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New Equations of the Resolution of The Navier-Stokes Equations

New Equations Derived from the Navier-Stokes Equations for the Description of the Motion of Viscous incompressible Fluids with a Proposed Solution*

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Abstract This paper represents an attempt to give a solution of the Navier-Stokes equations under the assumptions (A) of the problem as described by the Clay Mathematics Institute [2]. After elimination of the pressure, we obtain the fundamental equations function of the velocity vector u and vorticity vector $\Omega = curl(u)$, then we deduce the new equations for the description of the motion of viscous incompressible fluids, derived from the Navier-Stokes equations, given by:

$$\nu\Delta\Omega - \frac{\partial\Omega}{\partial t} = 0$$

$$\Delta p = -\sum_{i=1}^{i=3}\sum_{j=1}^{j=3}\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

Then, we give a proof of the solution of the Navier-Stokes equations u and p that are smooth functions and u verifies the condition of bounded energy.

Keywords Navier-Stokes equations · incompressible fluids · heat equation · Poisson equation.

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* The idea of the title was inspired from the title of the supplement of the book of O.A. Ladyzhenskaya [1].

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To the memory of my Father who taught me calculus.

1 Introduction

As it was described in the paper cited above, the Euler and Navier-Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t), u_2(x,t), \dots, u_n(x,t))^T \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$ defined for position $x \in \mathbb{R}^n$ and time $t \geq 0$.

Here we are concerned with incompressible fluids filling all of $\mathbb{R}^n$. The Navier-Stokes equations are given by:

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad i \in \{1,.,n\} \ (x \in \mathbb{R}^n, \ t \geq 0) \ (1)$$

$$divu = \sum_{i=1}^{i=n} \frac{\partial u_i}{\partial x_i} = 0 \ \ (x \in \mathbb{R}^n, \ t \geq 0) \ (2)$$

with the initial conditions:

$$u(x,0) = u^o(x) \ \ (x \in \mathbb{R}^n) \tag{3}$$

where $u^o(x)$ a given vector function of class C^∞, $f_i(x,t)$ are the components of a given external force (e.g gravity), ν is a positive coefficient (viscosity), and Δ is the Laplacian in the space variables. Euler equations are equations (1) (2) (3) with $\nu = 0$.

2 The Navier-Stokes Equations

We try to present a solution to the Navier-Stokes equations following assumptions (A) as described in [2] that summarized here:

* (A) **Existence and smooth solutions** $\in \mathbb{R}^3$ **the Navier-Stokes equations:**

- Take $\nu > 0$. Let $u^0(x)$ a smooth function such that $div(u^0(x)) = 0$ and satisfying:

$$||\partial^\delta_{x_j} u^0(x)|| \leq C_{\delta K}(1 + ||x||)^{-K} \ on \ \mathbb{R}^3 \quad \forall \delta, \ K \tag{4}$$

- Take $f \equiv 0$. Then show that there are functions $p(x,t), u(x,t)$ of class C^∞ on $\mathbb{R}^3 \times [0, +\infty)$ satisfying (1),(2),(3),(4) and:

$$\int_{\mathbb{R}^3} ||u(x,t)||^2 dx < C \ , \forall t \geq 0, \ \text{(bounded energy)} \tag{5}$$

We consider the Navier-Stokes equations in this case, we take $\nu > 0$ and $f_i \equiv 0$, then equations (1) are written for $n = 3$ as :

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} + u_3\frac{\partial u_1}{\partial z} - \nu\Delta u_1 = -\frac{\partial p}{\partial x} \tag{6}$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} + u_3\frac{\partial u_2}{\partial z} - \nu\Delta u_2 = -\frac{\partial p}{\partial y} \tag{7}$$

$$\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x} + u_2\frac{\partial u_3}{\partial y} + u_3\frac{\partial u_3}{\partial z} - \nu\Delta u_3 = -\frac{\partial p}{\partial z} \tag{8}$$

