Solution of a Sangaku "Tangency" Problem via Geometric Algebra

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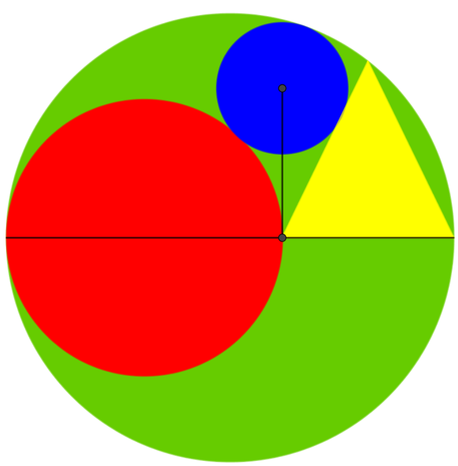
Abstract

Because the shortage of worked-out examples at introductory levels is an obstacle to widespread adoption of Geometric Algebra (GA), we use GA to solve one of the beautiful *sangaku* problems from 19th-Century Japan. Among the GA operations that prove useful is the rotation of vectors via the unit bivector **i**.



"The center of the red circle and the base of the isosceles triangle lie along the same diameter of the green circle. The blue circle is tangent to the other three figures. Prove that the line connecting its center to the point of contact between the red circle and the triangle is perpendicular to the above-mentioned diameter."

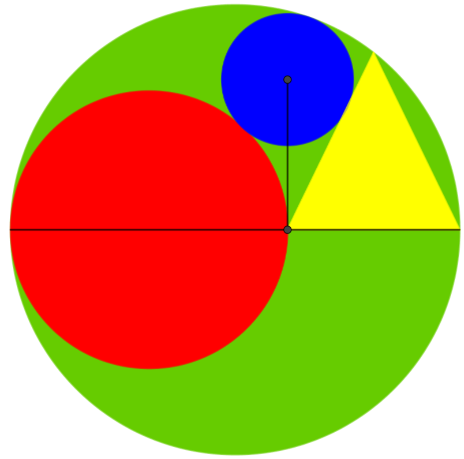


Figure 1: The center of the red circle and the base of the isosceles triangle lie along the same diameter of the green circle. The blue circle is tangent to the other three figures. Prove that the line connecting its center to the point of contact between the red circle and the triangle is perpendicular to the above-mentioned diameter.

1 Problem Statement

> *In Fig. 1, the center of the red circle and the base of the isosceles triangle lie along the same diameter of the green circle. The blue circle is tangent to the other three figures. Prove that the line connecting its center to the point of contact between the red circle and the triangle is perpendicular to the above-mentioned diameter.*

2 Formulation of the Problem in Geometric-Algebra Terms

Fig. 2 defines the vectors that we will use. Note the notation used to distinguish between points and vectors: for example, $\mathbf{c}_1$ is the vector from the origin to the point c_1. Also, c_1^2 denotes $\|\mathbf{c}_1\|^2$.

In GA terms, we are to prove that $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = 0$. Other formulations are possible; for example, that $\mathbf{c}_3\hat{\mathbf{b}} = \hat{\mathbf{b}}\mathbf{c}_3$.

3 Solution Strategy

We will derive an equation that that is satisfied by two circles. For one of them, $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = 0$.

