Distance measure of Pythagorean fuzzy sets

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Abstract

The Pythagorean fuzzy set (PFS), as an extension of intuitionistic fuzzy set, is more capable of expressing and handling the uncertainty under uncertain environments. Whereas, how to measure the distance between Pythagorean fuzzy sets appropriately is still an open issue. Therefore, a novel distance measure between Pythagorean fuzzy sets is proposed based on the Jensen–Shannon divergence in this paper. The new distance measure has the following merits: *i*) it meets the axiomatic definition of distance measure; *ii*) it can better indicate the discrimination degree of PFSs. Then, numerical examples are demonstrated that the PFSJS distance measure is feasible and reasonable.

Keywords: Pythagorean fuzzy sets, Distance measure, Jensen–Shannon divergence

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1. Introduction

In recent years, the problems of modeling and dealing with uncertainty have attracted great attentions from researchers in decision theory [1]. Nevertheless, the fuzzy set theory [2] presented by Zadeh in 1965 plays an important role in

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decision theory. For the sake of handling the uncertainty of information more accurately, the fuzzy sets had been extended to intuitionistic fuzzy sets (IFSs) by Atanassov [3] in 1986. Later on, Yager extended the intuitionistic fuzzy sets into the Pythagorean fuzzy sets (PFSs) [4, 5] in 2013.

It is well known that a lot of distance measures have been proposed for the fuzzy sets and IFSs in the last few decades [6]. In this paper, we try to measure the distance of PFSs by the divergence between PFSs to measure the difference of PFSs. Since the square root of Jensen-Shannon divergence is a true metric in the space of probability distributions [7], a novel distance measure between PFSs is proposed based on the Jensen–Shannon divergence. It has been proven in this paper that the new distance measure has some merits, in which it not only meets the axiomatic definition of distance measure, but also can better indicate the discrimination degree of PFSs. After that, numerical examples are demonstrated that the proposed distance measure is more reasonable.

The rest of this paper is organized as follows. Section 2 introduces the preliminaries of this paper briefly. In Section 3, some new distance measures of PFSs are defined. Section 4 illustrates the proposed distance measure. Finally, Section 5 gives a conclusion.

2. Preliminaries

2.1. Pythagorean fuzzy sets

Definition 2.1 *[3] Let X be a finite universe of discourse. An intuitionistic fuzzy set (IFS) A in X is defined by the mathematical form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{1}$$

where the functions

$$\mu_A(x) : X \to [0,1] \quad and \quad \nu_A(x) : X \to [0,1] \tag{2}$$

with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1. \tag{3}$$

The $\mu_A(x)$ is the degree of membership of $x \in X$; $\nu_A(x)$ is the degree of non-membership of $x \in X$.

For an IFS A in X, a hesitancy function of $x \in X$ is defined by

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{4}$$

which reflects the hesitancy degree of $x \in X$.

Definition 2.2 *[4, 5] Let X be a finite universe of discourse. A Pythagorean fuzzy set (PFS) A in X is defined by the mathematical form*

$$A = \{\langle x, A_Y(x), A_N(x)\rangle | x \in X\}, \tag{5}$$

where the functions

$$A_Y(x) : X \to [0,1] \quad and \quad A_N(x) : X \to [0,1] \tag{6}$$

with the condition

$$0 \leq A_Y^2(x) + A_N^2(x) \leq 1. \tag{7}$$

The $A_Y(x)$ is the degree of membership of the element $x \in X$; $A_N(x)$ is the degree of non-membership of the element $x \in X$. Here, $C(x)$ is denoted as the commitment of the membership function, in which $C^2(x) = A_Y^2(x) + A_N^2(x)$.

For the PFS A in X, a hesitancy function of $x \in X$ is defined by

$$H(x) = \sqrt{1 - C^2(x)}, \tag{8}$$

which reflects the uncertainty or a lack of commitment associated with the membership and non-membership grades of $x \in X$.

2.2. Jensen–Shannon divergence measure

Definition 2.3 *[8] Let X be a discrete random variable, P and Q be two probability distributions of X, in which $P = \{p_1, p_2, \dots, p_n\}$ and $Q = \{q_1, q_2, \dots, q_n\}$. The Jensen-Shannon divergence between P and Q is defined by*

$$JS(P, Q) = \frac{1}{2}\left[S\left(P, \frac{P+Q}{2}\right) + S\left(Q, \frac{P+Q}{2}\right)\right], \tag{9}$$

where $S(P, Q) = \sum_i p_i \ log \frac{p_i}{q_i}$ $(1 \le i \le n)$ is the Kullback-Leibler divergence and $\sum_i p_i = \sum_i q_i = 1$.