Let:

$$A(u) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{pmatrix} \tag{9}$$

The equations (6-7-8) can be written under vectorial form:

$$\frac{\partial u}{\partial t} + A(u).u = \nu\Delta u - gradp \tag{10}$$

Let Ω the vector $curl(u)$, then:

$$\Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{vmatrix} \partial_x \\ \partial_y \\ \partial_z \end{vmatrix} \wedge \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{pmatrix} \partial_y u_3 - \partial_z u_2 \\ \partial_z u_1 - \partial_x u_3 \\ \partial_x u_2 - \partial_y u_1 \end{pmatrix} \tag{11}$$

Taking the curl of the both members of (10), then, equation (10) becomes as follows:

$$\boxed{A(u).\Omega - A(\Omega).u = \nu\Delta\Omega - \frac{\partial\Omega}{\partial t}} \tag{12}$$

where:

$$A(\Omega) = \begin{pmatrix} \frac{\partial \omega_1}{\partial x} & \frac{\partial \omega_1}{\partial y} & \frac{\partial \omega_1}{\partial z} \\ \frac{\partial \omega_2}{\partial x} & \frac{\partial \omega_2}{\partial y} & \frac{\partial \omega_2}{\partial z} \\ \frac{\partial \omega_3}{\partial x} & \frac{\partial \omega_3}{\partial y} & \frac{\partial \omega_3}{\partial z} \end{pmatrix} \tag{13}$$

The equations (12) are the fundamental equations of this study. These are nonlinear partial differential equations of the third order. Their resolutions are the solutions of the Navier-Stokes equations.

3 The Study of The Fundamental Equations (12)

3.1 A New Fundamental Equations of the Navier-Stokes Equations

We re-write the equations (12):

$$A(u).\Omega - A(\Omega).u = \nu\Delta\Omega - \frac{\partial\Omega}{\partial t}$$

We can also write it :

$$A(-u).(-\Omega) - A(-\Omega).(-u) = \nu\Delta\Omega - \frac{\partial\Omega}{\partial t} \tag{14}$$

As u and Ω are not independent variables, we have $curl(-u) = -curl(u) = -\Omega$, we obtain :

$$A(-u).(-\Omega) - A(-\Omega).(-u) = \nu\Delta(-\Omega) - \frac{\partial(-\Omega)}{\partial t} \tag{15}$$

Comparing the last two equations (14-15), we arrive to:

$$\nu\Delta\Omega - \frac{\partial\Omega}{\partial t} = \nu\Delta(-\Omega) - \frac{\partial(-\Omega)}{\partial t} = -\left(\nu\Delta\Omega - \frac{\partial\Omega}{\partial t}\right)$$

Hence:

$$\nu\Delta\Omega - \frac{\partial\Omega}{\partial t} = 0 \tag{16}$$

From the equation (12), we get necessary that:

$$A(u).\Omega - A(\Omega).u = 0 \tag{17}$$

The first new fundamental equation is (16), from it we will obtain $u(x,t)$. Taking the divergence of the both members of equation (10), we obtain the known equation determining $p(x,t)$:

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_j}{\partial x_i} \tag{18}$$

It is therefore the new fundamental differential system:

$$\boxed{\begin{cases} \nu\Delta\Omega - \dfrac{\partial\Omega}{\partial t} = 0 \Longrightarrow \ u \\[2ex] \Delta p = -\displaystyle\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_j}{\partial x_i} \Longrightarrow \ p \end{cases}} \tag{19}$$

4 Resolution of the equations (19)

From the first equation of (19), we can write that:

$$curl(\nu\Delta u - \frac{\partial u}{\partial t}) = 0 \tag{20}$$

then:

Case 1- $\nu\Delta u - \dfrac{\partial u}{\partial t} \equiv 0\,(x \in \mathbb{R}^n,\ t \geq 0)$;

Case 2- $\nu\Delta u - \dfrac{\partial u}{\partial t} = K(t)$ with K is a vector function depending only of t.