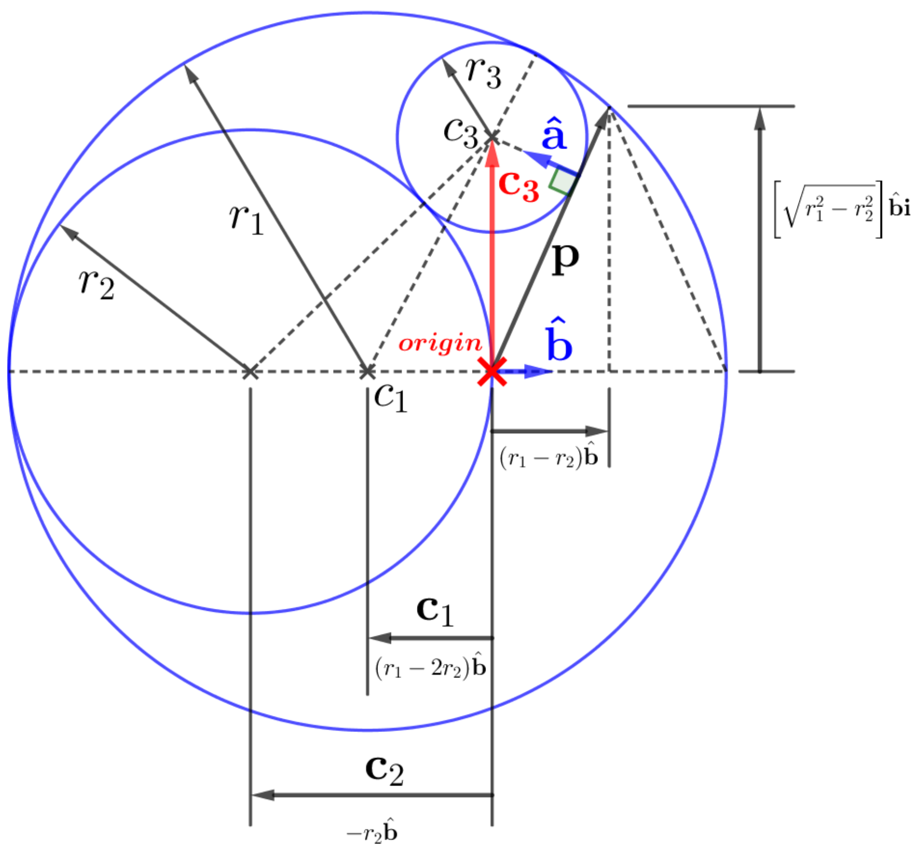


Figure 2: The vectors and frame of reference that we will use in our solution.

4 Observations

Two key observations are that

$$r_3 = \mathbf{c}_3 \cdot \hat{\mathbf{a}} \tag{4.1}$$

and that, in turn,

$$\begin{aligned}\hat{\mathbf{a}} &= \hat{\mathbf{p}}\mathbf{i} \\ &= -\left[\frac{\sqrt{r_1^2 - r_2^2}}{\sqrt{2r_1\left(r_1 - r_2\right)}}\right]\hat{\mathbf{b}} \; + \; \left[\frac{r_1 - r_2}{\sqrt{2r_1\left(r_1 - r_2\right)}}\right]\hat{\mathbf{b}}\mathbf{i}\;.\end{aligned} \tag{4.2}$$

We also see that by expressing the distance between c_1 and c_3 as $r_1 - r_3$ and $\|\mathbf{c}_3 - \mathbf{c}_1\|$, we obtain

$$\left(\mathbf{c}_3 - \mathbf{c}_1\right)^2 = \left(r_1 - r_3\right)^2,$$

which after simplification becomes

$$c_3^2 + 2\left(2r_2 - r1\right)\mathbf{c}_3 \cdot \hat{\mathbf{b}} + 4r_2\left(r_2 - r_1\right) = r_3^2 - r_3 r_1\;. \tag{4.3}$$

Similarly, because $\|\mathbf{c}_3 - \mathbf{c}_2\| = r_2 + r_3$,

$$c_3^2 + 2r_2\mathbf{c}_3 \cdot \hat{\mathbf{b}} = r_3^2 + 2r_3 r_2\;. \tag{4.4}$$

5 Derivation of the Equation that We Seek

We begin by subtracting Eq. (4.4) from Eq. (4.3), then solving for r_3:

$$r_3 = \left(\frac{r_1 - r_2}{r_1 + r_2}\right)\left(2r_2 + \mathbf{c}_3 \cdot \hat{\mathbf{b}}\right). \tag{5.1}$$

Substituting that expression for r_3 in Eq. (4.4), then simplifying,

$$c_3^2 - \left(\frac{r_1 - r_2}{r_1 + r_2}\right)^2\left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 - 4r_1r_2\left[\frac{r_1 - r_2}{\left(r_1 + r_2\right)^2}\right]\mathbf{c}_3 \cdot \hat{\mathbf{b}} = \frac{8r_1r_2^2\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2}.$$

Now, we write c_3^2 as $\left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 + \left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2$, obtaining

$$\left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2 + \left[\frac{4r_1r_2}{\left(r_1 + r_2\right)^2}\right]\left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 - 4r_1r_2\left[\frac{r_1 - r_2}{\left(r_1 + r_2\right)^2}\right]\mathbf{c}_3 \cdot \hat{\mathbf{b}} = \frac{8r_1r_2^2\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2}. \tag{5.2}$$

An expression for $\left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2$ in terms of $\mathbf{c}_3 \cdot \hat{\mathbf{b}}$ by equating the expressions for r_3 given by Eqs. (4.1) and (5.1),

$$\mathbf{c}_3 \cdot \hat{\mathbf{a}} = \left(\frac{r_1 - r_2}{r_1 + r_2}\right)\left(2r_2 + \mathbf{c}_3 \cdot \hat{\mathbf{b}}\right),$$

then expressing $\hat{\mathbf{a}}$ via Eq. (4.2):

$$\mathbf{c}_3 \cdot \left\{-\left[\frac{\sqrt{r_1^2 - r_2^2}}{\sqrt{2r_1\left(r_1 - r_2\right)}}\right]\hat{\mathbf{b}} + \left[\frac{r_1 - r_2}{\sqrt{2r_1\left(r_1 - r_2\right)}}\right]\hat{\mathbf{b}}\mathbf{i}\right\} = \left(\frac{r_1 - r_2}{r_1 + r_2}\right)\left(2r_2 + \mathbf{c}_3 \cdot \hat{\mathbf{b}}\right). \tag{5.3}$$

