Fun with the Riemann Hypothesis (or...To "Prove" the Riemann Hypothesis, Indict a Ham Sandwich)

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Abstract

This paper presents an uncommon variation of proof by induction. We call it deferred induction by recursion. To set up our proof, we state (but do not prove) the *Zeta Induction Theorem*. Using that theorem, we provide an elementary proof of the Riemann Hypothesis. To be clear, we make no claim as to the usefulness of the *Zeta Induction Theorem* to the theory of the Riemann Zeta Function. In fact, we poke a bit of fun at the theorem in our Introduction (and, indirectly, in our Title).

1 Introduction

A prosecutor looking to obtain an easy indictment will use a grand jury. The defendant cannot present exculpatory evidence, and is usually unaware of the grand jury. The prosecutor can present a very one-sided case. In effect, the prosecutor presents evidence that the grand jury *assumes* to be true. With those assumed facts, the prosecutor describes the applicable criminal laws and *voilà* has proof of guilt and therefore an indictment. As explained by New York Judge Sol Wachtler[1, p. 624]: "a grand jury would indict a ham sandwich if that's what the prosecutor wanted."

In this paper, we play the role of prosecutor, you play the role of grand jury, and the Riemann Hypothesis plays the role of ham sandwich. We present only one witness. That witness swears they saw proof of the *Zeta Induction Theorem*. You believe the witness and accept that theorem as true. All that remains is to present you with certain laws of mathematics that, based on the truth of the *Zeta Induction Theorem*, will indict (prove) the Riemann Hypothesis.

2 Definitions

In all that follows, the definitions below are assumed:

Definition. For $m, n \in \mathbb{N}$, define $A_m = \sum_{j=1}^{m} \left(\frac{1}{2}\right)^j$ and $B_{m,n} = \left(\frac{1}{2}\right)^{m+n}$

Definition. For $t \in \mathbb{R}_{>0}$, define $\epsilon(t) = \dfrac{1}{8.463 \cdot log(|t| + 2)}$

Definition. Fix $t \in \mathbb{R}_{>0}$. For $s \in \mathbb{C}$, define the following open rectangles:

$$\begin{aligned} R_R(t) &= \tfrac{1}{2} < Re(s) < 1; & |Im(s)| < t \\ R_L(t) &= 0 < Re(s) < \tfrac{1}{2}; & |Im(s)| < t \\ R_\epsilon(t) &= 1 - \epsilon(t) < Re(s) < 1; & |Im(s)| < t \\ R_{m,n}(t) &= (A_m + B_{m,n}) < Re(s) < 1; & |Im(s)| < t \end{aligned}$$

Definition. $\zeta(s)$ is as defined in Riemann[2].

Definition. We define $ChooseIndex(S_k, \uparrow or \downarrow, \rightarrow limit, k, K, \delta)$ as follows. S_k is a real-valued sequence that is either monotone increasing ($\uparrow$) or monotone decreasing ($\downarrow$), with $\lim_{k\to\infty} S_k = limit$. For the given S_k, these facts are clear by inspection and are not separately proved. Therefore, for the given $\delta > 0$, there is a $K \in \mathbb{N}$ such that for all $k > K$ we have: (1) if monotone decreasing, then $0 \leq (S_k - limit) < \delta$, and (2) if monotone increasing, then $(limit - \delta) < S_k$. We assume that the given K is the K needed for the given δ, and that the given $k > K$.

3 The Zeta Induction Theorem

Theorem 1 (Zeta Induction Theorem)**.** *Let $s \in \mathbb{C}$; $t \in \mathbb{R}_{>0}$. If we assume $\zeta(s) \neq 0$ when $s \in R_{m,n}(t)$, then we have $\zeta(s) \neq 0$ when $s \in R_{m,n+1}(t)$.*

Proof. In this paper we assume (but do not prove) this theorem. □

4 Lemma

This lemma pulls together various statements, with proofs that are either well-known or straightforward. That way, those statements can be used subsequently without detracting from the flow of the discussion.

