Considerations on the Newton's binomial expansion

$$\left(x+y\right)^n = x^n + y^n + xy\sum_{j=0}^{n-2}\left(x^j + y^j\right)\left(x+y\right)^{n-2-j}$$

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Contents

Abstract

The binomial theorem, established by Isaac Newton, is of undeniable importance and finds its use in numerous fields. This study presents alternative formulations to Newton's formula as well as some results that can be derived from it concerning Fermat's Last Theorem.

Chapter 1

Another way to express Newton's binomial formula

1.1 Subject of the chapter

The binomial formula can be rewritten. This new formulation in turn allows for further calculations that highlight certain properties that the original formula does not reveal.

1.2 Another formula

Let n be a given natural number, x and y being two non-zero real numbers. In all that follows, this natural number n is assumed to be greater than or equal to 3. We can write

$$\frac{(x+y)^n - x^n}{(x+y)-x} = \sum_{j=0}^{n-1} (x+y)^{n-1-j}\, x^j = \frac{(x+y)^n - x^n}{y}$$

and likewise

$$\frac{(x+y)^n - y^n}{(x+y)-y} = \sum_{j=0}^{n-1} (x+y)^{n-1-j}\, y^j = \frac{(x+y)^n - y^n}{x}$$

Let's add these two quantities

$$\frac{(x+y)^n - x^n}{y} + \frac{(x+y)^n - y^n}{x} = \sum_{j=0}^{n-1} (x+y)^{n-1-j} \left(x^j + y^j\right)$$

and we get the formula

$$(x+y)^{n+1} - \left(x^{n+1} + y^{n+1}\right) = xy \sum_{j=0}^{n-1} (x+y)^{n-1-j} \left(x^j + y^j\right)$$

that for convenience we write

$$(x+y)^n - (x^n + y^n) = xy \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) \tag{1.1}$$

Newton's formula, which we recall here

$$(x+y)^n = \sum_{j=0}^{n} C_n^j x^{n-j} y^j \tag{1.2}$$

wherein

$$C_n^j = \frac{n!}{(n-j)!j!} \tag{1.3}$$

therefore allows us to establish the equality

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=1}^{n-1} C_n^j x^{n-j-1} y^{j-1}$$

or lastly

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{n-2} C_n^{j+1} x^{n-2-j} y^j$$

1.3 Study of the new formula

Let us write

$$A_n(x,y) = \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) \tag{1.4}$$

Let us first note that

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{p-2} (x+y)^{n-2-j} \left(x^j + y^j\right) + \sum_{j=p-1}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right)$$

with $p \in \mathbb{N}^*$ and $p < n$, or likewise

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{p-2} (x+y)^{(n-p)+(p-2-j)} \left(x^j + y^j\right) + \sum_{j=p-1}^{n-2} (x+y)^{n-2-(j-(p-1)+p-1)} \left(x^{j-(p-1)+p-1} + y^{j-(p-1)+p-1}\right)$$

and

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) =$$

$$(x+y)^{n-p} \sum_{j=0}^{p-2} (x+y)^{p-2-j} \left(x^j + y^j\right)$$
$$+ \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-2-(j+p-1)} \left(x^{j+p-1} + y^{j+p-1}\right)$$

and

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) =$$

$$(x+y)^{n-p} \sum_{j=0}^{p-2} (x+y)^{p-2-j} \left(x^j + y^j\right)$$
$$+ \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-p-1-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

and finally

$$A_n(x,y) = (x+y)^{n-p} A_p(x,y)$$
$$+ \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-p-1-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

Let us now consider the case wherein $n = p + 1$, then

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + \sum_{j=0}^{0} (x+y)^{-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

or likewise

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + \left(x^{p-1} + y^{p-1}\right)$$

but

$$x^{p-1} + y^{p-1} = (x+y)^{p-1} - xyA_{p-1}(x,y)$$

and so

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + (x+y)^{p-1} - xyA_{p-1}(x,y) \qquad (1.5)$$

Let us now focus more specifically on $A_n(x, y)$ and develop this quantity from the formula 1.4 on page 2. Then

$$\begin{aligned}A_n(x,y) &= 3(x+y)^{n-2} + \sum_{j=0}^{n-4}(x+y)^{n-4-j}\left(x^{j+2}+y^{j+2}\right)\\ &= 3(x+y)^{n-2} + \left(x^2+y^2\right)^{n-4} + \sum_{j=0}^{n-5}(x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right)\\ &= 3(x+y)^{n-2} + (x+y)^{n-2} - 2xy(x+y)^{n-4} + \sum_{j=0}^{n-5}(x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right)\\ &= 4(x+y)^{n-2} - 2xy(x+y)^{n-4} + \sum_{j=0}^{n-5}(x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right)\end{aligned}$$

As we continue our calculations in the same manner, we obtain

$$\begin{aligned}A_n(x,y) = 5(x+y)^{n-2}\\ - 5xy(x+y)^{n-4} + \sum_{j=0}^{n-6}(x+y)^{n-6-j}\left(x^{j+4}+y^{j+4}\right)\end{aligned}$$

$$\begin{aligned}A_n(x,y) = 6(x+y)^{n-2} - 9xy(x+y)^{n-4} + 2x^2y^2(x+y)^{n-6}\\ + \sum_{j=0}^{n-7}(x+y)^{n-7-j}\left(x^{j+5}+y^{j+5}\right)\end{aligned}$$

$$\begin{aligned}A_n(x,y) = 7(x+y)^{n-2} - 14xy(x+y)^{n-4} + 7x^2y^2(x+y)^{n-6}\\ + \sum_{j=0}^{n-8}(x+y)^{n-8-j}\left(x^{j+6}+y^{j+6}\right)\end{aligned}$$

$$\begin{aligned}A_n(x,y) = 8(x+y)^{n-2} - 20xy(x+y)^{n-4} + 16x^2y^2(x+y)^{n-6} - 2x^3y^3(x+y)^{n-8}\\ + \sum_{j=0}^{n-9}(x+y)^{n-9-j}\left(x^{j+7}+y^{j+7}\right)\end{aligned}$$

$$\begin{aligned}A_n(x,y) = 9(x+y)^{n-2} - 27xy(x+y)^{n-4} + 30x^2y^2(x+y)^{n-6} - 9x^3y^3(x+y)^{n-8}\\ + \sum_{j=0}^{n-10}(x+y)^{n-10-j}\left(x^{j+8}+y^{j+8}\right)\end{aligned}$$

$$\begin{aligned}A_n(x,y) = 10(x+y)^{n-2} - 35xy(x+y)^{n-4} + 50x^2y^2(x+y)^{n-6} - 25x^3y^3(x+y)^{n-8}\\ + 2x^4y^4(x+y)^{n-10}\\ + \sum_{j=0}^{n-11}(x+y)^{n-11-j}\left(x^{j+9}+y^{j+9}\right)\end{aligned}$$

$$A_n(x,y) = 11(x+y)^{n-2} - 44xy(x+y)^{n-4} + 77x^2y^2(x+y)^{n-6} - 55x^3y^3(x+y)^{n-8} + 11x^4y^4(x+y)^{n-10} + \sum_{j=0}^{n-12}(x+y)^{n-12-j}\left(x^{j+10}+y^{j+10}\right)$$

It is of course possible to push these calculations as far as we wish. By giving n the values 3, 4, 5, 6,..., we deduce the new respective developments of $A_3(x,y)$, $A_4(x,y)$, $A_5(x,y)$, $A_6(x,y)$, etc...

Let us now assume that the following formulas respectively hold true up to ranks $2k$ and $2k+1$, with $k \in \mathbb{N}^*$

$$A_{2k}(x,y) = \sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j (xy)^j (x+y)^{2(k-1-j)} \tag{1.6}$$

$$A_{2k+1}(x,y) = (x+y)\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j (xy)^j (x+y)^{2(k-1-j)} \tag{1.7}$$

The coefficients D_{2k}^{j} and D_{2k+1}^{j} are, if possible, to be made explicit (and will indeed be subsequently).

Let's go back to the equation 1.5 on page 3 and rewrite it in the form

$$A_{2k+2}(x,y) = (x+y)A_{2k+1}(x,y) + (x+y)^{2k} - xyA_{2k}(x,y)$$

Let us now develop this relation

$$A_{2k+2}(x,y) = (x+y)^2\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j (xy)^j (x+y)^{2(k-1-j)} + (x+y)^{2k} - xy\sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

$$\Longleftrightarrow$$

$$A_{2k+2}(x,y) = (x+y)^2\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j (xy)^j (x+y)^{2(k-1-j)} + (x+y)^{2k} + \sum_{j=0}^{k-1} D_{2k}^{j}(-1)^{j+1} (xy)^{j+1} (x+y)^{2(k-1-j)}$$

Let us carry on with our calculations. We obtain in an equivalent manner

$$A_{2k+2}(x,y)=\sum_{j=0}^{k-1}D_{2k+1}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)}$$
$$+(x+y)^{2k}$$
$$+\sum_{j=0}^{k-1}D_{2k}^{j}(-1)^{j+1}(xy)^{j+1}(x+y)^{2(k-1-j)}$$
$$\Longleftrightarrow$$
$$A_{2k+2}(x,y)=\sum_{j=0}^{k-1}D_{2k+1}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)}$$
$$+(x+y)^{2k}$$
$$+\sum_{j=1}^{k}D_{2k}^{j-1}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)}$$
$$\Longleftrightarrow$$
$$A_{2k+2}(x,y)=\sum_{j=1}^{k-1}\left(D_{2k+1}^{j}+D_{2k}^{j-1}\right)(-1)^{j}(xy)^{j}(x+y)^{2(k-j)}$$
$$+D_{2k+1}^{0}(x+y)^{2k}+(x+y)^{2k}+D_{2k}^{k-1}(xy)^{k}$$

and we can write

$$A_{2k+2}(x.y)=\sum_{j=0}^{k}D_{2k+2}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)}$$

with

$$D_{2k+2}^{0}=D_{2k+1}^{0}+1$$
$$D_{2k+2}^{k}=D_{2k}^{k-1}$$

and

$$(\forall j\in\mathbb{N})\,(1\leq j\leq k-1)\left(D_{2k+2}^{j}=D_{2k+1}^{j}+D_{2k}^{j-1}\right)$$

Similarly, we have

$$A_{2k+3}(x,y)=(x+y)\,A_{2k+2}(x,y)+(x+y)^{2k}-xyA_{2k+1}(x,y)$$

Let us make it more explicit

$$A_{2k+3}(x,y)=(x+y)\sum_{j=0}^{k}D_{2k+2}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)}$$
$$+(x+y)^{2k+1}$$
$$-xy\,(x+y)\sum_{j=0}^{k-1}D_{2k+1}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-1-j)}$$

hence

$$A_{2k+3}(x,y) = \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)+1} + (x+y)^{2k+1} + \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j+1} (xy)^{j+1} (x+y)^{2(k-1-j)+1}$$

which is equivalent to

$$A_{2k+3}(x,y) = (x+y)^{2k+1} + \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)+1} + \sum_{j=1}^{k-1} D_{2k+1}^{j-1} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)+1}$$

and also

$$A_{2k+3}(x,y) = \left(D_{2k+2}^{0} + 1\right)(x+y)^{2k+1} + \sum_{j=1}^{k} \left(D_{2k+2}^{j} + D_{2k+1}^{j-1}\right) (-1)^{j} (xy)^{j} (x+y)^{2(k-j)+1}$$

and we can finally write

$$A_{2k+3}(x,y) = (x+y) \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$

with

$$D_{2k+2}^{0} = D_{2k+1}^{0} + 1$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k)\left(D_{2k+3}^{j} = D_{2k+2}^{j} + D_{2k+1}^{j-1}\right)$$