4.1 Resolution of the equations (19) case 1

Let the change of variables:

$$x = \nu X \tag{21}$$
$$y = \nu Y \tag{22}$$
$$z = \nu Z \tag{23}$$
$$t = \nu T \tag{24}$$
$$u(x,y,z,t) = U(X,Y,Z,T) \tag{25}$$
$$p(x,y,z,t) = P(X,Y,Z,T) \tag{26}$$

Then:

$$\partial_x u dx + \partial_y u dy + \partial_z u dz + \partial_t u dt = \partial_X U dX + \partial_Y U dY + \partial_Z U dZ + \partial_T U dT$$
$$\nu(\partial_x u dX + \partial_y u dY + \partial_z u dZ + \partial_t u dT) = \partial_X U dX + \partial_Y U dY + \partial_Z U dZ + \partial_T U dT$$
$$\partial_x u = \frac{1}{\nu}\partial_X U,\ \partial_y u = \frac{1}{\nu}\partial_Y U,\ \partial_z u = \frac{1}{\nu}\partial_Z U,\ \partial_t u = \frac{1}{\nu}\partial_T U \tag{27}$$

Then the equation:

$$\frac{\partial u}{\partial t} - \nu\Delta u = 0$$

becomes:

$$\boxed{\frac{\partial U}{\partial T} - \Delta U = 0} \tag{28}$$

This is the heat equation!

4.1.1 Resolution of the Equation (28)

Noting that $U^0(X,Y,Z) = U^0(\boldsymbol{X}) = U(X,Y,Z,0) = u(x,y,z,0) = u^0(x,y,z)$, then the solution of (28) with $T \geq 0$ satisfying:

$$U \in \mathbb{R}^3 \text{ and of class } C^\infty(\mathbb{R}^3 \times [0,+\infty)) \tag{29}$$

$$U(\boldsymbol{X},0) = U^0(\boldsymbol{X}) \tag{30}$$

is given by [3]:

$$U(\boldsymbol{X},T) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{U^0(\alpha,\beta,\gamma)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{31}$$

where $dV = d\alpha d\beta d\gamma$ and $U(\boldsymbol{X},T)$ is unique with $U(\boldsymbol{X},0) = U^0(\boldsymbol{X})$, then u is unique.

We denote:

$$\boldsymbol{X} = (X,Y,Z)^T \tag{32}$$

$$\Gamma = (\alpha,\beta,\gamma)^T \tag{33}$$

Then, we can write the norm of $U(\boldsymbol{X},T)$ as:

$$||U(\boldsymbol{X},T)|| \leq \frac{e^{-\dfrac{X^2+Y^2+Z^2}{4T}}}{2\sqrt{\pi T}} \int_{\mathbb{R}^3} ||U^0(\alpha,\beta,\gamma)|| e^{-\dfrac{(||\Gamma||^2 - 2\Gamma.\boldsymbol{X})}{4T}} dV \tag{34}$$

The presence of the term $e^{-\dfrac{X^2+Y^2+Z^2}{4T}}$ implies that if $||X|| \longrightarrow +\infty$, $||U(\boldsymbol{X},T)|| \longrightarrow 0$ fast enough [4]. Then, for t fixed, $||u(x,y,z,t)|| \longrightarrow 0$ when $\sqrt{x^2+y^2+z^2} \longrightarrow +\infty$, hence, from now, we assume that we are dealing only with such rapidly decreasing velocities.

4.1.2 Expression of U

We have:

$$U_1 = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{U_1^0(\alpha,\beta,\gamma)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{35}$$

$$U_2 = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{U_2^0(\alpha,\beta,\gamma)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{36}$$

$$U_3 = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{U_3^0(\alpha,\beta,\gamma)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{37}$$

4.1.3 Checking $div(U)=0$

Let us calculate $\partial_X U_1$, we get:

$$\frac{\partial U_1}{\partial X} = \frac{-1}{4\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{(X-\alpha)U_1^0(\alpha,\beta,\gamma)}{T\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{38}$$

We can write the above expression as follows:

$$\frac{\partial U_1}{\partial X} = \frac{-1}{2\sqrt{\pi T}} \int_{\mathbb{R}^2} d\beta d\gamma \int_{\alpha=-\infty}^{\alpha=+\infty} U_1^0(\alpha,\beta,\gamma) \frac{\partial}{\partial \alpha}\left(e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \right) d\alpha \tag{39}$$