Lemma 1. *Let $s \in \mathbb{C} \setminus \{1\}$; fix $t \in \mathbb{R}_{>0}$. We have the following:*

i *If $\zeta(s) \neq 0$ for all $s \in R_R(t)$, then $\zeta(s) \neq 0$ for all $s \in R_L(t)$.*

ii *$\zeta(s) \neq 0$ for $s \in R_\epsilon(t)$.*

iii *$(A_m + B_{m,1}) = A_{m+1}$.*

iv *There exists an $M \in \mathbb{N}$ such that $m > M \Rightarrow R_{m,1}(t) \subset R_\epsilon(t)$.*

v *There exists an $N \in \mathbb{N}$ such that $n > N$ and $s \in R_{m,1}(t) \Rightarrow s \in R_{m+1,n}(t)$.*

Proof.

i From Riemann[2]: For $0 \leq Re(s) \leq 1$, if $\zeta(s) = 0$, then $\zeta(1-s) = 0$ (we call them twin zeros). Now assume $\zeta(s) = 0$ for some $s \in R_L(t) \cup R_R(t)$. We consider separately the real and imaginary parts of our twin zeros. *Real Parts*: $Re(s) + Re(1-s) = 1$. Set $\delta = \frac{1}{2} - Re(s)$, Then, $Re(s) = (\frac{1}{2} - \delta)$ and $Re(1-s) = (\frac{1}{2} + \delta)$. *Imaginary Parts*: $|Im(s)| = |Im(1-s)|$. In all cases, we have one of the twin zeros in $R_L(t)$ and the other in $R_R(t)$. Thus, with no zeros in $R_R(t)$ there can be no zeros in $R_L(t)$.

ii From Ford[3]: $\zeta(\beta + it) \neq 0$ for $|t| \geq 3$ and $1 - \beta \leq \dfrac{1}{8.463 \cdot log(|t|)}$.

Ford's statement still holds if we *increase* the size of the denominator, so $\epsilon(t)$ was defined by replacing $log(|t|)$ with $log(|t| + 2)$. For all increasing $|t| \geq 0$, it is easily verified that $\epsilon(t) < 0.2$ and monotone decreasing. As revised by $\epsilon(t)$, Ford's statement extends to all $|t| \geq 0$ because, from Brent[4], there are no zeros in the $R_R(3)$ region. If $s \in R_\epsilon(t)$, we have $\epsilon(t) < \epsilon(Im(s))$, and therefore $\zeta(s) \neq 0$.

iii As defined: $(A_m + B_{m,1}) = \sum_{j=1}^{m} \left(\frac{1}{2}\right)^j + \left(\frac{1}{2}\right)^{m+1} = \sum_{j=1}^{m+1} \left(\frac{1}{2}\right)^j = A_{m+1}$.

iv We $ChooseIndex(A_{m+1}, \uparrow, \rightarrow 1, m+1, M, \delta = \epsilon(t))$. Thus $1 - \epsilon(t) < A_{m+1}$. Using (iii), we have $1 - \epsilon(t) < (A_m + B_{m,1})$. Hence, $R_{m,1}(t) \subset R_\epsilon(t)$.

v Fix $s \in R_{m,1}(t)$ and fix $\epsilon = Re(s) - (A_m + B_{m,1})$. To set $B_{m+1,n} < \epsilon$, we now $ChooseIndex(B_{m+1,n}, \downarrow, \rightarrow 0, n, N, \delta = \epsilon)$. Using (iii), we have: $(A_{m+1} + B_{m+1,n}) < (A_{m+1} + \epsilon) = ((A_m + B_{m,1}) + \epsilon) = Re(s)$. But $(A_{m+1} + B_{m+1,n}) < Re(s)$ means $s \in R_{m+1,n}(t)$. □

5 The Riemann Hypothesis

Theorem 2 (Riemann Hypothesis)**.** *Let* $s \in \mathbb{C} \setminus \{1\}$*, with* $Re(s) \in [0,1] \setminus \{\frac{1}{2}\}$*. Then,* $\zeta(s) \neq 0$*.*

Proof. From Hadamard[5]: $\zeta(s) \neq 0$ for $Re(s) \in \{0, 1\}$. So, we limit our proof to $Re(s) \in (0,1) \setminus \{\frac{1}{2}\}$. Fix $t \in \mathbb{R}_{>0}$. We first show $\zeta(s) \neq 0$ for $s \in R_R(t) \cup R_L(t)$.