This concludes our mathematical induction and we can write at last as a conclusion

$$(\forall k \in \mathbb{N}^{*})\left(A_{2k} = \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-j)}\right) \tag{1.8}$$

with

$$D_{2k}^{0} = D_{2k-1}^{0} + 1 \Longleftrightarrow D_{2k}^{0} = 2k \tag{1.9}$$

and

$$D_{2k}^{k-1} = D_{2k-2}^{k-2} = \cdots = D_{4}^{1} = 2 \tag{1.10}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k-1)\left(D_{2k}^{j} = D_{2k-1}^{j} + D_{2k-2}^{j-1}\right) \tag{1.11}$$

and as well

$$(\forall k \in \mathbb{N}^{*})\left(A_{2k+1} = (x+y)\sum_{j=0}^{k-1} D_{2k+1}^{j}\,(-1)^{j}\,(xy)^{j}\,(x+y)^{2(k-1-j)}\right) \tag{1.12}$$

with

$$D_{2k+1}^{0} = D_{2k}^{0} + 1 \Longleftrightarrow D_{2k}^{0} = 2k+1 \tag{1.13}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k)\left(D_{2k+1}^{j} = D_{2k}^{j} + D_{2k-1}^{j-1}\right) \tag{1.14}$$

1.4 Values taken by the coefficients D_h^j wherein $(h \in \mathbb{N})$ and $(h \geq 3)$

We have, as we have just established

$$(\forall h \in \mathbb{N})\,(h \geq 3)\left(D_h^0 = h\right)$$

Let us now take $j = 1$. We have

$$D_h^1 = D_{h-1}^1 + D_{h-2}^0$$

We can then write

$$\left.\begin{array}{r} D_h^1 = D_{h-1}^1 + D_{h-2}^0 \\ D_{h-1}^1 = D_{h-2}^1 + D_{h-3}^0 \\ \ldots \\ \ldots \\ \ldots \\ D_5^1 = D_4^1 + D_3^0 \end{array}\right\} \Longrightarrow D_h^1 = \sum_{j=0}^{h-5} D_{h-2-j}^0 + D_4^1$$

but

$$D_{h-2-j}^0 = h-2-j$$

and, according to the relation 1.10 established on page 7

$$D_4^1 = 2$$

hence we get

$$D_h^1 = \sum_{j=0}^{h-5}(h-2-j) + 2 = ((h-2)+(h-3)+(h-4)+\cdots+3)+2$$

and therefore

$$2D_h^1 = h\,(h+3)$$

and finally

$$(\forall h \in \mathbb{N}^{*})\,(h \geq 3)\left(D_h^1 = \frac{h\,(h-3)}{2}\right) \tag{1.15}$$

Clearly

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^1 \in \mathbb{N}\right)$$

By similar calculations, we find for any natural integer $h \geq 3$

$$D_h^2 = \frac{h\,(h-4)\,(h-5)}{6} \tag{1.16}$$

$$D_h^3 = \frac{h\,(h-5)\,(h-6)\,(h-7)}{24} \tag{1.17}$$

There as well

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^2 \in \mathbb{N}\right)$$

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^3 \in \mathbb{N}\right)$$

We then note that the relations 1.11 and 1.14 established on page 8 as well as those (see relations 1.15, 1.16 and 1.17) established on pages 8 and 9 allow us to affirm

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\left(\forall j \in \left\{0, 1, \cdots, \frac{h-4}{2}\right\}\right)\left(D_h^j \in \mathbb{N}\right)$$

Let us now assume, h being chosen even and for all $j \in \left\{0, 1, \cdots, \frac{h-2}{2}\right\}$ the formula

$$D_h^j = \frac{h\,(h-(j+2))!}{(j+1)!\,(h-2\,(j+1))!} \tag{1.18}$$

true up to rank h, for any even natural number less than or equal to h.

Let us also assume true for all $j \in \left\{0, 1, \cdots, \frac{h-4}{2}\right\}$, up to rank $h-1$, for any odd natural number less than or equal to $h-1$, the formula

$$D_{h-1}^j = \frac{(h-1)\,((h-1)-(j+2))!}{(j+1)!\,((h-1)-2\,(j+1))!} \tag{1.19}$$

then

$$D_{h-1}^{j-1} = \frac{(h-1)\,(h-1-(j+1)!)}{j!\,(h-1-2j)!} = \frac{(h-1)\,(h-(j+2)!)}{j!\,(h-1-2j)!}$$

The relation 1.11 established on page 8 allows us to write

$$D_{h+1}^{j}=\frac{h\left(h-(j+2)\right)!}{(j+1)!\left(h-2\left(j+1\right)\right)!}+\frac{\left(h-1\right)\left(h-(j+2)\right)!}{j!\left(h-1-2j\right)!}$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j}=\frac{\left(h-(j+2)\right)!}{j!}\left(\frac{h}{(j+1)\left(h-2\left(j+1\right)\right)!}+\frac{\left(h-1\right)}{\left(h-1-2j\right)!}\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j}=\frac{\left(h-(j+2)\right)!}{j!}\left(\frac{h\left(h-1-2j\right)+\left(h-1\right)\left(j+1\right)}{\left(h-1-2j\right)!\left(j+1\right)}\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j}=\frac{\left(h-(j+2)\right)!}{(j+1)!\left(h-1-2j\right)!}\left(h\left(h-1\right)-2jh+\left(h-1\right)j+\left(h-1\right)\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j}=\frac{\left(h-(j+2)\right)!}{(j+1)!\left(h-1-2j\right)!}\left(h^{2}-1-\left(h+1\right)j\right)$$

and finally

$$D_{h+1}^{j}=\frac{\left(h-(j+2)\right)!}{(j+1)!\left(h-1-2j\right)!}\left(h+1\right)\left(h-1-j\right)$$

We can therefore write

$$D_{h+1}^{j}=\frac{\left(h+1\right)\left(h-(j+1)\right)!}{(j+1)!\left(h+1-2\left(j+1\right)\right)!}\tag{1.20}$$

We would carry out the calculations in the same way assuming h to be odd.

We verify that

$$\left(\forall h\in\mathbb{N}^{*}\right)\left(h\geq 3\right)\left(D_{h}^{0}=h\right)$$

and, denoting $2\mathbb{N}$ the set of even natural integers

$$\left(\forall h=2k\in 2\mathbb{N}^{*}\right)\left(h\geq 4\right)\left(D_{2k}^{k-1}=2\right)$$

We have therefore established at the end of this mathematical induction

$$\left(\forall k\in\mathbb{N}^{*}\right)\left(\forall j\in\{0,1,2,\cdots,k-1\}\right)$$
$$\left(D_{2k+1}^{j}=\frac{\left(2k+1\right)\left(2k-(j+1)\right)!}{(j+1)!\left(2k+1-2\left(j+1\right)\right)!}\right)$$
$$\left(D_{2(k+1)}^{j}=\frac{2\left(k+1\right)\left(2\left(k+1\right)-1-(j+1)\right)!}{(j+1)!\left(2\left(k+1\right)-2\left(j+1\right)\right)!}\right)\tag{1.21}$$

Let us Note that for any natural integer h

$$h-2\left(j+1\right)+\left(j+1\right)=h-\left(j+1\right)$$

We can then write

$$D_{h}^{j}=\frac{h\left(h-(j+1)\right)!}{\left(h-(j+1)\right)(j+1)!\left(h-2\left(j+1\right)\right)!}$$

and also

$$D_{h}^{j}=\frac{h}{h-(j+1)}C_{h-(j+1)}^{j+1}$$

1.5 Study on the coefficients D_h^j

For the following odd natural numbers $h = 2k+1$, we verify by calculation the relations

$$k = 1 \Longleftrightarrow h = 2k+1 = 3$$
$$D_3^0 = 3C_0^0$$

$$k = 2 \Longleftrightarrow h = 2k+1 = 5$$
$$D_5^0 = 5C_1^0$$
$$D_5^1 = 5C_1^1$$

$$k = 3 \Longleftrightarrow h = 2k+1 = 7$$
$$D_7^0 = 7C_2^0$$
$$D_7^1 = 7C_2^1$$
$$D_7^2 = 7C_2^2$$

$$k = 4 \Longleftrightarrow h = 2k+1 = 9$$
$$D_9^0 = 9C_3^0$$
$$D_9^1 = 9C_3^1$$
$$D_9^2 = 9C_3^2 + 3C_0^0$$
$$D_9^1 = 9C_3^3$$

$$k = 5 \Longleftrightarrow h = 2k+1 = 11$$
$$D_{11}^0 = 11C_4^0$$
$$D_{11}^1 = 11C_4^1$$
$$D_{11}^2 = 11\left(C_4^2 + C_1^0\right)$$
$$D_{11}^3 = 11\left(C_4^3 + C_1^1\right)$$
$$D_{11}^4 = 11C_4^4$$

$$k = 6 \Longleftrightarrow h = 2k+1 = 13$$
$$D_{13}^0 = 13C_5^0$$
$$D_{13}^1 = 13C_5^1$$
$$D_{13}^2 = 13\left(C_5^2 + 2C_2^0\right)$$
$$D_{13}^3 = 13\left(C_5^3 + 2C_2^1\right)$$
$$D_{13}^4 = 13\left(C_5^4 + 2C_2^2\right)$$
$$D_{13}^5 = 13C_5^5$$

$k = 7 \Longleftrightarrow h = 2k + 1 = 15$

$$
\begin{aligned}
D_{15}^{0} &= 15C_{6}^{0}\\
D_{15}^{1} &= 15C_{6}^{1}\\
D_{15}^{2} &= 15\left(C_{6}^{2} + 3C_{3}^{0}\right)\\
D_{15}^{3} &= 15\left(C_{6}^{3} + 3C_{3}^{1}\right)\\
D_{15}^{4} &= 15\left(C_{6}^{4} + 3C_{3}^{2} + 3C_{0}^{0}\right)\\
D_{15}^{5} &= 15\left(C_{6}^{5} + 3C_{3}^{3}\right)\\
D_{15}^{6} &= 15C_{6}^{6}
\end{aligned}
$$

$k = 8 \Longleftrightarrow h = 2k + 1 = 17$

$$
\begin{aligned}
D_{17}^{0} &= 17C_{7}^{0}\\
D_{17}^{1} &= 17C_{7}^{1}\\
D_{17}^{2} &= 17\left(C_{7}^{2} + 5C_{4}^{0}\right)\\
D_{17}^{3} &= 17\left(C_{7}^{3} + 5C_{4}^{1}\right)\\
D_{17}^{4} &= 17\left(C_{7}^{4} + 5C_{4}^{2} + C_{0}^{0}\right)\\
D_{17}^{5} &= 17\left(C_{7}^{5} + 5C_{4}^{3} + C_{1}^{1}\right)\\
D_{17}^{6} &= 17\left(C_{7}^{6} + 5C_{4}^{4}\right)\\
D_{17}^{7} &= 17C_{7}^{7}
\end{aligned}
$$

$k = 9 \Longleftrightarrow h = 2k + 1 = 19$

$$
\begin{aligned}
D_{19}^{0} &= 19C_{8}^{0}\\
D_{19}^{1} &= 19C_{8}^{1}\\
D_{19}^{2} &= 19\left(C_{8}^{2} + 7C_{5}^{0}\right)\\
D_{19}^{3} &= 19\left(C_{8}^{3} + 7C_{5}^{1}\right)\\
D_{19}^{4} &= 19\left(C_{8}^{4} + 7C_{5}^{2} + 3C_{2}^{0}\right)\\
D_{19}^{5} &= 19\left(C_{8}^{5} + 7C_{5}^{3} + 3C_{2}^{1}\right)\\
D_{19}^{5} &= 19\left(C_{8}^{6} + 7C_{5}^{4} + 2C_{2}^{2}\right)\\
D_{19}^{6} &= 19\left(C_{8}^{7} + 7C_{5}^{5}\right)\\
D_{19}^{8} &= 19C_{8}^{8}
\end{aligned}
$$

$k = 10 \Longleftrightarrow h = 2k+1 = 21$

$$\begin{aligned}
D_{21}^{0} &= 21C_9^0\\
D_{21}^{1} &= 21C_9^1\\
D_{21}^{2} &= 21\left(C_9^2 + 19C_6^0\right)\\
D_{21}^{3} &= 21\left(C_9^3 + 19C_6^1\right)\\
D_{21}^{4} &= 21\left(C_9^4 + 19C_6^2 + 14C_3^0\right)\\
D_{21}^{5} &= 21\left(C_9^5 + 19C_6^3 + 14C_3^1\right)\\
D_{21}^{6} &= 21\left(C_9^6 + 19C_6^4 + 14C_3^2 + 3C_0^0\right)\\
D_{21}^{7} &= 21\left(C_9^7 + 19C_6^5 + 14C_3^3\right)\\
D_{21}^{8} &= 21\left(C_9^7 + 19C_6^6\right)\\
D_{21}^{9} &= 21C_9^9
\end{aligned}$$

We are led to assume that for any odd natural integer $2k+1$, greater than or equal to 3, each coefficient D_{2k+1}^{j} can be expressed as follows

$$D_{2k+1}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} \tag{1.22}$$

with

$$0 \leq k-1-3l \leq k-1 \tag{1.23}$$

and we will write by convention

$$(\forall j)\,(j-2l<0)\left(C_{k-1-3l}^{j-2l} = 0\right) \tag{1.24}$$

In order to show the validity of this formula for any non-zero natural integer k, we will try to develop, as far as possible, the coefficients F_{2k+1}^{l} as a function of k and l.

