where

$$S_k = \tfrac{1}{2}\left(\tfrac{\alpha+1}{\sqrt{\alpha}} + \tfrac{\alpha-1}{\sqrt{\alpha}}E_k\right), \qquad \widetilde{S}_kS_k = S_k\widetilde{S}_k = 1, \qquad 1 \le k \le 6. \tag{96}$$

Note that the three factors S_k, $1 \le k \le 6$, mutually commute. This leads to (isotropic) scaling of points

$$\mathbf{x}' = \alpha\,\widetilde{S}\mathbf{x}S, \qquad \mathbf{x}'_\varepsilon = \alpha\,\mathbf{x}_\varepsilon. \tag{97}$$

Note that the overall factor α could be omitted in $\mathbf{x}' = \alpha\,S\mathbf{x}\widetilde{S}$, due to the homogeneity of the point representation $\mathbf{x}$, but we include it for convenience, such that $\mathbf{x}' \cdot \mathbf{e}_\infty = \mathbf{x} \cdot \mathbf{e}_\infty = -1$.

The following multivector elements are invariant under scaling (97)

$$\mathbf{e}_o, \mathbf{e}_{o1}, \mathbf{e}_{o2}, \mathbf{e}_{o3}, \mathbf{e}_{o4}, \mathbf{e}_{o5}, \mathbf{e}_{o6}, \mathbf{I}_{oa}, \mathbf{I}^{\triangleright}_{oa}, \mathbf{I}_{ob}, \mathbf{I}^{\triangleright}_{o}, \mathbf{I}_o. \tag{98}$$

For bivectors this means

$$\mathbf{I}^{\triangleright}_{oa} = \alpha^2\widetilde{S}\mathbf{I}^{\triangleright}_{oa}S, \tag{99}$$

for trivectors

$$\{\mathbf{I}_{oa}, \mathbf{I}_{ob}\} = \alpha^3\widetilde{S}\{\mathbf{I}_{oa}, \mathbf{I}_{ob}\}S, \tag{100}$$

for 5-vectors

$$\mathbf{I}^{\triangleright}_{o} = \alpha^5\widetilde{S}\mathbf{I}^{\triangleright}_{o}S, \tag{101}$$

and for the 6-vector

$$\mathbf{I}_o = \alpha^6\,\widetilde{S}\mathbf{I}_o\,S. \tag{102}$$

Because quadrics (67) are constructed from outer products of points, the scaling operators S_k, $k = 1, \ldots, 6$, of (96) act (even individually) via outermorphisms covariantly on quadrics. And the expressions for intersections of quadrics (72), also remain covariant under scaling (97), because scaling (97) maps $\mathbf{I} \to \alpha^9\mathbf{I}$ to a scalar multiple of itself, and because the representation of points and quadrics is homogeneous.

Furthermore, due to the relationships (7) and (8) one can show that the scalor (97) maps the six infinity vectors $\{\mathbf{e}_{\infty i}, i = 1, \ldots, 6\} \to \alpha^2\{\mathbf{e}_{\infty i}, i = 1, \ldots, 6\}$ to scalar multiples of themselves. This together with (98), that means that scaling can be covariantly applied to axis aligned quadrics, due to the homogeneity of the QCGA representation. This also applies to intersection operations involving axis aligned quadrics.

Isotropic scaling does apply to embedded CGA objects, because it simply maps $\mathbf{e}_\infty \to \alpha^2\mathbf{e}_\infty$, $\mathbf{I}^{\triangleright}_{\infty a} \to \alpha^4\mathbf{I}^{\triangleright}_{\infty a}$, $\mathbf{I}_{\infty a} \to \alpha^6\mathbf{I}_{\infty a}$, $\mathbf{I}_{\infty b} \to \alpha^6\mathbf{I}_{\infty b}$, $\mathbf{I}^{\triangleright}_{\infty} \to \alpha^{10}\mathbf{I}^{\triangleright}_{\infty}$, and $\mathbf{I}_\infty \to \alpha^{12}\mathbf{I}_\infty$, to scalar multiples of themselves, respectively, which is no problem due to the homogeneity of the representation.

6.4. Versor combinations and alternative hybrid approach

Translators permit rotations and scaling relative to arbitrary centers of rotation and scaling, by first translating the respective center position to the origin, rotating or scaling with the above versors R, S, followed by back translation to the center.

Alternatively, it is possible to combine quadric conformal geometric algebra with double conformal geometric algebra (DCGA) [4]. For that the coefficients of the quadric a,b,c,d,e,f,g,h,i,j, can be extracted with (71) from $\mathbf{q}$, and then used to define the same quadric in DCGA. The rotors for reflection, rotation, translation and scaling are substantially simpler in DCGA. So if preferred, versor operations can be executed within DCGA, and the coefficients $a',b',c',d',e',f',g',h',i',j'$, of the resulting new quadric can be extracted from its DCGA bivector representation and transferred back to QCGA to define the same quadric in QCGA.

7. Conclusion

This paper introduced small modifications to quadric conformal geometric algebra (QCGA), in order to enable the formulation of versor transformations for rotations, translations and scaling of three-dimensional quadrics. To our knowledge this is the first time a set of versors for achieving all three types of geometric transformations for quadrics in a compatible way has been proposed. In principle now quadrics can be defined in QCGA by outer products of points, or directly by their implicit equation parameters, they can be intersected and transformed with the help of versors without leaving the QCGA algebra framework.

Implementation questions of QCGA have initially been discussed in [1]. The early proceedings contribution to AGACSE 2018 by Breuils, Fuchs and Nozick [2] shows further progress, regarding implementations.

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Soli Deo Gloria[7]. The author requests the Creative Peace License[6] to be adhered to regarding the content of this work. I am grateful to V. Nozick for always keeping the existence question of versors for geometric transformations in QCGA open, even if initial evidence seemed to speak against this possibility. I further gratefully acknowledge the fruitful collaboration with S. Breuils and V. Nozick, both at Universite Paris Est (France), Laurent Fuchs at Universite de Poitiers (France), S. J. Sangwine at University of Essex (UK) and A. Sugimoto at National Institute of Informatics (Japan). Finally, I warmly thank the organizers of AGACSE 2018 for preparing and conducting a fantastic conference.

[7] J.C. Maxwell had inscribed at the gate of his Cavendish laboratory: The works of the Lord are great, Studied by all who have pleasure in them. [Psalm 111:2, New King James Version]. D. Capkova writes in Colloquium Comenius and Descartes, Foundation Comenius Museum, Naarden, 1997, p. 15, that: "Comenius' warning against onesided rationalism and against universal application of rationalism speaks to our time very urgently."

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