$JS(P, Q)$ can also be formed as

$$\begin{aligned} JS(P, Q) &= H\left(\frac{P+Q}{2}\right) - \frac{1}{2}H(P) - \frac{1}{2}H(Q), \\ &= \frac{1}{2}\left[\sum_i p_i \ log\left(\frac{2p_i}{p_i+q_i}\right) + \sum_i q_i \ log\left(\frac{2q_i}{p_i+q_i}\right)\right], \end{aligned} \tag{10}$$

where $H(A) = -\sum_i p_i \ log \ p_i$ $(1 \le i \le n)$ is the Shannon entropy.

Then, the square root of Jensen-Shannon divergence can be defined by

$$D_{JS} = \sqrt{JS(P, Q)}. \tag{11}$$

Property 1 *[7] Let P, Q and K be three arbitrary probability distributions of X, then*

(1) $D_{JS}(P, Q) \ge 0$, where $D_{JS}(P, Q) = 0$ iff $P = Q$, for $P, Q \in X$,

(2) $D_{JS}(P, Q) = D_{JS}(Q, P)$, for $P, Q \in X$,

(3) $D_{JS}(P, K) + D_{JS}(K, Q) \ge D_{JS}(P, Q)$, for $P, Q, K \in X$,

(4) $D_{JS}(P, Q) \le 1$, for $P, Q \in X$.

3. New distance measures between PFSs

3.1. Normalized Hamming and Euclidean distances for PFSs

We extend Hamming and Euclidean distances for PFSs as the normalized distances. More specifically, the normalized distances for PFSs are proposed that is concordant with the mathematical notion of normalization.

Let $X = \{x_1, x_2, ..., x_n\}$ be a finite universe of discourse, and A and B be two PFSs in X, in which $A = \{\langle x_i, A_Y(x_i), A_N(x_i)\rangle | x_i \in X\}$ and $B = \{\langle x_i, B_Y(x_i), B_N(x_i)\rangle | x_i \in X\}$.

Definition 3.1 *The normalized Hamming distance measure between PFSs A and B is defined by*

$$\widetilde{D}_{Hm}(A,B) = \frac{1}{2n} \cdot \sum_{i=1}^{n} (|\mu_A^2(x_i) - \mu_B^2(x_i)| + |\nu_A^2(x_i) - \nu_B^2(x_i)| + |\pi_A^2(x_i) - \pi_B^2(x_i)|). \tag{12}$$

Definition 3.2 *The normalized Euclidean distance measure between PFSs A and B is defined by*

$$\widetilde{D}_E(A,B) = \sqrt{\frac{1}{2n} \cdot ((\mu_A^2(x_i) - \mu_B^2(x_i))^2 + (\nu_A^2(x_i) - \nu_B^2(x_i))^2 + (\pi_A^2(x_i) - \pi_B^2(x_i))^2)}. \tag{13}$$

Definition 3.3 *The normalized Chen's distance measure between PFSs A and B is defined by*

$$\widetilde{D}_C(A,B) = [\frac{1}{2n} \cdot ((\mu_A^2(x) - \mu_B^2(x))^\beta + (\nu_A^2(x) - \nu_B^2(x))^\beta + (\pi_A^2(x) - \pi_B^2(x))^\beta)]^{\frac{1}{\beta}}, \tag{14}$$

where β is a distance parameter, satisfying $\beta \geq 1$. If $\beta = 1$, it degenerates to the Hamming distance as Eq. (12). If $\beta = 2$, it degenerates to the Euclidean distance as Eq. (13).

It is clear that the above distance measures satisfy the metric conditions (cf. [9]).

3.2. A new distance measure for PFSs

In Pythagorean fuzzy set (PFS) theory, how to measure the difference between PFSs is still an open issue. Clearly, the square root of Jensen-Shannon divergence is a true metric in the space of probability distributions. Therefore, a new distance measure for PFSs is proposed based on Jensen–Shannon divergence in this section. Then, some essential properties of the proposed distance measure are inferred. It has been proven that the new distance not only satisfies the axiomatic definition of distance measure, but also is capable of distinguishing PFSs more better and can avoid counterintuitive outcomes.

Definition 3.4 *Let X be a finite universe of discourse, and A and B be two Pythagorean fuzzy sets in $PFSs(X)$: $[0,1] \times [0,1] \rightarrow [0,1]$, in which $A = \{\langle x, A_Y(x), A_N(x)\rangle | x \in X\}$ and $B = \{\langle x, B_Y(x), B_N(x)\rangle | x \in X\}$. The Pythagorean fuzzy divergence measure, denoted as $PJS(A,B)$ between two PFSs A and B is defined by*

$$PJS(A,B) = \frac{1}{2}\left[S\left(A, \frac{A+B}{2}\right) + S\left(B, \frac{A+B}{2}\right)\right], \tag{15}$$

with

$$S(A,B) = \sum_{\kappa} A_{\kappa}^2(x)\ log\frac{A_{\kappa}^2(x)}{B_{\kappa}^2(x)}, \quad \kappa \in \{Y, N, H\} \tag{16}$$

$$A_H^2(x) = 1 - A_Y^2(x) - A_N^2(x), \tag{17}$$

where $S(A,B)$ is the Kullback-Leibler divergence and $\sum_{\kappa} A_{\kappa}^2(x) = \sum_{\kappa} B_{\kappa}^2(x) = 1$.