For any natural integer k, we verify the relations

$$D_{2k+1}^{0} = (2k+1)\,C_{k-1}^{0}$$

$$D_{2k+1}^{1} = (2k+1)\,C_{k-1}^{1}$$

We can always write with $k \geq 4$

$$D_{2k+1}^{2} = (2k+1)\,C_{k-1}^{2} + \left(D_{2k+1}^{2} - (2k+1)\,C_{k-1}^{2}\right)C_{k-4}^{0}$$

But, in accordance with the relations 1.3 and 1.21 established on pages 2 and 10

$$D_{2k+1}^{2} - (2k+1)\,C_{k-1}^{2} = (2k+1)\left(\frac{(2k-3)!}{3!\,(2k+1-6)!} - \frac{(k-1)!}{2!\,(k-3)!}\right)$$

and similarly

$$D_{2k+1}^{2} - (2k+1)\,C_{k-1}^{2} = (2k+1)\left(\frac{(2k-3)!}{3!\,(2k-5)!} - \frac{(k-1)!}{2!\,(k-3)!}\right)$$

and

$$D_{2k+1}^{2}-(2k+1)\,C_{k-1}^{2}=(2k+1)\left(\frac{(2k-3)\,(2k-4)}{3!}-\frac{(k-1)\,(k-2)}{2!}\right)$$

and

$$D_{2k+1}^{2}-(2k+1)\,C_{k-1}^{2}=(2k+1)\left(\frac{(2k-3)\,(k-2)}{3}-\frac{(k-1)\,(k-2)}{2}\right)$$

and also

$$D_{2k+1}^{2}-(2k+1)\,C_{k-1}^{2}=(2k+1)\left(\frac{2\,(2k-3)\,(k-2)-3\,(k-1)\,(k-2)}{6}\right)$$

and finally

$$D_{2k+1}^{2}-(2k+1)\,C_{k-1}^{2}=\frac{(2k+1)\,(k-2)\,(k-3)}{6}$$

Let us write

$$F_{2k+1}^{1}=D_{2k+1}^{2}-(2k+1)\,C_{k-1}^{2}=\frac{(2k+1)\,(k-2)\,(k-3)}{3!}$$

In the same way, we would find

$$D_{2k+1}^{3}=(2k+1)\left(C_{k-1}^{3}+F_{2k+1}^{1}C_{k-4}^{1}\right)$$

and

$$D_{2k+1}^{4}=(2k+1)\left(C_{k-1}^{4}+F_{2k+1}^{1}C_{k-4}^{2}+F_{2k+1}^{2}C_{k-7}^{0}\right)$$

which gives us

$$F_{2k+1}^{2}=\left(\left(D_{2k+1}^{4}-(2k+1)\,C_{k-1}^{4}\right)-\left(D_{2k+1}^{2}-(2k+1)\,C_{k-1}^{2}\right)C_{k-4}^{2}\right)$$

By calculations similar to the previous ones, the coefficients D_{2k+1}^{j} and C_{k-j}^{l} being made explicit, we find

$$F_{2k+1}^{2}=\frac{(2k+1)\,(k-3)\,(k-4)\,(k-5)\,(k-6)}{5!}$$

We are then led to assume that, for any natural integer $k\geq 1$, the equality

$$F_{2k+1}^{l}=\frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}\tag{1.25}$$

with the natural integer l such that

$$0\leq l\leq\lfloor\frac{k}{3}\rfloor$$

Let us now calculate the difference of the coefficients F_{2k+1}^{l} and F_{2k-1}^{l}

$$F_{2k+1}^{l}-F_{2k-1}^{l}=\frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}-\frac{(2k-1)\,(k-2-l)!}{(2l+1)!\,(k-2-3l)!}$$

Then

$$F^l_{2k+1}-F^l_{2k-1}=\frac{(k-2-l)!}{(2l+1)!\,(k-2-3l)!}\left(\frac{(2k+1)\,(k-1-l)-(2k-1)\,(k-1-3l)}{(k-1-3l)}\right)$$

and

$$F^l_{2k+1}-F^l_{2k-1}=\frac{(k-2-l)!}{(2l+1)!\,(k-1-3l)!}\left((2k+1)\,(k-1-l)-(2k-1)\,(k-1-3l)\right)$$

and also

$$F^l_{2k+1}-F^l_{2k-1}=\frac{(k-2-l)!}{(2l+1)!\,(k-1-3l)!}\left(2\,(2l+1)\,(k-1)\right)$$

and finally

$$F^l_{2k+1}-F^l_{2k-1}=\frac{2\,(k-1)\,(k-2-l)!}{(2l)!\,(k-1-3l)!} \tag{1.26}$$

Our hypothesis 1.22 stated on page 13 now leads us to use a mathematical induction to show the existence of the relation

$$(\forall k\in\mathbb{N}^*)\,(k\geq 1)\,(\forall j\in\mathbb{N})\,(0\leq j\leq k-1)\left(D^j_{2k+1}=\sum_{l=0}^{\lfloor\frac{k}{3}\rfloor}F^l_{2k+1}C^{j-2l}_{k-1-3l}\right)$$

wherein each coefficient F^l_{2k+1} is expressed by the formula 1.25 established on page 14.

Let us assume that the following relation is true up to rank $2k-1$, for any natural number $j\leq k-2$

$$D^j_{2k-1}=\sum_{l=0}^{\lfloor\frac{k-1}{3}\rfloor}F^l_{2k-1}C^{j-2l}_{k-2-3l} \tag{1.27}$$

with

$$F^l_{2k-1}=\frac{(2k-1)\,(k-2-l)!}{(2l+1)!\,(k-2-3l)!}$$

Let us calculate now the difference

$$D^j_{2k-1}-D^{j-1}_{2k-3}=D^j_{2k-2}$$

and also

$$D^j_{2k-2}=\sum_{l=0}^{\lfloor\frac{k-1}{3}\rfloor}F^l_{2k-1}C^{j-2l}_{k-2-3l}-\sum_{l=0}^{\lfloor\frac{k-2}{3}\rfloor}F^l_{2k-3}C^{j-2l-1}_{k-3-3l} \tag{1.28}$$

Then, we are faced with two cases

1.5.1 Case 1: $\lfloor\frac{k-1}{3}\rfloor=\lfloor\frac{k-2}{3}\rfloor=m$

Nous avons We have

$$\lfloor\frac{k-1}{3}\rfloor=m\Longleftrightarrow k-1=3m+\rho>3m$$

The only values that ρ can take a priori are 0, 1 and 2.

1.5.1.1 $\rho = 0$

$$\begin{aligned}\rho = 0 &\Longrightarrow k-1 = 3m \\ &\Longleftrightarrow k-2 = 3m-1 < 3m\end{aligned}$$

1.5.1.2 $\rho = 1$

$$\begin{aligned}\rho = 1 &\Longrightarrow k-1 = 3m+1 \\ &\Longleftrightarrow k-2 = 3m\end{aligned}$$

1.5.1.3 $\rho = 2$

$$\begin{aligned}\rho = 2 &\Longrightarrow k-1 = 3m+2 \\ &\Longrightarrow k-2 = 3m+1\end{aligned}$$

Clearly, ρ cannot be equal to 0. We also note that in this case 1

$$2k+1 \not\equiv 0 \quad (3) \tag{1.29}$$

Let us recall that

$$\left(C^{j-2l}_{k-2-3l} = C^{j-2l-1}_{k-3-3l} + C^{j-2l}_{k-3-3l}\right) \Longleftrightarrow \left(C^{j-2l-1}_{k-3-3l} = C^{j-2l}_{k-2-3l} - C^{j-2l}_{k-3-3l}\right) \tag{1.30}$$

We then have (see the relation 1.28 established on page 15)

$$D^j_{2k-2} = \sum_{l=0}^{m} \left(F^l_{2k-1} C^{j-2l}_{k-2-3l} - F^l_{2k-3} C^{j-2l-1}_{k-3-3l}\right) \tag{1.31}$$

which is equivalent to

$$D^j_{2k-2} = \sum_{l=0}^{m} \left(F^l_{2k-1} C^{j-2l}_{k-2-3l} - F^l_{2k-3}\left(C^{j-2l}_{k-2-3l} - C^{j-2l}_{k-3-2l}\right)\right)$$

and

$$D^j_{2k-2} = \sum_{l=0}^{m} \left(\left(F^l_{2k-1} - F^l_{2k-3}\right) C^{j-2l}_{k-2-3l} + F^l_{2k-3} C^{j-2l}_{k-3-2l}\right)$$

with $m = \lfloor \frac{k-2}{3} \rfloor = \lfloor \frac{k-1}{3}, \rfloor$. In particular, among the natural numbers $2k+1$, where k satisfies this property, we find all prime integers strictly greater than 3.

1.5.2 Case 2: $\lfloor \frac{k-1}{3} \rfloor = \lfloor \frac{k-2}{3} \rfloor + 1 = m$

We have

$$\lfloor \frac{k-1}{3} \rfloor = m \Longleftrightarrow k-1 = 3m+\rho$$

As before,

1.5.2.1 $\rho = 0$

$$\begin{aligned}\rho = 0 &\Longrightarrow k-1 = 3m \\ &\Longleftrightarrow k-2 = 3m-1\end{aligned}$$

1.5.2.2 $\rho = 1$

$$\begin{aligned}\rho = 1 &\Longrightarrow k-1 = 3m+1 \\ &\Longleftrightarrow k-2 = 3m\end{aligned}$$

1.5.2.3 $\rho = 2$

$$\begin{aligned}\rho = 2 &\Longrightarrow k-1 = 3m+2 \\ &\Longleftrightarrow k-2 = 3m+1\end{aligned}$$

And in this case, ρ can only be equal to 0. We also note

$$\begin{aligned}2k+1 \equiv 0 &\Longleftrightarrow k \equiv 1 \quad (3) \\ &\Longleftrightarrow k-1 \equiv 0 \quad (3)\end{aligned}$$

We then have (see the relation 1.28 established on page 15)

$$\begin{aligned}D^j_{2k-2} &= \sum_{l=0}^{m} F^l_{2k-1} C^{j-2l}_{k-2-3l} - \sum_{l=0}^{m-1} F^l_{2k-3} C^{j-2l-1}_{k-3-3l} \\ &= F^m_{2k-1} C^{j-2(m+1)}_{k-2-3(m+1)} + \sum_{l=0}^{m-1} F^l_{2k-1} C^{j-2l}_{k-2-3l} - \sum_{l=0}^{m-1} F^l_{2k-3} C^{j-2l-1}_{k-3-3l} \\ &= F^m_{2k-1} C^{j-2(m+1)}_{k-2-3(m+1)} + \sum_{l=0}^{m-1} \left(\left(F^l_{2k-1} - F^l_{2k-3} \right) C^{j-2l}_{k-2-3l} + F^l_{2k-3} C^{j-2l}_{k-3-3l} \right)\end{aligned}$$

with $m = \lfloor \frac{k-1}{3} \rfloor$ and $m-1 = \lfloor \frac{k-2}{3} \rfloor$.