Now we do an integration by parts, we get:

$$\frac{\partial U_1}{\partial X} = \frac{-1}{2\sqrt{\pi T}} \int_{\mathbb{R}^2} d\beta d\gamma \left[U_1^0(\alpha,\beta,\gamma).e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \right]_{\alpha=-\infty}^{\alpha=+\infty} +$$
$$\frac{1}{2\sqrt{\pi T}} \int_{\mathbb{R}^2} d\beta d\gamma \int_{\alpha=-\infty}^{\alpha=+\infty} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \frac{\partial U_1^0(\alpha,\beta,\gamma)}{\partial \alpha}.d\alpha \tag{40}$$

Taking into account the assumption that:

$$||\partial_{X_j}^{\delta} U^0(\boldsymbol{X})|| \leq \nu C_{\delta K}(1+\nu||\boldsymbol{X}||)^{-K} \ on\ \mathbb{R}^3 \quad \forall \delta,\, K \tag{41}$$

where X_j denotes one of the coordinates X, Y, Z, and choosing $K > 1$ and $\delta = 0$, we obtain :

$$||U^0(\boldsymbol{X})|| \leq C_{0K}(1+\nu||\boldsymbol{X}||)^{-K} \tag{42}$$

and the first term of the right member of (40) is zero. Then:

$$\frac{\partial U_1}{\partial X} = \frac{1}{2\sqrt{\pi T}} \int_{\mathbb{R}^2} d\beta d\gamma \int_{\alpha=-\infty}^{\alpha=+\infty} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \frac{\partial U_1^0(\alpha,\beta,\gamma)}{\partial \alpha}.d\alpha \tag{43}$$

or:

$$\frac{\partial U_1}{\partial X} = \frac{1}{2\sqrt{\pi T}} \int_{\mathbb{R}^3} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \frac{\partial U_1^0(\alpha,\beta,\gamma)}{\partial \alpha}.dV \tag{44}$$

As a result:

$$div(U) = \sum_{X_j} \frac{\partial U_j}{\partial X_j} = \frac{1}{2\sqrt{\pi T}} \int_{\mathbb{R}^3} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \sum_{\alpha_j} \frac{\partial U_j^0(\alpha,\beta,\gamma)}{\partial \alpha}.dV = 0 \tag{45}$$

because $U^0(\alpha,\beta,\gamma)$ satisfies $div(U^0) = \sum_{\alpha_j} \dfrac{\partial U_j^0(\alpha,\beta,\gamma)}{\partial \alpha_j} = 0$.

4.1.4 Estimation of $\int_{\mathbb{R}^3} ||U(\boldsymbol{X},T)||^2 dV$

We have:

$$||U(\boldsymbol{X},T)||^2 = \sum_i U_i^2 = \frac{1}{4\pi T}\left\| \int_{\mathbb{R}^3} U^0(\alpha,\beta,\gamma).e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \right\|^2$$

$$\leq \frac{1}{4\pi T}\int_{\mathbb{R}^3} \left\|U^0(\alpha,\beta,\gamma)\right\|^2 .e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}} dV \tag{46}$$

Using the condition (42):

$$||U^0(\boldsymbol{X})|| \leq C_{0K}(1+\nu||\boldsymbol{X}||)^{-K}$$

We obtain as a result:

$$||U(\boldsymbol{X},T)||^2 \leq \frac{C_{0K}^2}{4\pi T}\int_{\mathbb{R}^3} \frac{e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}}}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} d\alpha d\beta d\gamma \tag{47}$$

Let us now majorize $\int_{\mathbb{R}^3} ||u(\boldsymbol{x},t)||^2 dxdydz$:

$$\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz = \int_{\mathbb{R}3} ||U(\boldsymbol{X},T)||^2 dxdydz = \nu^3 \int_{\mathbb{R}3} ||U(\boldsymbol{X},T)||^2 dXdYdZ$$

$$\leq \frac{\nu^3 C_{0K}^2}{4\pi T}\int_{\mathbb{R}^3}\left[\int_{\mathbb{R}^3} \frac{e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}}}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} d\alpha d\beta d\gamma\right] dXdYdZ \tag{48}$$