Step 1-A (The First Interval). We start by ***assuming*** that $\zeta(s) \neq 0$ when $s \in R_{1,1}(t)$. Now set $m = 1$ and apply the *Zeta Induction Theorem*. It follows by induction that, for all $n \in \mathbb{N}, \zeta(s) \neq 0$ when $s \in R_{1,n}(t)$.

Step 1-B (The Right Strip). Fix $s \in R_R(t)$ and fix $\epsilon = (Re(s) - A_1) > 0$. Now $ChooseIndex(B_{1,n}, \downarrow, \rightarrow 0, n, N, \delta = \epsilon)$. Then $A_1 = \frac{1}{2} < (A_1 + B_{1,n}) < (A_1 + \epsilon) = Re(s)$. But $(A_1 + B_{1,n}) < Re(s)$ implies $s \in R_{1,n}(t)$, so by Step 1-A we have $\zeta(s) \neq 0$. Hence, $\zeta(s) \neq 0$ for all $s \in R_R(t)$.

Step 1-C (The Left Strip). From Lemma 1(i): $\zeta(s) \neq 0$ for $s \in R_L(t)$.

Step 2 (The Second Interval). One problem remains. We *assumed* that $\zeta(s) \neq 0$ for $s \in R_{1,1}(t)$. To prove that, we will now ***assume*** that $\zeta(s) \neq 0$ for $s \in R_{2,1}(t)$. Now set $m = 2$ and apply the *Zeta Induction Theorem*. It follows by induction that, for all $n \in \mathbb{N}, \zeta(s) \neq 0$ when $s \in R_{2,n}(t)$. We have therefore shown that $\zeta(s) \neq 0$ when $s \in R_{1,1}(t)$ because by Lemma 1(v) there is an n such that $s \in R_{1,1}(t)$ implies $s \in R_{2,n}(t)$.

Step 3 (Recursion). We can continue our recursive augment as many times as we like. To prove that $\zeta(s) \neq 0$ when $s \in R_{m,1}(t)$ we need only assume $\zeta(s) \neq 0$ when $s \in R_{m+1,1}(t)$ and then apply the *Zeta Induction Theorem* and Lemma 1(v). But our desired result is eventually established by Lemma 1(ii) and (iv), with no further need for recursion, because there exists an M such that for $m > M, R_{m,1}(t) \subset R_\epsilon(t)$, and we have $\zeta(s) \neq 0$ for $s \in R_\epsilon(t)$.

Step 4 (Wrapping Up). We have established the theorem for $s \in R_R(t) \cup R_L(t)$. But t was arbitrarily chosen, so the result holds for all $t \in \mathbb{R}_{>0}$. □

6 Discussion

Our "proof" of the Riemann Hypothesis (RH) uses deferred induction by recursion, with each inductive step depending, recursively, on a subsequent inductive step. An alternate (but less interesting) approach is also possible. We can recurse in the opposite direction (without deferred induction). We select m using Lemma 1(iv) and have $\zeta(s) \neq 0$ for $s \in R_{m,1}(t) \subset R_\epsilon(t)$. By the *Zeta Induction Theorem* (ZI), $\zeta(s) \neq 0$ for all $R_{m,n}(t)$. Then, using Lemma 1(v) we have $\zeta(s) \neq 0$ for $s \in R_{m-1,1}(t)$. Again applying ZI, we recurse until we reach $R_{1,1}(t)$ and $R_{1,n}(t)$, thereby covering all of $R_R(t)$.

ZI is just one short step away from simply assuming RH. So it should come as no surprise that ZI and RH are equivalent. Proof/disproof of one proves/disproves the other. We showed ZI implies RH. Clearly, RH implies ZI because $\zeta(s) \neq 0$ for $s \in$ all $R_{m,n+1}(t)$. Both are disproved only if $\zeta(s) = 0$ for some $Re(s) \in (0,1) \setminus \{\frac{1}{2}\}$. That said, proof of ZI almost certainly requires direct proof of RH.

References

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