Let's return to Case 1 and take our hypothesis 1.27 stated on page 15

$$\sum_{l=0}^{m} F^l_{2k-3} C^{j-2l}_{k-3-3l} = D^j_{2k-3}$$

then, in accordance with the relation 1.31 set out on page 16

$$D^j_{2k-2} - D^j_{2k-3} = D^{j-1}_{2k-4}$$

and finally, we get the equality

$$D^{j-1}_{2k-4} = \sum_{l=0}^{\lfloor \frac{k-2}{3} \rfloor} F^l_{2k-4} C^{j-2l}_{k-2-3l} \tag{1.32}$$

with, in accordance with the relation 1.26 established on page 15

$$F_{2k-4}^{l} = F_{2k-1}^{l} - F_{2k-3}^{l}$$

We still have to establish that the equality 1.32 on page 17 is true when $k \geq 4$ describes $\mathbb{N}$. We first ensure by a simple calculation that it is indeed verified when k successively takes the values 4, 5 and $6 \cdots$, while j describes its domain.

We then assume that this equality holds true for any given natural integer less or equal to $2k$, for all $j \leq (k-1)$, that is

$$D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l}$$

We can notice that the calculations made to get the formula giving $D_{h=2k}^{j}$ as a function of the coefficients F_{2k}^{l} and the binomial coefficients C_{k-3l}^{j+1-2l} are generalizable to any value of h in $\mathbb{N}$. We just have to verify by mathematical induction the correctness of the formulation of the odd index h coefficients $D_{h=2k+1}^{j}$ to obtain a valid result whatever the parity of the index h.

Let us go back to the initial hypothesis on the odd index coefficients (see our hypothesis 1.27 stated on page 15) and let us make use of what we just established. We verify

$$D_{2k+1}^{j} = D_{2k}^{j} + D_{2k-1}^{j-1}$$

with

$$D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l}$$

and

$$D_{2k-1}^{j-1} = \sum_{l=0}^{\lfloor \frac{k-1}{3} \rfloor} F_{2k-1}^{l} C_{k-2-3l}^{j-2l}$$

According to the calculations we have just made on pages 16 and 17, we have

$$\begin{aligned}
D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k-2}^{l} C_{k-1-3l}^{j-2l} + F_{2k-1}^{l} C_{k-2-3l}^{j-2l} \\
\Longleftrightarrow D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} \left(F_{2k+1}^{l} - F_{2k-1}^{l}\right) C_{k-1-3l}^{j-2l} + F_{2k-1}^{l} C_{k-2-3l}^{j-2l} \\
\Longleftrightarrow D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} - \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k-1}^{l} C_{k-1-3l}^{j-2l-1} \\
\Longleftrightarrow D_{2k}^{j} &= D_{2k+1}^{j} - D_{2k-1}^{j-1}
\end{aligned}$$

This result is in agreement with the equality 1.14 established on page 8.

As we know how to express F^l_{2k-2} and F^l_{2k-1} against l and k, we can now calculate F^l_{2k+1}. We thus find

$$D^j_{2k+1} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F^l_{2k+1} C^{j-2l}_{k-1-3l}$$

with

$$F^l_{2k+1} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}$$

Our mathematical induction is therefore complete for every coefficient D^j_h, with of even or odd index h.

Let us now summarize all the results we obtained in the previous pages (see the equations 1.8 and 1.12 on pages 7 and 8)

$$(\forall n \in \mathbb{N})\,(n \geq 3)\left((x^n + y^n) = x^n + y^n + xy \sum_{j=1}^{n-2} A_n\,(x,y)\right)$$

with $n = 2k$ (see the equation 1.8 on page 7)

$$A_{2k}\,(x,y) = \sum_{j=0}^{k-1} D^j_{2k}\,(-1)^j\,(xy)^j\,(x+y)^{2(k-1-j)}$$

and

$$(\forall k \in \mathbb{N}^*)\,(k > 1)\,(\forall j \in \mathbb{N})\,(j \leq k-1)\left(D^j_{2k} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F^l_{2k} C^{j+1-2l}_{k-3l}\right)$$

and

$$F^l_{2k} = \frac{2k\,(k-1-l)!}{(2l)!\,(k-3l)!}$$

and for $n = 2k+1$ (see the equation 1.12 on page 8)

$$A_{2k+1}\,(x,y) = (x+y) \sum_{j=0}^{k-1} D^j_{2k+1}\,(-1)^j\,(xy)^j\,(x+y)^{2(k-1-j)}$$

and

$$(\forall k \in \mathbb{N}^*)\,(\forall j \in \mathbb{N})\,(j \leq k-1)\left(D^j_{2k+1} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F^l_{2k+1} C^{j-2l}_{k-1-3l}\right)$$

and

$$F^l_{2k+1} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}$$

1.6 Study of $A_{2k+1}(x,y)$ wherein $k \in \mathbb{N}^*$

We will show in this section how further factorize the quantity $A_{2k+1}(x,y)$. Using the previous results, we can write

$$A_{2k+1}(x,y) = (x+y)\sum_{j=0}^{k-1}\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-j)}$$

We then have, for each k, and for all j and all l

$$\begin{aligned} & F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-j)} \\ & \quad = F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^{j-2l+2l} (xy)^{j-2l+2l} (x+y)^{2(k-1-3l+3l-(j-2l)-2l)} \\ & = \left(F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^{j-2l} (xy)^{j-2l} (x+y)^{2(k-1-3l-(j-2l))}\right) (-1)^{2l} (xy)^{2l} (x+y)^{2l} \end{aligned}$$

We can therefore write $A_{2k+1}(x,y)$ in the following manner

$$\begin{aligned} & A_{2k+1}(x,y) \\ & = (x+y) \\ & \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} (-1)^{2l} (x+y)^{2l} \\ & \sum_{j=0}^{k-1} C_{k-1-3l}^{j-2l} (-1)^{j-2l} (xy)^{j-2l} (x+y)^{2(k-1-3l-(j-2l))} \end{aligned}$$

If j varies from 0 to $k-1$, then $j-2l$ varies from 0 to $k-1-2l$, and as we necessarily have

$$j - 2l \le k - 1 - 3l$$

we get

$$\begin{aligned} & A_{2k+1}(x,y) \\ & = (x+y)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} (-1)^{2l} (x+y)^{2l} \sum_{j=0}^{k-1-3l} C_{k-1-3l}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-3l-j)} \end{aligned}$$

but

$$\begin{aligned} & \sum_{j=0}^{k-1-3l} C_{k-1-3l}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-3l-j)} \\ & = \left((x+y)^{2} - xy\right)^{k-1-3l} \\ & = \left(x^{2} + xy + y^{2}\right)^{k-1-3l} \end{aligned}$$

and lastly

$$A_{2k+1}(x,y) = (x+y)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} (-1)^{2l} (x+y)^{2l} \left(x^{2} + xy + y^{2}\right)^{k-1-3l}$$

If, in addition, we assume that $2k+1$ is an odd natural number strictly greater than 3 and not a multiple of 3, then (see the equality 1.29 on page 16)

$$k-1 \not\equiv 0 \quad (3)$$

and therefore $k-1-3l$ does not vanish for any value of l. Consequently $A_{2k+1}(x,y)$ is always divisible by x^2+xy+y^2 and we can write for any natural number $n=2k+1>3$

$$A_{2k+1}(x,y) =$$
$$(x+y)\left(x^2+xy+y^2\right)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2l}\left(x^2+xy+y^2\right)^{k-2-3l} \quad (1.33)$$

1.7 Various ways to express the Binomial expansion

We are getting now close to the end of this study, the purpose of which was to express the Newton binomial expansion in other manners. As enounced (see the equation 1.2 on page 2) and later established (see the equation 1.1 on page 1), we have

$$\begin{aligned}
(x+y)^n &= \sum_{j=0}^{n} C_n^j x^{n-j} y^j \\
&= x^n + y^n + \sum_{j=1}^{n-1} C_n^j x^{n-j} y^j \\
&= x^n + y^n + xy\sum_{j=0}^{n-2} C_n^{j+1} x^{n-2-j} y^j \\
&= x^n + y^n + xy\sum_{j=0}^{n-2} (x+y)^{n-2-j}\left(x^j+y^j\right)
\end{aligned}$$

Moreover, depending on whether the natural integer n is even or odd, the binomial expansion can be equally expressed as follows

$n=2k$ **even**

$$(x+y)^{2k} = x^{2k} + y^{2k} + xy\sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

with

$$D_{2k}^{j} = \frac{2k\,(2k-1-(j+1))!}{(j+1)!\,(2k-2\,(j+1))!}$$

as established above (see the equation 1.8 on page 7).

$n = 2k + 1$ **odd**

$$(x+y)^{2k+1} = x^{2k+1} + y^{2k+1} + xy\,(x+y) \sum_{j=0}^{k-1} D_{2k+1}^{j} \,(-1)^j\,(xy)^j\,(x+y)^{2(k-1-j)}$$

with

$$D_{2k+1}^{j} = \frac{(2k+1)\,(2k-(j+1))!}{(j+1)!\,(2k+1-2\,(j+1))!}$$

as established above (see the equation 1.12 on page 8).

$n = 2k + 1 > 3$ **and** $n \not\equiv 0 \ (3)$

$$\begin{aligned} (x+y)^n &= x^n + y^n + \\ & xy\,(x+y)\left(x^2+xy+y^2\right) \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}\,(-1)^{2l}\,(x+y)^{2l}\left(x^2+xy+y^2\right)^{k-2-3l} \end{aligned} \tag{1.34}$$

with

$$\begin{aligned} F_{2k+1}^{l} &= \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!} \\ &\equiv 0 \quad (n = 2k+1) \end{aligned} \tag{1.35}$$

as established above (see the equation 1.33 on page 21).

Let us that the set of these natural integers n contains all the odd prime integers distinct from 3.

Outlining these results concludes this study. The following chapter is devoted to the study of some additional properties of the binomial coefficients.

Chapter 2

Some additional considerations on the binomial coefficients

2.1 Subject of the chapter

The binomial coefficients have multiple properties and this chapter is devoted to highlighting some of them.

2.2 The classic binomial expansion rearranged

Let us consider the binomial expansion for any integer exponent $m > 2$

$$(1+1)^m = \sum_{k=0}^{m} C_m^k$$

recalling that

$$(\forall m \in \mathbb{N}^*) \left(C_m^k = C_m^{m-k}\right) \tag{2.1}$$

Let us rearrange it as follows

$$\begin{aligned} (1+1)^m &= \sum_{k=0}^{\lfloor \frac{m}{3} \rfloor} C_m^{3k+0} + \sum_{k=1}^{\lfloor \frac{m}{3} \rfloor} C_m^{3k+1} + \sum_{k=2}^{\lfloor \frac{m}{3} \rfloor} C_m^{3k+2} \\ &= \mathrm{a}_{m,0} + \mathrm{a}_{m,1} + \mathrm{a}_{m,2} \\ &= 2^m \end{aligned} \tag{2.2}$$

with

$$\mathrm{a}_{m,0} = \sum_{k=0}^{\lfloor \frac{m}{3} \rfloor} C_m^{3k+0} \tag{2.3}$$

$$\mathrm{a}_{m,1} = \sum_{k=1}^{\lfloor \frac{m}{3} \rfloor} C_m^{3k+1} \tag{2.4}$$

$$\mathrm{a}_{m,2} = \sum_{k=2}^{\lfloor \frac{m}{3} \rfloor} C_m^{3k+2} \tag{2.5}$$

$\mathrm{a}_{m,0}$, $\mathrm{a}_{m,1}$ and $\mathrm{a}_{m,2}$ are the elements of three distinct sequences indexed by $\mathbb{N}$. Let us look at the values taken by each of the elements of these three sequences. We will have to consider the values taken by m in $\mathbb{Z}/3\mathbb{Z}$.