Thus,

$$\begin{aligned}\left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2 = &\left[\frac{\sqrt{2r_1\left(r_1 - r_2\right)}}{r_1 + r_2}\right]^2\left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 \\ &+ 4r_2\left[\frac{2r_1\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2} + \sqrt{\frac{2r_1}{r_1 + r_2}}\right]\mathbf{c}_3 \cdot \hat{\mathbf{b}} \\ &+ \frac{8r_1r_2\left(R_1 - R_2\right)}{\left(r_1 + r_2\right)^2}.\end{aligned} \tag{5.4}$$

Substituting that expression for $\left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2$ in Eq. (5.2),

$$\begin{aligned}0 = &\left\{\left[\frac{\sqrt{2r_1\left(r_1 - r_2\right)}}{r_1 + r_2} + \sqrt{\frac{2r_1}{r_1 + r_2}}\right]^2 + \frac{4r_1r_2}{\left(r_1 + r_2\right)^2}\right\}\left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 \\ &+ 4r_2\left\{\frac{3r_1\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2} + \sqrt{\frac{2r_1}{r_1 + r_2}}\right\}\mathbf{c}_3 \cdot \hat{\mathbf{b}}.\end{aligned} \tag{5.5}$$

The two roots are

1. $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = 0$, with $\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right) = \dfrac{2r_2\sqrt{2r_1\left(r_1 - r_2\right)}}{r_1 + r_2}$ and $r_3 = \dfrac{2r_2\left(r_1 - r_2\right)}{r_1 + r_2}$; and
2. $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = -4r_2\left(r_1 - r_2\right)\left[\dfrac{r_1 + \sqrt{2r_1\left(r_1 + r_2\right)}}{2\left(r_1 - r_2\right)\sqrt{2r_1\left(r_1 + r_2\right)} + 3r_1^2 + r_2^2}\right]$,
   $\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right) = -\dfrac{2r_2\left(r_1 + r_2\right)\sqrt{2r_1\left(r_1 - r_2\right)} + 4r_1r_2\sqrt{r_1^2 - r_2^2}}{2\left(r_1 - r_2\right)\sqrt{2r_1\left(r_1 + r_2\right)} + 3r_1^2 + r_2^2}$, and
   $r_3 = \dfrac{2r_1r_2\left(r_1 - r_2\right)}{2\left(r_1 - r_2\right)\sqrt{2r_1\left(r_1 + r_2\right)} + 3r_1^2 + r_2^2}$.

The first solution is in red in Fig. 3; the second is in magenta. The first demonstrates that which was to be proved, but the second is extraneous: the magenta circle is tangent to the extension of the side of the isosceles triangle, but not to the triangle itself.

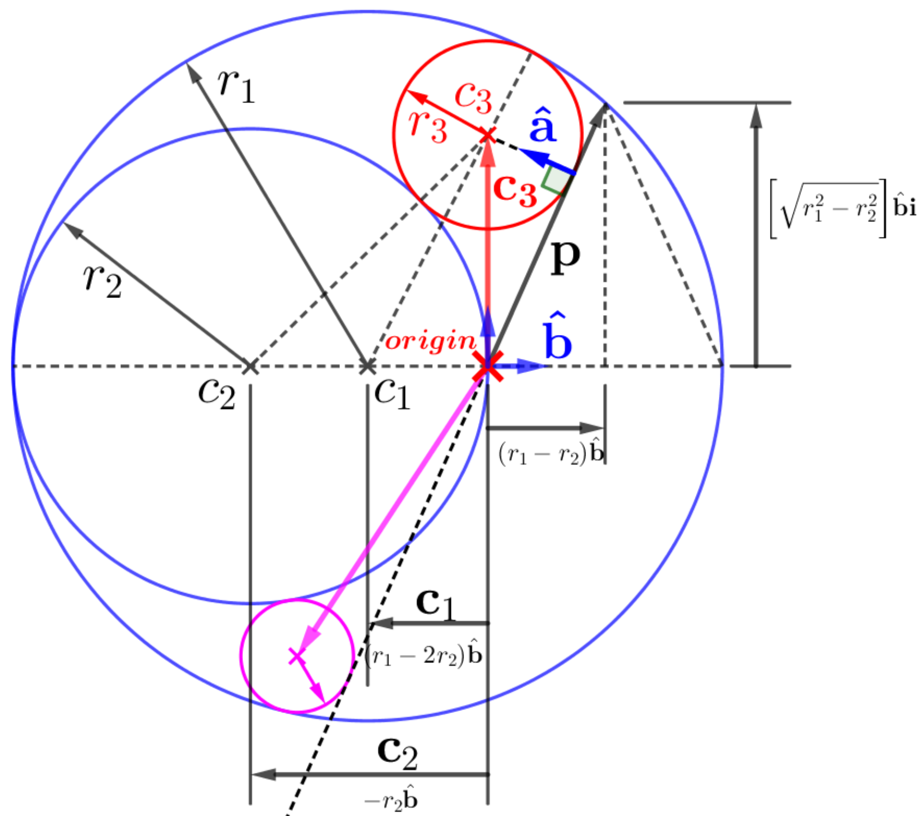


Figure 3: The two solutions to Eq. (5.5). For our purposes, the magenta circle is extraneous: it is tangent to the extension of the side of the isosceles triangle, but not to the triangle itself.