As the integral $\int_{\mathbb{R}^3} e^{-X^2-Y^2-Z^2} dXdYdZ < +\infty$, we can permute the two triple integrals of the above equation. Let:

$$\tau_0 = \frac{\nu^3 C_{0K}^2}{4\pi} \tag{49}$$

we obtain:

$$\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz \leq \frac{\tau_0}{T}\int_{\mathbb{R}^3}\left[\int_{\mathbb{R}^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}} dXdYdZ\right].\frac{d\alpha d\beta d\gamma}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} \tag{50}$$

Let:

$$I = \int_{\mathbb{R}^3} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}} dXdYdZ \tag{51}$$

and let the following change of variables:

$$\begin{cases} \overline{X} = \frac{X-\alpha}{\sqrt{2T}} \Longrightarrow dX = \sqrt{2T}d\overline{X} \quad and\ \overline{X}^2 = \frac{(X-\alpha)^2}{2T} \\ \overline{Y} = \frac{Y-\beta}{\sqrt{2T}} \Longrightarrow dY = \sqrt{2T}d\overline{Y} \quad and\ \overline{Y}^2 = \frac{(Y-\beta)^2}{2T} \\ \overline{Z} = \frac{Z-\gamma}{\sqrt{2T}} \Longrightarrow dZ = \sqrt{2T}d\overline{Z} \quad and\ \overline{Z}^2 = \frac{(Z-\gamma)^2}{2T} \end{cases} \tag{52}$$

I is written as:

$$I = (\sqrt{2T})^3 \left[\int_{-\infty}^{+\infty} e^{-\overline{X}^2} d\overline{X}\right]^3 = 2T\sqrt{2T}\left[2\int_0^{+\infty} e^{-\xi^2} d\xi\right]^3 = 2T\sqrt{T}.\pi\sqrt{\pi} = 2\pi T\sqrt{\pi T} \tag{53}$$

using the formula $2\displaystyle\int_0^{+\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$. Then the equation (50) becomes:

$$\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz \leq 2\tau_0\pi\sqrt{\pi T}\int_{\mathbb{R}^3} \frac{d\alpha d\beta d\gamma}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} \tag{54}$$

Let us now:

$$B = \int_{\mathbb{R}^3} \frac{d\alpha d\beta d\gamma}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} \tag{55}$$

and we use the spherical coordinates:

$$\begin{cases} \alpha = rsin\theta cos\varphi \\ \beta = rsin\theta sin\varphi \\ \gamma = rcos\theta \end{cases} \tag{56}$$

the form of the volume $d\alpha d\beta d\gamma = r^2 sin\theta dr d\theta d\varphi$ and B becomes:

$$B = \int_{\theta=0}^{\theta=\pi} sin\theta d\theta \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_0^r \frac{r^2 dr}{(1+\nu r)^{2K}} = 4\pi\int_0^r \frac{r^2 dr}{(1+\nu r)^{2K}} \tag{57}$$

We take $K = 2$, the integral B is convergent when $r \to +\infty$. Let:

$$F = lim_{r\to+\infty}\int_0^r \frac{r^2 dr}{(1+\nu r)^4} = \int_0^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} = \int_0^1 \frac{r^2 dr}{(1+\nu r)^4} + \int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} \tag{58}$$

But :

$$\int_0^1 \frac{r^2 dr}{(1+\nu r)^4} < \int_0^1 r^2 dr = \left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3} \tag{59}$$

We calculate now $\displaystyle\int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4}$. Let the change of variables:

$$\xi = 1+\nu r \Rightarrow r = \frac{\xi-1}{\nu} \Rightarrow dr = \frac{d\xi}{\nu} \tag{60}$$

then:

$$\int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} = \frac{1}{\nu^3}\int_{1+\nu}^{+\infty} \frac{\xi^2-2\xi+1}{\xi^4} d\xi = l(\nu)\ avec \quad l(\nu) = \frac{3\nu 2+9\nu+5}{\nu^3(1+\nu)^3} \tag{61}$$