2.3 The sequences $\mathrm{a}_{m,k}$

2.3.1 $m \equiv 0 \Longleftrightarrow m-1 \equiv 2\ (3)$

Then

$$\begin{aligned}(1+1)^m = &C_m^0 + C_m^3 + \cdots + C_m^{3k} + \cdots + C_m^m \\ &+ C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^{m-2} \\ &+ C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^{m-1}\end{aligned}$$

with

$$\mathrm{a}_{m,0} = C_m^0 + C_m^3 + \cdots + C_m^{3k} + \cdots + C_m^m \tag{2.6}$$

$$\mathrm{a}_{m,1} = C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^{m-2} \tag{2.7}$$

$$\mathrm{a}_{m,2} = C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^{m-1} \tag{2.8}$$

2.3.1.1 The sequence $\mathrm{a}_{m,0}$

$$\begin{aligned}\mathrm{a}_{m,0} &= C_m^0 + C_m^3 + \cdots + C_m^{3K} + \cdots + C_m^{m-3} + C_m^m \\ &= 1 \\ &+ \left(C_{m-1}^2 + C_{m-1}^3\right) + \cdots \\ &+ \left(C_{m-1}^{3K-1} + C_{m-1}^{3K}\right) + \cdots \\ &+ \left(C_{m-1}^{m-1-3} + C_{m-1}^{m-1-2}\right) \\ &+ 1 \\ &= 1 + \left(C_{m-1}^2 + \cdots + C_{m-1}^{3K-1} + \cdots + C_{m-1}^{m-1-3}\right) \\ &+ \left(C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-2}\right) + 1 \\ &= \left(C_{m-1}^2 + \cdots + C_{m-1}^{3K-1} + \cdots + C_{m-1}^{m-1-3} + C_{m-1}^{m-1}\right) \\ &+ \left(C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-2}\right)\end{aligned}$$

In the case wherein $m-1 \equiv 2\ (3)$ (see relations 2.17 and 2.19 on page 28)

$$\mathrm{a}_{m-1,0} = C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-2}$$

$$\mathrm{a}_{m-1,2} = C_{m-1}^2 + C_{m-1}^5 + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1}$$

and therefore

$$\mathrm{a}_{m,0} = \mathrm{a}_{m-1,2} + \mathrm{a}_{m-1,0} \tag{2.9}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$\mathrm{a}_{m-1,2} = \mathrm{a}_{m-1,0}$$

2.3.1.2 The sequence $\mathrm{a}_{m,1}$

$$\begin{aligned}
\mathrm{a}_{m,1} &= C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^{m-2} \\
&= \left(C_{m-1}^0 + C_{m-1}^1\right) \\
&+ \left(C_{m-1}^3 + C_{m-1}^4\right) \\
&+ \cdots \\
&+ \left(C_{m-1}^{3K} + C_{m-1}^{3K+1}\right) \\
&+ \cdots \\
&+ \left(C_{m-1}^{m-1-2} + C_{m-1}^{m-1-1}\right) \\
&= \left(C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + \cdots + C_{m-1}^{m-1-2}\right) \\
&+ \left(C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + \cdots + C_{m-1}^{m-1-1}\right)
\end{aligned}$$

In the case wherein $m - 1 \equiv 2 \ (3)$ (see relations 2.17 and 2.18 on page 28)

$$\mathrm{a}_{m-1,0} = C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-2}$$

$$\mathrm{a}_{m-1,1} = C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1-1}$$

and therefore

$$\mathrm{a}_{m,1} = \mathrm{a}_{m-1,0} + \mathrm{a}_{m-1,1} \tag{2.10}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$\mathrm{a}_{m-1,0} = \mathrm{a}_{m-1,1}$$

2.3.1.3 The sequence $\mathrm{a}_{m,2}$

$$\begin{aligned}
\mathrm{a}_{m,2} &= C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^{m-1} \\
&= \left(C_{m-1}^1 + C_{m-1}^2\right) \\
&+ \left(C_{m-1}^4 + C_{m-1}^5\right) \\
&+ \cdots \\
&+ \left(C_{m-1}^{3K+1} + C_{m-1}^{3K+2}\right) \\
&+ \cdots \\
&+ \left(C_{m-1}^{m-1-1} + C_{m-1}^{m-1}\right) \\
&= \left(C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1-1}\right) \\
&+ \left(C_{m-1}^2 + C_{m-1}^5 + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1}\right)
\end{aligned}$$

In the case wherein $m - 1 \equiv 2 \ (3)$ (see relations 2.18 and 2.19 on page 28)

$$\mathrm{a}_{m-1,1} = C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1-1}$$

$$\mathrm{a}_{m-1,2} = C_{m-1}^2 + C_{m-1}^5 + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1}$$

and therefore

$$\mathrm{a}_{m,2} = \mathrm{a}_{m-1,1} + \mathrm{a}_{m-1,2}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$\mathrm{a}_{m-1,1} = \mathrm{a}_{m-1,2}$$

2.3.2 $m \equiv 1 \Longleftrightarrow m-1 \equiv 0 \ (3)$

then

$$\begin{aligned}(1+1)^m = & C_m^0 + C_m^3 + \cdots + C_m^{3k} + \cdots + C_m^{m-1} \\ & + C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^m \\ & + C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^{m-2}\end{aligned}$$

with

$$\mathrm{a}_{m,0} = C_m^0 + C_m^3 + \cdots + C_m^{3k} + \cdots + C_m^{m-1} \tag{2.11}$$

$$\mathrm{a}_{m,1} = C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^m \tag{2.12}$$

$$\mathrm{a}_{m,2} = C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^{m-2} \tag{2.13}$$

2.3.2.1 The sequence $\mathrm{a}_{m,0}$

$$\begin{aligned}\mathrm{a}_{m,0} &= C_m^0 + C_m^3 + \cdots + C_m^{3K} + \cdots + C_m^{m-1} \\ &= 1 \\ &+ \left(C_{m-1}^2 + C_{m-1}^3\right) \\ &+ \cdots \\ &+ \left(C_{m-1}^{3K-1} + C_{m-1}^{3K}\right) \\ &+ \cdots \\ &+ \left(C_{m-1}^{m-1-1} + C_{m-1}^{m-1}\right) \\ &= \left(C_{m-1}^2 + \cdots + C_{m-1}^{3K-1} + \cdots + C_{m-1}^{m-1-1}\right) \\ &+ \left(1 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1}\right)\end{aligned}$$

In the case wherein $m-1 \equiv 0 \ (3)$ (see relations 2.6 and 2.8 on page 24)

$$\mathrm{a}_{m-1,0} = C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m--1}$$

$$\mathrm{a}_{m-1,2} = C_{m-1}^2 + C_{m-1}^5 + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1-1}$$

and therefore

$$\mathrm{a}_{m,0} = \mathrm{a}_{m-1,0} + \mathrm{a}_{m-1,2} \tag{2.14}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$\mathrm{a}_{m-1,0} = \mathrm{a}_{m-1,2}$$

2.3.2.2 The sequence $\mathrm{a}_{m,1}$

$$\begin{aligned}
\mathrm{a}_{m,1} &= C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^{m-3} + C_m^m \\
&= \left(C_{m-1}^0 + C_{m-1}^1\right) \\
&+ \left(C_{m-1}^3 + C_{m-1}^4\right) \\
&+ \cdots \\
&+ \left(C_{m-1}^{3K} + C_{m-1}^{3K+1}\right) \\
&+ \cdots \\
&+ \left(C_{m-1}^{m-1-3} + C_{m-1}^{m-1-2}\right) \\
&+ 1 \\
&= \left(C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-3} + 1\right) \\
&+ \left(C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1-2}\right)
\end{aligned}$$

In the case wherein $m-1 \equiv 0 \ (3)$ (see relations 2.6 and 2.7 on page 24)

$$\mathrm{a}_{m-1,0} = C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m--1}$$

$$\mathrm{a}_{m-1,1} = C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1-2}$$

and therefore

$$\mathrm{a}_{m,1} = \mathrm{a}_{m-1,0} + \mathrm{a}_{m-1,1} \tag{2.15}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$\mathrm{a}_{m-1,0} = \mathrm{a}_{m-1,1}$$

2.3.2.3 The sequence $\mathrm{a}_{m,2}$

$$\begin{aligned}
\mathrm{a}_{m,2} &= C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^{m-2} \\
&= \left(C_{m-1}^1 + C_{m-1}^2\right) \\
&+ \left(C_{m-1}^4 + C_{m-1}^5\right) + \cdots \\
&+ \left(C_{m-1}^{3K+1} + C_{m-1}^{3K+2}\right) + \cdots \\
&+ \left(C_{m-1}^{m-1-2} + C_{m-1}^{m-1-1}\right) \\
&= \left(C_{m-1}^1 + C_{m-1}^4 \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1-2}\right) \\
&+ \left(C_{m-1}^2 + C_{m-1}^5 + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1-1}\right)
\end{aligned}$$

In the case wherein $m-1 \equiv 0 \ (3)$ (see relations 2.7 and 2.8 on page 24)

$$\mathrm{a}_{m-1,1} = C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1-2}$$

$$\mathrm{a}_{m-1,2} = C_{m-1}^2 + C_{m-1}^5 + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1-1}$$

and therefore

$$\mathrm{a}_{m,2} = \mathrm{a}_{m-1,1} + \mathrm{a}_{m-1,2} \tag{2.16}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$\mathrm{a}_{m-1,1} = \mathrm{a}_{m-1,2}$$

2.3.3 $m \equiv 2 \Longleftrightarrow m-1 \equiv 1 \ (3)$

then

$$\begin{aligned}(1+1)^m &= C_m^0 + C_m^3 + \cdots + C_m^{3k} + \cdots + C_m^{m-2} \\ &+ C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^{m-1} \\ &+ C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^m\end{aligned}$$

with

$$\mathrm{a}_{m,0} = C_m^0 + C_m^3 + \cdots + C_m^{3k} + \cdots + C_m^{m-2} \tag{2.17}$$

$$\mathrm{a}_{m,1} = C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^{m-1} \tag{2.18}$$

$$\mathrm{a}_{m,2} = C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^m \tag{2.19}$$

2.3.3.1 The sequence $\mathrm{a}_{m,0}$

$$\begin{aligned}\mathrm{a}_{m,0} &= C_m^0 + C_m^3 + \cdots + C_m^{3K} + \cdots + C_m^{m-2} \\ &= 1 \\ &+ \left(C_{m-1}^2 + C_{m-1}^3\right) \\ &+ \cdots \\ &+ \left(C_{m-1}^{3K-1} + C_{m-1}^{3K}\right) \\ &+ \cdots \\ &+ \left(C_{m-1}^{m-1-2} + C_{m-1}^{m-1-1}\right) \\ &= \left(C_{m-1}^2 + \cdots + C_{m-1}^{3K-1} + \cdots + C_{m-1}^{m-1-2}\right) \\ &+ \left(1 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-1}\right)\end{aligned}$$

In the case wherein $m-1 \equiv 1 \ (3)$ (see relations 2.11 and 2.13 on page 26)

$$\mathrm{a}_{m-1,0} = C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-1}$$

$$\mathrm{a}_{m-1,2} = C_{m-1}^2 + C_{m-1}^5 + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1-2}$$

and therefore

$$\mathrm{a}_{m,0} = \mathrm{a}_{m-1,0} + \mathrm{a}_{m-1,2} \tag{2.20}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$\mathrm{a}_{m-1,0} = \mathrm{a}_{m-1,2}$$

2.3.3.2 The sequence $\mathrm{a}_{m,1}$

$$\begin{aligned}
\mathrm{a}_{m,1} &= C_m^1 + C_m^4 + \cdots + C_m^{3K+1} + \cdots + C_m^{m-1-3} + C_m^{m-1} \\
&= \left(C_{m-1}^0 + C_{m-1}^1\right) + \cdots \\
&+ \left(C_{m-1}^3 + C_{m-1}^4\right) + \cdots \\
&+ \left(C_{m-1}^{3K} + C_{m-1}^{3K+1}\right) + \cdots \\
&+ \left(C_{m-1}^{m-1-4} + C_{m-1}^{m-1-3}\right) \\
&+ \left(C_{m-1}^{m-1-1} + C_{m-1}^{m-1}\right) \\
&= \left(C_{m-1}^0 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-1}\right) \\
&+ \left(C_{m-1}^1 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1}\right) \\
&= \left(C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-4} + C_{m-1}^{m-1-1}\right) \\
&+ \left(C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1}\right)
\end{aligned}$$

In the case wherein $m - 1 \equiv 1\ (3)$ (see relations 2.11 and 2.12 on page 26)

$$\mathrm{a}_{m-1,0} = C_{m-1}^0 + C_{m-1}^3 + \cdots + C_{m-1}^{3K} + \cdots + C_{m-1}^{m-1-1}$$

$$\mathrm{a}_{m-1,1} = C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1}$$

and therefore

$$\mathrm{a}_{m,1} = \mathrm{a}_{m-1,0} + \mathrm{a}_{m-1,1} \tag{2.21}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$\mathrm{a}_{m-1,0} = \mathrm{a}_{m-1,1}$$