$PJS(A,B)$ can also be formed as

$$PJS(A,B) = H\left(\frac{A+B}{2}\right) - \frac{1}{2}H(A) - \frac{1}{2}H(B),$$
$$= \frac{1}{2}\left[\sum_{\kappa} A_{\kappa}^2(x)\ log\left(\frac{2A_{\kappa}^2(x)}{A_{\kappa}^2(x)+B_{\kappa}^2(x)}\right) + \sum_{\kappa} B_{\kappa}^2(x)\ log\left(\frac{2B_{\kappa}^2(x)}{A_{\kappa}^2(x)+B_{\kappa}^2(x)}\right)\right], \tag{18}$$

with

$$H(A) = -\sum_{\kappa} A_{\kappa}^2(x)\ log\ A_{\kappa}^2(x), \quad \kappa \in \{Y, N, H\} \tag{19}$$

where $H(A)$ is the Shannon entropy.

On the basis of Definition 3.4, a new distance measure for PFSs can be defined based on the square of the Pythagorean fuzzy divergence.

Definition 3.5 *Let X be a finite universe of discourse, and A and B be two Pythagorean fuzzy sets. A new distance measure for PFSs, denoted as $D_{\chi}(A,B)$ between A and B is defined by*

$$D_{\chi}(A,B) = \sqrt{PJS(A,B)}. \tag{20}$$

The properties of the proposed distance measure can be inferred as follows:

Property 2 *Let A, B and C be three arbitrary probability distributions of X, then*

P1. $D_{\chi}(A,B) = 0$ *iff* $A = B$, *for* $A, B \in X$,

P2. $D_{\chi}(A,B) = D_{JS}(B,A)$, *for* $A, B \in X$,

P3. $D_{\chi}(A,B) + D_{JS}(B,C) \geq D_{\chi}(A,C)$, *for* $A, B, C \in X$,

P4. $0 \leq D_{\chi}(A,B) \leq 1$, *for* $A, B \in X$.

Definition 3.6 *Given two PFSs A and B in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$, where $A = \{\langle x_i, A_Y(x_i), A_N(x_i)\rangle | x_i \in X\}$ and $B =$*

$\{\langle x_i, B_Y(x_i), B_N(x_i)\rangle | x_i \in X\}$. The normalized new distance measure between A and B is defined by

$$
\begin{aligned}
\widetilde{D}_{\chi}(A,B) &= \frac{1}{n}\cdot\sum_{i=1}^{n} D_{\chi}(A,B) \\
&= \frac{1}{2n}\left[\sum_{\kappa} A_{\kappa}^2(x_i)\ log\left(\frac{2A_{\kappa}^2(x_i)}{A_{\kappa}^2(x_i)+B_{\kappa}^2(x_i)}\right) + \sum_{\kappa} B_{\kappa}^2(x_i)\ log\left(\frac{2B_{\kappa}^2(x_i)}{A_{\kappa}^2(x_i)+B_{\kappa}^2(x_i)}\right)\right],
\end{aligned}
\tag{21}
$$

where $A_H^2(x) = 1 - A_Y^2(x) - A_N^2(x)$ and $\kappa \in \{Y, N, H\}$.

4. Numerical example

Example 1 Suppose three PFSs A, B and C in the universe of discourse X. Let us randomly set $\alpha = 0.50$ and $\beta = 0.60$. These PFSs are given as follows:

$$A = \langle x, 0.50, 0.60\rangle;$$

$$B = \langle x, 0.60, 0.50\rangle;$$

$$C = \langle x, 0.40, 0.60\rangle.$$

By Eqs. (18)-(20), the distances can be measured as

$$D_{\chi}(A,B) = 0.0693;$$

$$D_{\chi}(A,C) = 0.0593;$$

$$D_{\chi}(C,B) = 0.1141.$$

5. Conclusion

In Pythagorean fuzzy sets (PFSs) theory, how to measure the distance between PFSs properly is still an open issue. To address this issue, in this paper, we first proposed the normalized distance measures. Then, a novel distance measure was proposed based on the Jensen–Shannon divergence. The new distance measure had some advantages than other distance measure methods. To be specific, first, the PFSJS distance measure satisfied the axiomatic definition of distance measure. Secondly, the PFSJS distance measure indicated the discrimination degree of PFSs more better. In summary, this study considers divergence measure of PFSs, which is a main contribution in this work.

Conflict of Interest

The author states that there are no conflicts of interest.

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