As a result:

$$B < 4\pi(\frac{1}{3} + l(\nu)) \tag{62}$$

Hence the important result:

$$\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz < 8\tau_0\pi^2\sqrt{\pi T}\left(\frac{1}{3} + l(\nu)\right) \tag{63}$$

or:

$$\boxed{\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz < +\infty \quad \forall t} \tag{64}$$

let:

$$\boxed{\int_{\mathbb{R}3} ||U(\boldsymbol{X},T)||^2 dXdYdZ < +\infty \quad \forall T} \tag{65}$$

because:

$$\int_{\mathbb{R}3} ||U(\boldsymbol{X},T)||^2 dXdYdZ = \frac{1}{\nu^3}\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz$$

4.1.5 The expression of partial derivatives of $U(X,T)$

We begin with the first partial derivative ∂_X of the first component of $U(X,T)$: it is given by the equation (44):

$$\frac{\partial U_1}{\partial X} = \frac{1}{2\sqrt{\pi T}}\int_{\mathbb{R}^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}}\frac{\partial U_1^0(\alpha,\beta,\gamma)}{\partial\alpha}.dV$$

Let us calculate $\dfrac{\partial^2 U_1}{\partial X^2}$. We obtain:

$$\begin{aligned}
\frac{\partial^2 U_1}{\partial X^2} &= \frac{-1}{4T\sqrt{\pi T}}\int_{\mathbb{R}^3}(X-\alpha)e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}}\frac{\partial U_1^0(\alpha,\beta,\gamma)}{\partial\alpha}.dV \\
&= \frac{-1}{2\sqrt{\pi T}}\int_{\mathbb{R}^3}\frac{\partial}{\partial\alpha}e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}}.\frac{\partial U_1^0(\alpha,\beta,\gamma)}{\partial\alpha}.dV \\
= \frac{-1}{2\sqrt{\pi T}}&\int_{\mathbb{R}^2} d\beta d\gamma\left[\frac{\partial}{\partial\alpha}U_1^0(\alpha,\beta,\gamma).e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}}\right]_{\alpha=-\infty}^{\alpha=+\infty} + \\
\frac{1}{2\sqrt{\pi T}}&\int_{\mathbb{R}^2} d\beta d\gamma\int_{\alpha=-\infty}^{\alpha=+\infty} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}}\frac{\partial^2 U_1^0(\alpha,\beta,\gamma)}{\partial\alpha^2}.d\alpha
\end{aligned} \tag{66}$$

Taking into account the assumption (41), we obtain:

$$\frac{\partial^2 U_1}{\partial X^2} = \frac{1}{2\sqrt{\pi T}}\int_{\mathbb{R}^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}}\frac{\partial^2 U_1^0(\alpha,\beta,\gamma)}{\partial\alpha^2}.d\alpha d\beta d\gamma \tag{67}$$

Using the same assumption cited above, we obtain that $\left\|\frac{\partial^2 U_1}{\partial X^2}\right\| \longrightarrow 0$ when $||\boldsymbol{X}|| \longrightarrow +\infty$. Then for t fixed $||\partial_x u(x,y,z,t)|| \longrightarrow 0$ if $\sqrt{x^2+y^2+z^2} \longrightarrow +\infty$. We easily verify this property for the derivatives of $u(x,y,z,t)$ concerning the spatial coordinates of all orders, with t fixed.

4.1.6 The expression of $p(x,y,z,t)$

We rewrite equation (10):

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \nu \Delta u_i = -\frac{\partial p}{\partial x_i}$$

It can be written under vectorial form:

$$\nabla p = \nu \Delta u - \frac{\partial u}{\partial t} - A(u).u \tag{68}$$

with the matrix $A(u)$ given by (9). As $\nu \Delta u - \frac{\partial u}{\partial t} = 0$, then the equation (68) becomes:

$$\nabla p = -A(u).u \tag{69}$$

As $u \in \mathbb{R}^3$ and of class $C^\infty(\mathbb{R}^3 \times [0,+\infty))$, $\partial_i p$ are of class $C^\infty(\mathbb{R}^3 \times [0,+\infty)) \Longrightarrow p(x,y,z,t)$ is also of class $C^\infty(\mathbb{R}^3 \times [0,+\infty))$.