2.3.3.3 The sequence $\mathrm{a}_{m,2}$

$$\begin{aligned}
\mathrm{a}_{m,2} &= C_m^2 + C_m^5 + \cdots + C_m^{3K+2} + \cdots + C_m^{m-3} + C_m^m \\
&= \left(C_{m-1}^1 + C_{m-1}^2\right) \\
&+ \left(C_{m-1}^4 + C_{m-1}^5\right) \\
&+ \cdots \\
&+ \left(C_{m-1}^{3K+1} + C_{m-1}^{3K+2}\right) \\
&+ \cdots \\
&+ \left(C_{m-1}^{m-1-3} + C_{m-1}^{m-1-2}\right) \\
&+ 1 \\
&= \left(C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1-3} + 1\right) \\
&+ \left(C_{m-1}^2 + C_{m-1}^5 + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1-2} + 1\right)
\end{aligned}$$

In the case wherein $m - 1 \equiv 1\ (3)$ (see relations 2.12 and 2.13 on page 26)

$$\mathrm{a}_{m-1,1} = C_{m-1}^1 + C_{m-1}^4 + \cdots + C_{m-1}^{3K+1} + \cdots + C_{m-1}^{m-1}$$

$$a_{m-1,2} = C_{m-1}^{2} + C_{m-1}^{5} + \cdots + C_{m-1}^{3K+2} + \cdots + C_{m-1}^{m-1-2}$$

and therefore

$$a_{m,2} = a_{m-1,1} + a_{m-1,2} \tag{2.22}$$

with (due to the symmetry of the binomial coefficients expressed in the relation 2.1 on page 23)

$$a_{m-1,0} = a_{m-1,2}$$

The relations 2.9, 2.14 and 2.20 on pages 24 and 28 (they are identical for each m), which we recall here

$$a_{m,0} = a_{m-1,0} + a_{m-1,2}$$

$$a_{m,1} = a_{m-1,1} + a_{m-1,0}$$

$$a_{m,2} = a_{m-1,2} + a_{m-1,1}$$

allow us to write

$$a_{m,1} - a_{m,0} = a_{m-1,1} - a_{m-1,2} \tag{2.23}$$

$$a_{m,1} - a_{m,2} = a_{m-1,0} - a_{m-1,2} \tag{2.24}$$

$$a_{m,0} - a_{m,2} = a_{m-1,0} - a_{m-1,1} \tag{2.25}$$

It is then easy to verify that for the following values

$m = 0 \equiv 0 \ (3)$

$$a_{0,0} = 1$$
$$a_{0,1} = 0$$
$$a_{0,2} = 0$$

$m = 1 \equiv 1 \ (3)$

$$a_{1,0} = 1 = a_{0,0} + a_{0,2}$$
$$a_{1,1} = 1 = a_{0,1} + a_{0,0}$$
$$a_{1,2} = 0 = a_{0,2} + a_{0,1}$$

$m = 2 \equiv 2 \ (3)$

$$a_{2,0} = 1 = a_{1,0} + a_{1,2}$$
$$a_{2,1} = 2 = a_{1,1} + a_{1,0}$$
$$a_{2,2} = 1 = a_{1,2} + a_{1,1}$$

$m = 3 \equiv 0 \ (3)$

$$a_{3,0} = 2 = a_{2,0} + a_{2,2}$$
$$a_{3,1} = 3 = a_{2,1} + a_{2,0}$$
$$a_{3,2} = 3 = a_{2,2} + a_{2,1}$$

$m = 4 \equiv 1 \ (3)$

$$\begin{aligned}
\mathrm{a}_{4,0} &= 5 = \mathrm{a}_{3,0} + \mathrm{a}_{3,2} \\
\mathrm{a}_{4,1} &= 5 = \mathrm{a}_{3,1} + \mathrm{a}_{3,0} \\
\mathrm{a}_{4,2} &= 6 = \mathrm{a}_{3,2} + \mathrm{a}_{3,1}
\end{aligned}$$

and we can write

$$(\forall k_1 \in \{0,1,2\})\,(\forall k_2 \in \{0,1,2\})\,(k_1 \neq k_2)\,(|\mathrm{a}_{m,k_1} - \mathrm{a}_{m,k_2}| \leq 1) \tag{2.26}$$

Also, for any given $k \in \{0,1,2\}$, at least one of the three $\mathrm{a}_{m,k}$ is distinct from the other two (see relations 2.23, 2.25 and 2.24 on page 30).

2.4 The sequences $\mathrm{A}_{n,k}$

Let us now consider the relation

$$\sum_{m=0}^{n} \sum_{k=0}^{m} (1+1)^m = \sum_{m=0}^{n} 2^m = 2^{n+1} - 1 \tag{2.27}$$

and let us take the identity 2.2 on page 23. We can form three new sequences $\mathrm{A}_{n,0}$, $\mathrm{A}_{n,1}$ and $\mathrm{A}_{n,2}$

$$\mathrm{A}_{n,0} = \sum_{m=0}^{n} \mathrm{a}_{m,0} = \mathrm{A}_{n-1,0} + \mathrm{a}_{n,0}$$

$$\mathrm{A}_{n,1} = \sum_{m=0}^{n} \mathrm{a}_{m,1} = \mathrm{A}_{n-1,1} + \mathrm{a}_{n,1}$$

$$\mathrm{A}_{n,2} = \sum_{m=0}^{n} \mathrm{a}_{m,2} = \mathrm{A}_{n-1,2} + \mathrm{a}_{n,2}$$

and write, in accordance with the relation 2.27 on page 31

$$S_{\mathrm{A}_n} = \mathrm{A}_{n,0} + \mathrm{A}_{n,1} + \mathrm{A}_{n,2} = 2^{n+1} - 1$$

$$S_{\mathrm{A}_{n-1}} = \mathrm{A}_{n-1,0} + \mathrm{A}_{n-1,1} + \mathrm{A}_{n-1,2} = 2^{n} - 1$$

which leads, unsurprisingly,

$$S_{\mathrm{A}_n} - S_{\mathrm{A}_{n-1}} = \mathrm{a}_{n,0} + \mathrm{a}_{n,1} + \mathrm{a}_{n,2} = 2^n$$

Let us also note that for any integer k

$$2^{2k} \equiv 1 \Longleftrightarrow 2^{2k} - 1 \equiv 0 \quad (3) \tag{2.28}$$

$$2^{2k+1} \equiv 2 \Longleftrightarrow 2^{2k+1} - 1 \equiv 1 \quad (3) \tag{2.29}$$

2.5 The sequences $\mathrm{b}_{m,k}$

Let us construct from the sequences $\mathrm{a}_{m,k}$, wherein $k \in \{0,1,2\}$

$$\mathrm{b}_{m,k} = \mathrm{a}_{m,k} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k} \tag{2.30}$$

Each term of each of these sequences $\mathrm{b}_{m,k}$ takes its value in the set $\{0,1\}$. Moreover, for a given m, one of the three terms $\mathrm{b}_{m,k}$ is distinct from the other two and we can form, a priori, at most $2^3 - 2 = 6$ distinct triplets $(\mathrm{b}_{m,0}, \mathrm{b}_{m,1}, \mathrm{b}_{m,2})$. We have

$$\mathrm{b}_{m-1,0} = \mathrm{a}_{m-1,0} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m-1,k} \iff \mathrm{a}_{m-1,0} = \mathrm{b}_{m-1,0} + \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m-1,k}$$

and likewise

$$\mathrm{b}_{m-1,2} = \mathrm{a}_{m-1,2} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m-1,k} \iff \mathrm{a}_{m-1,2} = \mathrm{b}_{m-1,2} + \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m-1,k}$$

and therefore

$$\begin{aligned}
\mathrm{b}_{m,0} &= \mathrm{a}_{m,0} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k} \\
&= \mathrm{a}_{m-1,0} + \mathrm{a}_{m-1,2} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k} \\
&= \mathrm{b}_{m-1,0} + \mathrm{b}_{m-1,2} + 2 \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m-1,k} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k}
\end{aligned}$$

We obtain in the same way

$$\begin{aligned}
\mathrm{b}_{m,1} &= \mathrm{a}_{m,1} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k} \\
&= \mathrm{a}_{m-1,1} + \mathrm{a}_{m-1,0} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k} \\
&= \mathrm{b}_{m-1,1} + \mathrm{b}_{m-1,0} + 2 \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m-1,k} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{b}_{m,2} &= \mathrm{a}_{m,2} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k} \\
&= \mathrm{a}_{m-1,2} + \mathrm{a}_{m-1,1} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k} \\
&= \mathrm{b}_{m-1,2} + \mathrm{b}_{m-1,1} + 2 \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m-1,k} - \min_{\forall k \in \{0,1,2\}} \mathrm{a}_{m,k}
\end{aligned}$$

and therefore

$$\mathrm{b}_{m,1} - \mathrm{b}_{m,0} = \mathrm{b}_{m-1,1} - \mathrm{b}_{m-1,2} \tag{2.31}$$

$$\mathrm{b}_{m,1} - \mathrm{b}_{m,2} = \mathrm{b}_{m-1,0} - \mathrm{b}_{m-1,2} \tag{2.32}$$

$$\mathrm{b}_{m,0} - \mathrm{b}_{m,2} = \mathrm{b}_{m-1,0} - \mathrm{b}_{m-1,1} \tag{2.33}$$

Let us study the following different cases

$b_{m,1} = b_{m,0} = 0 \Longleftrightarrow b_{m,2} = 1$

$$0 - 0 = b_{m-1,1} - b_{m-1,2}$$

$$0 - 1 = b_{m-1,0} - b_{m-1,2}$$

$$0 - 1 = b_{m-1,0} - b_{m-1,1}$$

which leads to

$$(b_{m-1,1} = b_{m-1,2} = 1) \Longleftrightarrow (b_{m-1,0} = 0)$$

and

$$(b_{m,1} = b_{m,0}) \Longleftrightarrow (b_{m-1,1} = b_{m-1,2}) \tag{2.34}$$

$b_{m,1} = b_{m,0} = 1 \Longleftrightarrow b_{m,2} = 0$

$$1 - 1 = b_{m-1,1} - b_{m-1,2}$$

$$1 - 0 = b_{m-1,0} - b_{m-1,2}$$

$$1 - 0 = b_{m-1,0} - b_{m-1,1}$$

which leads to

$$(b_{m-1,1} = b_{m-1,2} = 0) \Longleftrightarrow (b_{m-1,0} = 1)$$

and

$$(b_{m,1} = b_{m,0}) \Longleftrightarrow (b_{m-1,1} = b_{m-1,2}) \tag{2.35}$$

$b_{m,1} = b_{m,2} = 0 \Longleftrightarrow b_{m,0} = 1$

$$0 - 1 = b_{m-1,1} - b_{m-1,2}$$

$$0 - 0 = b_{m-1,0} - b_{m-1,2}$$

$$1 - 0 = b_{m-1,0} - b_{m-1,1}$$

which leads to

$$(b_{m-1,2} = b_{m-1,0} = 1) \Longleftrightarrow (b_{m-1,1} = 0)$$

and

$$(b_{m,1} = b_{m,2}) \Longleftrightarrow (b_{m-1,2} = b_{m-1,0}) \tag{2.36}$$

$b_{m,1} = b_{m,2} = 1 \Longleftrightarrow b_{m,0} = 0$

$$1 - 0 = b_{m-1,1} - b_{m-1,2}$$

$$1 - 1 = b_{m-1,0} - b_{m-1,2}$$

$$0 - 1 = b_{m-1,0} - b_{m-1,1}$$

which leads to

$$(b_{m-1,2} = b_{m-1,0} = 0) \Longleftrightarrow (b_{m-1,1} = 1)$$

and

$$(b_{m,1} = b_{m,2}) \Longleftrightarrow (b_{m-1,2} = b_{m-1,0}) \tag{2.37}$$

$b_{m,0} = b_{m,2} = 0 \Longleftrightarrow b_{m,1} = 1$

$$1 - 0 = b_{m-1,1} - b_{m-1,2}$$

$$1 - 0 = b_{m-1,0} - b_{m-1,2}$$

$$0 - 0 = b_{m-1,0} - b_{m-1,1}$$

which leads to

$$(b_{m-1,0} = b_{m-1,1} = 1) \Longleftrightarrow (b_{m-1,2} = 0)$$

and

$$(b_{m,0} = b_{m,2}) \Longleftrightarrow (b_{m-1,0} = b_{m-1,1}) \tag{2.38}$$

$b_{m,0} = b_{m,2} = 1 \Longleftrightarrow b_{m,1} = 0$

$$0 - 1 = b_{m-1,1} - b_{m-1,2}$$

$$0 - 1 = b_{m-1,0} - b_{m-1,2}$$

$$1 - 1 = b_{m-1,0} - b_{m-1,1}$$

which leads to

$$(b_{m-1,0} = b_{m-1,1} = 0) \Longleftrightarrow (b_{m-1,2} = 1)$$

and

$$(b_{m,0} = b_{m,2}) \Longleftrightarrow (b_{m-1,0} = b_{m-1,1}) \tag{2.39}$$

The equivalences 2.34, 2.35, 2.36 and 2.37 on page 33 and 2.38 and 2.39 on page 34 now allow us to write

$$(b_{m-1,1} = b_{m-1,0}) \Longleftrightarrow (b_{m-2,1} = b_{m-2,2})$$

$$(b_{m-1,1} = b_{m-1,2}) \Longleftrightarrow (b_{m-2,0} = b_{m-2,2})$$

$$(b_{m-1,0} = b_{m-1,2}) \Longleftrightarrow (b_{m-2,0} = b_{m-2,1})$$

et aussi

$$(b_{m-2,1} = b_{m-2,0}) \Longleftrightarrow (b_{m-3,1} = b_{m-3,2})$$

$$(b_{m-2,1} = b_{m-2,2}) \Longleftrightarrow (b_{m-3,0} = b_{m-3,2})$$

$$(b_{m-2,0} = b_{m-2,2}) \Longleftrightarrow (b_{m-3,0} = b_{m-3,1})$$

and finally

$$\begin{aligned}(b_{m,1} = b_{m,0}) &\Longleftrightarrow (b_{m-1,1} = b_{m-1,2}) \\ &\Longleftrightarrow (b_{m-2,0} = b_{m-2,2}) \\ &\Longleftrightarrow (b_{m-3,0} = b_{m-3,1})\end{aligned} \tag{2.40}$$

$$\begin{aligned}(b_{m,1} = b_{m,2}) &\Longleftrightarrow (b_{m-1,0} = b_{m-1,2}) \\ &\Longleftrightarrow (b_{m-2,0} = b_{m-2,1}) \\ &\Longleftrightarrow (b_{m-3,1} = b_{m-3,2})\end{aligned} \tag{2.41}$$

$$\begin{aligned}(b_{m,0} = b_{m,2}) &\Longleftrightarrow (b_{m-1,0} = b_{m-1,1}) \\ &\Longleftrightarrow (b_{m-2,1} = b_{m-2,2}) \\ &\Longleftrightarrow (b_{m-3,0} = b_{m-3,2})\end{aligned} \tag{2.42}$$

2.6 The sequences $\mathrm{B}_{n,k}$

Let us construct from the sequences $\mathrm{b}_{m,k}$ the sequences

$$\mathrm{B}_{n,k} = \mathrm{B}_{n-1,k} + \mathrm{b}_{n,k}$$

For the first values taken by n, we can manually check

$n = 0 \equiv 1 \ (3)$

$$\begin{aligned} \mathrm{B}_{0,0} &= 1 \\ \mathrm{B}_{0,1} &= 0 \\ \mathrm{B}_{0,2} &= 0 \end{aligned}$$

$n = 1 \equiv 1 \ (3)$

$$\begin{aligned} \mathrm{B}_{1,0} = 2 &= \mathrm{B}_{0,0} + \mathrm{b}_{1,0} = 1 + 1 \\ \mathrm{B}_{1,1} = 1 &= \mathrm{B}_{0,1} + \mathrm{b}_{1,1} = 0 + 1 \\ \mathrm{B}_{1,2} = 0 &= \mathrm{B}_{0,2} + \mathrm{b}_{1,2} = 0 + 0 \end{aligned}$$

$n = 2 \equiv 2 \ (3)$

$$\begin{aligned} \mathrm{B}_{2,0} = 2 &= \mathrm{B}_{1,0} + \mathrm{b}_{2,0} = 2 + 0 \\ \mathrm{B}_{2,1} = 2 &= \mathrm{B}_{1,1} + \mathrm{b}_{2,1} = 1 + 1 \\ \mathrm{B}_{2,2} = 0 &= \mathrm{B}_{1,2} + \mathrm{b}_{2,2} = 0 + 0 \end{aligned}$$

$n = 3 \equiv 0 \ (3)$

$$\begin{aligned} \mathrm{B}_{3,0} = 2 &= \mathrm{B}_{2,0} + \mathrm{b}_{3,0} = 2 + 0 \\ \mathrm{B}_{3,1} = 3 &= \mathrm{B}_{2,1} + \mathrm{b}_{3,1} = 2 + 1 \\ \mathrm{B}_{3,2} = 1 &= \mathrm{B}_{2,2} + \mathrm{b}_{3,2} = 0 + 1 \end{aligned}$$

$n = 4 \equiv 1 \ (3)$

$$\begin{aligned} \mathrm{B}_{4,0} = 2 &= \mathrm{B}_{3,0} + \mathrm{b}_{4,0} = 2 + 0 \\ \mathrm{B}_{4,1} = 3 &= \mathrm{B}_{3,1} + \mathrm{b}_{4,1} = 3 + 0 \\ \mathrm{B}_{4,2} = 2 &= \mathrm{B}_{3,2} + \mathrm{b}_{4,2} = 1 + 1 \end{aligned}$$

$n = 5 \equiv 2 \ (3)$

$$\begin{aligned} \mathrm{B}_{5,0} = 3 &= \mathrm{B}_{4,0} + \mathrm{b}_{5,0} = 2 + 1 \\ \mathrm{B}_{5,1} = 3 &= \mathrm{B}_{4,1} + \mathrm{b}_{5,1} = 3 + 0 \\ \mathrm{B}_{5,2} = 3 &= \mathrm{B}_{4,2} + \mathrm{b}_{5,2} = 2 + 1 \end{aligned}$$

$n = 6 \equiv 0 \ (3)$

$$\begin{aligned} \mathrm{B}_{6,0} = 4 &= \mathrm{B}_{5,0} + \mathrm{b}_{6,0} = 3 + 1 \\ \mathrm{B}_{6,1} = 3 &= \mathrm{B}_{5,1} + \mathrm{b}_{6,1} = 3 + 0 \\ \mathrm{B}_{6,2} = 3 &= \mathrm{B}_{5,2} + \mathrm{b}_{6,2} = 3 + 0 \end{aligned}$$

$n = 7 \equiv 1 \ (3)$

$$\begin{aligned} \mathrm{B}_{7,0} &= 5 = \mathrm{B}_{6,0} + \mathrm{b}_{7,0} = 4 + 1 \\ \mathrm{B}_{7,1} &= 4 = \mathrm{B}_{6,1} + \mathrm{b}_{7,1} = 3 + 1 \\ \mathrm{B}_{7,2} &= 3 = \mathrm{B}_{6,2} + \mathrm{b}_{7,2} = 3 + 0 \end{aligned}$$

$n = 8 \equiv 2 \ (3)$

$$\begin{aligned} \mathrm{B}_{8,0} &= 5 = \mathrm{B}_{7,0} + \mathrm{b}_{8,0} = 5 + 0 \\ \mathrm{B}_{8,1} &= 5 = \mathrm{B}_{7,1} + \mathrm{b}_{8,1} = 4 + 1 \\ \mathrm{B}_{8,2} &= 3 = \mathrm{B}_{7,2} + \mathrm{b}_{8,2} = 3 + 0 \end{aligned}$$

$n = 9 \equiv 0 \ (3)$

$$\begin{aligned} \mathrm{B}_{9,0} &= 5 = \mathrm{B}_{8,0} + \mathrm{b}_{9,0} = 5 + 0 \\ \mathrm{B}_{9,1} &= 6 = \mathrm{B}_{8,1} + \mathrm{b}_{9,1} = 5 + 1 \\ \mathrm{B}_{9,2} &= 4 = \mathrm{B}_{8,2} + \mathrm{b}_{9,2} = 3 + 1 \end{aligned}$$

$n = 10 \equiv 1 \ (3)$

$$\begin{aligned} \mathrm{B}_{10,0} &= 5 = \mathrm{B}_{9,0} + \mathrm{B}_{6,0} = 5 + 0 \\ \mathrm{B}_{10,1} &= 6 = \mathrm{B}_{9,1} + \mathrm{B}_{6,1} = 6 + 0 \\ \mathrm{B}_{10,2} &= 5 = \mathrm{B}_{9,2} + \mathrm{B}_{6,2} = 4 + 1 \end{aligned}$$

$n = 11 \equiv 2 \ (3)$

$$\begin{aligned} \mathrm{B}_{11,0} &= 6 = \mathrm{B}_{10,0} + \mathrm{b}_{11,0} = 5 + 1 \\ \mathrm{B}_{11,1} &= 6 = \mathrm{B}_{10,1} + \mathrm{b}_{11,1} = 6 + 0 \\ \mathrm{B}_{11,2} &= 6 = \mathrm{B}_{10,2} + \mathrm{b}_{11,2} = 5 + 1 \end{aligned}$$

Several remarks can be made following this verification. As for the sequences $\mathrm{b}_{m,k}$, we show below the values they take for $m \leq 11$ and $k \in \{0, 1, 2\}$

m	$\mathrm{b}_{m,0}$	$\mathrm{b}_{m,1}$	$\mathrm{b}_{m,2}$
0	1	0	0
1	1	1	0
2	0	1	0
3	0	1	1
4	0	0	1
5	1	0	1
6	1	0	0
7	1	1	0
8	0	1	0
9	0	1	1
10	0	0	1
11	1	0	1

This table tells us that the triplets $(\mathrm{b}_{m,0}, \mathrm{b}_{m,1}, \mathrm{b}_{m,2})$ are periodically repeated between $m = M$ and $m = M + 6$ and that these triplets over this period are distinct two by two. Also

$$\sum_{m=l}^{5+l} \mathrm{b}_{m,0} = \sum_{m=l}^{5+l} \mathrm{b}_{m,1} = \sum_{m=l}^{5+l} \mathrm{b}_{m,2} = 3 \tag{2.43}$$

with $l \in \{0, 1, 2, 3, 4, 5, 6\}$

Moreover, the values taken by the elements of the sequences $\mathrm{B}_{n,k}$, when m and k vary, show that the terms of the triplets $(\mathrm{B}_{n,0}, \mathrm{B}_{n,1}, \mathrm{B}_{n,2})$ are pairwise equal when $k = 5$ and $k = 11$.

Also, manual verification of the values taken by the elements of the sequences $\mathrm{B}_{n,k}$ when $m \leq 11$ indicates to us that the difference

$$(\forall k_1 \in \{0, 1, 2\})\,(\forall k_2 \in \{0, 1, 2\})\,(k_1 \neq k_2)\,(|\mathrm{B}_{n,k_1} - \mathrm{B}_{n,k_2}| \leq 2)$$

The relation 2.30 on page 32 and the equivalences 2.40, 2.41 and 2.42 established on page 34 now allow us to conclude

$$\begin{aligned}
&(\forall n \in \mathbb{N}) \\
&(\forall\,(\mathrm{A}_{n,0}, \mathrm{A}_{n,1}, \mathrm{A}_{n,2})) \\
&((\mathrm{A}_{n,0} = \mathrm{A}_{n,1} = \mathrm{A}_{n,2}) \Longleftrightarrow n \equiv 5 \quad (6))
\end{aligned} \tag{2.44}$$

and

$$\begin{aligned}
&(\forall n \in \mathbb{N}) \\
&(\forall k_1 \in \{0, 1, 2\}) \\
&(\forall k_2 \in \{0, 1, 2\}) \\
&(k_1 \neq k_2) \\
&(|\mathrm{B}_{n,k_1} - \mathrm{B}_{n,k_2}| \leq 2)
\end{aligned}$$

which implies, by construction

$$\begin{aligned}
&(\forall n \in \mathbb{N}) \\
&(\forall k_1 \in \{0, 1, 2\}) \\
&(\forall k_2 \in \{0, 1, 2\}) \\
&(k_1 \neq k_2) \\
&(|\mathrm{A}_{n,k_1} - \mathrm{A}_{n,k_2}| \leq 2)
\end{aligned} \tag{2.45}$$

We now turn to the study of Fermat's conjecture, a conjecture which was proved by Andrew Wiles (1993/1995).