With the variables X, Y, Z, T, the pressure verifies the equation:

$$\Delta P = -\sum_{i,j=1}^{3} \frac{\partial U_i}{\partial X_j} . \frac{\partial U_j}{\partial X_i} \tag{70}$$

we denote:

$$H = H(X,Y,Z,T) = \sum_{i,j=1}^{3} \frac{\partial U_i}{\partial X_j} . \frac{\partial U_j}{\partial X_i} \tag{71}$$

The equation (70) becomes:

$$\Delta P = -H \tag{72}$$

It is the Poisson equation.

Definition 1 The function :

$$\Phi(\boldsymbol{X}) = \frac{1}{4\pi ||\boldsymbol{X}||} \tag{73}$$

defined for $||\boldsymbol{X}|| \in \mathbb{R}^3$, $\boldsymbol{X} \neq \boldsymbol{O}$ is the fundamental solution of Laplace equation.

The solution of Poisson equation (72) is given by [5]:

$$P = P(X,Y,Z,T) = P(\boldsymbol{X},T) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{||\boldsymbol{X}-\boldsymbol{Q}||} H(\boldsymbol{Q})d\boldsymbol{Q} \tag{74}$$

where $\boldsymbol{Q} = (X', Y', Z')^T \in \mathbb{R}^3$ and $d\boldsymbol{Q} = dX'dY'dZ'$ the volume form.

From equation (69),, we can write for example, the first component of ∇p :

$$\frac{\partial p}{\partial x} = -\sum_j u_j \frac{\partial u_1}{\partial x_j} \tag{75}$$

Using the new variables, we obtain:

$$\frac{\partial P}{\partial x} = -\sum_j U_j \frac{\partial U_1}{\partial X_j} \Longrightarrow P = -\sum_i \int_0^X U_i(\alpha, Y, Z, T)\frac{\partial U_1(\alpha, Y, Z, T)}{\partial \alpha_i} d\alpha \tag{76}$$

Then:

$$|P| \leq \sum_i |X||U_i(X,Y,Z,T)| \left|\frac{\partial U_1(X,Y,Z,T)}{\partial X_i}\right| \leq 3||\boldsymbol{X}||.||\boldsymbol{U}||.\left\|\frac{\partial \boldsymbol{U}(X,Y,Z,T)}{\partial X_i}\right\| \tag{77}$$

As seen above, $||\boldsymbol{U}||$ and $\left\|\frac{\partial \boldsymbol{U}(X,Y,Z,T)}{\partial X_i}\right\|$ tend to zero if $||\boldsymbol{X} = \sqrt{X^2+Y^2+Z^2}|| \longrightarrow +\infty$. With the presence of the term $e^{-||\boldsymbol{X}||^2}$ in the expression of the vectors $\boldsymbol{U}$ and its first derivative $\partial_X \boldsymbol{U}$, $||\boldsymbol{X}||.||\boldsymbol{U}||.\left\|\frac{\partial \boldsymbol{U}(X,Y,Z,T)}{\partial X_i}\right\|$ tend to zero as $||\boldsymbol{X}|| \longrightarrow +\infty$. Then $|P| \longrightarrow 0$.

Again, from equation (69), we can write for the vector ∇p :

$$||\nabla p|| = \sqrt{\sum_j \left(\frac{\partial p}{\partial x_j}\right)^2} \leq ||A(u)||.||u|| \tag{78}$$

Taking $||A(u)|| = max\left\|\frac{\partial u_i}{\partial x_j}\right\|$, then:

$$\left|\frac{\partial p}{\partial x_i}\right| \leq ||\nabla p|| \leq max\left\|\frac{\partial u_i}{\partial x_j}\right\|.||u(x,y,z,t)|| \tag{79}$$

As seeing in paragraph (4.1.1), for t fixed, $||u(x,y,z,t)||$ and $||\partial_{x_i} u(x,y,z,t)||$ tend to zero as $\sqrt{x^2+y^2+z^2} \to +\infty$. We easily verify this property for the derivatives of p concerning the spatial coordinates of all orders, with t fixed.

Let us study $lim_{\boldsymbol{X}\longrightarrow+\infty}\dfrac{\partial P}{\partial T}$. With the variables X,Y,Z,T, we have for example:

$$\frac{\partial p}{\partial x}=-\sum_i u_i\frac{\partial u_1}{\partial x_i}\Longrightarrow\frac{\partial P}{\partial X}=-\sum_i U_i\frac{\partial U_1}{\partial X_i}\Longrightarrow$$

$$P=-\sum_i\int_0^X U_i(\alpha,\beta,\gamma,T)\frac{\partial U_1(\alpha,\beta,\gamma,T)}{\partial\alpha_i}d\alpha \tag{80}$$

We calculate $\partial_T P(X,Y,Z,T)$, we obtain:

$$\frac{\partial P}{\partial T}=-\sum_i\int_0^X\left(\frac{\partial U_i}{\partial T}.\frac{\partial U_1}{\partial\alpha_i}+U_i\frac{\partial^2 U_1}{\partial\alpha_i\partial T}\right)d\alpha \tag{81}$$

We suppose that $X>0$, then:

$$\left|\frac{\partial P}{\partial T}\right|\leq\sum_i\left(\left|X.\frac{\partial U_i}{\partial T}.\frac{\partial U_1}{\partial\alpha_i}\right|+\left|U_i.X.\frac{\partial^2 U_1}{\partial\alpha_i\partial T}\right|\right) \tag{82}$$

The presence of $e^{-\dfrac{X^2+Y^2+Z^2}{4T}}$ in the bounded expression of the six terms of the right member of the above inequality gives that $lim\left|\dfrac{\partial P}{\partial T}\right|\longrightarrow 0$ when $\sqrt{X^2+Y^2+Z^2}\longrightarrow+\infty$. We verify easily that the derivatives $\partial^\delta_{X,Y,Z,T}P$ of all orders, for T fixed, tend to zero as $\sqrt{X^2+Y^2+Z^2}\longrightarrow+\infty$.

We have given a proof of smooth solutions $u(x,y,z,t),p(x,y,z,t)$ of Navier-Stokes equations, defined for $(x,y,z)\in\mathbb{R}^3$ and $t\in[0,\tau)$ for any $\tau\in\mathbb{R}$.

4.2 Resolution of the equations (19) case 2

With the new variables X,Y,Z,T the equation of case 2 is written as:

$$\Delta\overline{U}-\frac{\partial\overline{U}}{\partial T}=\overline{K}(T) \tag{83}$$

with $\overline{K}(T)=\nu K(t)$. We put $\overline{U}=U-\displaystyle\int_0^T\overline{K}(\tau)d\tau$, then the new function U verifies:

$$\Delta U-\frac{\partial U}{\partial T}=0 \tag{84}$$

The solution of (83) is the function $\overline{U}=U-\displaystyle\int_0^T\overline{K}(\tau)d\tau$ where U is the solution of the case 1 studied above. The function $\overline{U}$ verifies the same remarks studies above as U.

5 Conclusion

In this work, we have obtained new fundamental equations derived from the classical Navier-Stokes equations. The first equation is the heat equation: the movement of fluids is like the propagation of the heat that can be acceptable. The expression of the solution founded (u, p) verifies the conditions (A) of existence and u, p are smooth functions of spatial coordinates and time solution.

References

1. Ladyzhenskaya O.A. 1969. *The Mathematical theory of viscous incompressible flow.* Gordan and Breach, 2^{sd} English edition.
2. Fefferman C.L. 2006. *Existence and smoothness of the Navier-Stokes equation*, Millennium Prize Problems, Clay Math. Inst., Cambridge, MA, pp57-67.
3. Godounov S. 1973. *Equations de la Physique Mathématique.* Edition Mir. 452p.
4. Fefferman C.L. 2008. *The Euler and Navier-Stokes equations.* pp 196-198. *The Princeton companion to mathematics.* Gowers T. (Editor), Princeton, ISBN 0691118809. 1027 pages.
5. Lawrence C. E. 1997. *Partial Differential Equations.* Graduate Studies in Mathematics, **Vol 19**, American Mathematical Society. 664 pages.