Chapter 3

Study of Fermat's conjecture

3.1 Subject of the chapter

This conjecture was proved by Andrew Wiles between 1993 and 1995. However, as it has been the case for other problems in the history of Mathematics, setting up to explore other avenues that could lead to other demonstrations is not without interest. This what we are going to try and show.

3.2 Reminder of the conjecture

Let the equation

$$x^n + y^n = z^n \tag{3.1}$$

with n prime integer, $n > 2 \in \mathbb{N}^*$. Pierre de Fermat (1607-1665) stated that no three non zero natural integers $x \in \mathbb{N}^*$, $y \in \mathbb{N}^*$ and $z \in \mathbb{N}^*$ could satisfy the relation 3.1. We shall assume

$$0 < x < y < z$$

This leads us to write

$$\begin{aligned} x + y \equiv z \quad (n) &\Longleftrightarrow x + y = kn + z \, (k \in \mathbb{N}^*) \\ &\Longrightarrow x > kn \end{aligned} \tag{3.2}$$

Leaving aside the case wherein $n = 3$, we are going to be interested in all the other cases wherein $n > 3$. It is possible, without loss of generality, to consider only the cases wherein n is prime.

3.3 First point

Let us therefore, if they exist, be $x \in \mathbb{N}^*$, $y \in \mathbb{N}^*$ and $z \in \mathbb{N}^*$ which satisfy the relation 3.1 (see in this page 38). It is then always possible to suppose that

x, y and z are pairwise coprime. We have

$$\begin{aligned} z^n &= x^n + y^n \\ &= x^n - (-1)^n y^n \\ &= (x - (-1)\, y) \sum_{j=0}^{n-1} x^{n-1-j} (-1)^j y^j \end{aligned}$$

In the case where $z \not\equiv 0 \ (n)$, $(x - (-1)\, y)$ and $\sum_{j=0}^{n-1} x^{n-1-j} (-1)^j y^j$ are two quantities that are coprime and we have

$$(\exists h \in \mathbb{N}^*)\, (x + y = h^n)$$

and

$$z^n \equiv 0 \Longrightarrow z \equiv 0 \quad (h)$$

Finally, the relation 1.34 and the formula 1.35 stated on page 22 allow us to write

$$\begin{aligned} (x+y)^n - z^n &= ((x+y) - z) \sum_{j=0}^{n-1} (x+y)^{n-1-j} z^j \\ &= n\lambda xy\, (x+y) \left(x^2 + xy + y^2\right) \end{aligned} \tag{3.3}$$

with $\lambda \in \mathbb{N}^*$.

It is obvious that the natural integers x, y, $(x+y)$ and $\left(x^2 + xy + y^2\right)$ are pairwise coprime.

3.4 Second point

Let n and p be two odd prime integers, distinct from each other or not. Let us place ourselves in $\mathbb{Z}/p\mathbb{Z}$, the set of integers modulo p, equipped with addition $(+)$ and multiplication $(*)$. This set, equipped with these two laws, is a commutative field and each of its elements u has an inverse $u^{-1} = u^{p-2}$. Let us also consider x, y and z, solutions, if they exist, of the equation 3.1 on page 38 as stated above (see section 3.2 on page 38).

We first notice that $x^2 + xy + y^2$ is always an odd natural integer. We can now choose p such that

$$x^2 + xy + y^2 \equiv 0 \quad (p) \tag{3.4}$$

Let us consider the relation 3.3 on this page 39 and review, when $p \neq n$, the following two cases:

3.4.0.1 $(x+y) - z \equiv 0 \ (p)$

We have

$$(x+y) - z \equiv 0 \Longleftrightarrow (x+y) \equiv z \quad (p)$$

and

$$\begin{aligned} x^2 + 2xy + y^2 &\equiv xy \quad (p) \\ &\equiv z^2 \quad (p) \end{aligned}$$

and in this case, it is evident that $z \not\equiv 0 \ (p)$. And we must have

$$\begin{aligned} x &\not\equiv 0 \quad (p) \\ y &\not\equiv 0 \quad (p) \\ (x+y) &\not\equiv 0 \quad (p) \\ z &\not\equiv 0 \quad (p) \end{aligned}$$

also

$$\begin{aligned} \sum_{j=0}^{n-1} (x+y)^{n-1-j} z^j &\equiv n (x+y)^{n-1} \quad (p) \\ &\equiv n z^{n-1} \quad (p) \\ &\not\equiv 0 \quad (p) \end{aligned}$$

3.4.0.2 $(x+y) - z \not\equiv 0 \ (p)$

We have

$$(x+y) - z \not\equiv 0 \Longleftrightarrow x + y \not\equiv z \quad (p)$$

and therefore necessarily

$$\sum_{j=0}^{n-1} (x+y)^{n-1-j} z^j \equiv 0 \quad (p)$$

Moreover

$$\begin{aligned} z \equiv 0 &\Longrightarrow z^n \equiv 0 \quad (p) \\ &\Longleftrightarrow x^n + y^n \equiv 0 \quad (p) \\ &\Longleftrightarrow (x+y)^n \equiv 0 \quad (p) \\ &\Longleftrightarrow x + y \equiv 0 \quad (p) \end{aligned}$$

But we have just shown that

$$x + y \not\equiv 0 \quad (p)$$

and therefore

$$z \not\equiv 0 \quad (p) \tag{3.5}$$

In both cases, $(x+y) - z$ and $\sum_{j=0}^{n-1} (x+y)^{n-1-j} z^j$ cannot simultaneously cancel out in $\mathbb{Z}/p\mathbb{Z}$. Also

$$(\forall x \in \mathbb{N}^*) (\forall y \in \mathbb{N}^*) (\forall z \in \mathbb{N}^*) (x^n + y^n = z^n) (z \not\equiv 0 \quad (p)) \tag{3.6}$$

3.5 Third point

Before we can proceed with the proof of Fermat's conjecture, we have two remarks to make. In what follows, we continue to place ourselves in $(\mathbb{Z}/p\mathbb{Z}, +, *)$, where p is a prime integer dividing $x^2 + xy + y^2$.

3.5.1 First remark

Recall that in $\mathbb{Z}/p\mathbb{Z}$

$$(\forall a \in \mathbb{Z}/p\mathbb{Z})\,(a \not\equiv 0 \quad (p))\,\big(a^{p-1} - 1 \equiv 0 \quad (p)\big)$$

Let a, b and c be three non-zero natural integers such that

$$a - b \equiv c \quad (p)$$

We have

$$a^{p-1} = a^{p-2}a \equiv 1 \quad (p)$$

and we can write

$$a - b \equiv c \quad (p) \Longleftrightarrow 1 - a^{p-2}b \equiv a^{p-2}c \quad (p)$$

We will agree to write, by abuse of notation

$$(\forall a \in \mathbb{N}^*)\left(a^{-1} = \frac{1}{a} \equiv a^{p-2} \quad (p)\right)$$

Let us write

$$\frac{x+y}{z} = (x+y)\,z^{-1} \equiv u_n \quad (p)$$

wherein $x \in \mathbb{N}^*$, $y \in \mathbb{N}^*$ and $z \in \mathbb{N}^*$, satisfy the relation 3.1 on page 38.

Then (see relations 3.3 and 3.4 on page 39)

$$u_n^n \equiv 1 \quad (p)$$

Clearly,

$$u_n^{p-1} \equiv 1 \quad (p)$$

and there exists $\nu \in \mathbb{Z}/p\mathbb{Z}$ such that

$$u_n^n \equiv u_n^\nu \equiv 1 \quad (p)$$

Note that in $\mathbb{Z}/p\mathbb{Z}$, ν is necessarily a divisor of $p-1$. Then

$$u_n^n \equiv 1 \quad (p) \Longleftrightarrow (\exists k_p \in \mathbb{N}^*)\,(n = k_p\nu)$$

but, n is assumed to be prime and consequently

$$n = \nu$$

We deduce that $n < p$ and that $p-1$ is a multiple of n.

3.5.2 Second remark

The integers x and y each being distinct from p

$$(\exists u_3 \in \mathbb{Z}/p\mathbb{Z})\,\big(u_3 = xy^{-1} \not\equiv 1 \quad (p)\big)\left(\sum_{j=0}^{2} u_3^j \equiv 0 \Longleftrightarrow u_3^3 \equiv 1 \quad (p)\right)$$

3.6 A proof of the conjecture

Let

$$u_3 = xy^{-1} \tag{3.7}$$

$$\begin{aligned}
\sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^j &= \sum_{j=0}^{n-1}\left(\frac{y}{z}\right)^j\left(1+\frac{x}{y}\right)^j \\
&\equiv \sum_{j=0}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{j} C_j^k u_3^k \quad (p) &(3.8)\\
&\equiv \sum_{j=0}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{\lfloor\frac{j}{3}\rfloor}(C_j^{3k+0}u_3^0 + C_j^{3k+1}u_3^1 + C_j^{3k+2}u_3^2) \quad (p) &(3.9)
\end{aligned}$$

and also

$$\begin{aligned}
\sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^j \equiv &\sum_{j=0}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{\lfloor\frac{j}{3}\rfloor}(C_j^{3k+0}u_3^0)\\
&+\sum_{j=1}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{\lfloor\frac{j}{3}\rfloor}(C_j^{3k+1}u_3^1)\\
&+\sum_{j=2}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{\lfloor\frac{j}{3}\rfloor}(C_j^{3k+2}u_3^2) \quad (p)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^j \equiv &\, u_3^0\sum_{j=0}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{\lfloor\frac{j}{3}\rfloor}(C_j^{3k+0})\\
&+u_3^1\sum_{j=1}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{\lfloor\frac{j}{3}\rfloor}(C_j^{3k+1})\\
&+u_3^2\sum_{j=2}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{\lfloor\frac{j}{3}\rfloor}(C_j^{3k+2}) \quad (p)
\end{aligned}$$

since $u_3^{3k+r} \equiv u_3^{3k}u_3^r \equiv u_3^r \ (p)$ with $r \in \{0,1,2\}$. And we can write

$$\sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^j \equiv \mathcal{A}_0u_3^0 + \mathcal{A}_1u_3^1 + \mathcal{A}_2u_3^2 \quad (p) \tag{3.10}$$

with, by taking the notations used in the relations 2.3, 2.4 and 2.5 on page 23

$$\begin{aligned}
\mathcal{A}_0 &\equiv \sum_{j=0}^{n-1}\left(\frac{y}{z}\right)^j\sum_{k=0}^{\lfloor\frac{j}{3}\rfloor}C_j^{3k+0} \quad (p)\\
&\equiv \sum_{j=0}^{n-1}\left(\frac{y}{z}\right)^j \mathrm{a}_{j,0} \quad (p) &(3.11)
\end{aligned}$$

$$\mathcal{A}_1 \equiv \sum_{j=1}^{n-1} \left(\frac{y}{z}\right)^j \sum_{k=0}^{\lfloor \frac{j}{3} \rfloor} C_j^{3k+1} \quad (p)$$
$$\equiv \sum_{j=1}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,1} \quad (p) \tag{3.12}$$

$$\mathcal{A}_2 \equiv \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \sum_{k=0}^{\lfloor \frac{j}{3} \rfloor} C_j^{3k+2} \quad (p)$$
$$\equiv \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,2} \quad (p) \tag{3.13}$$

Also, it is obvious that for each of the sums 3.11, 3.12 and 3.13, when r takes the values 0, 1 and 2

$$(\forall j)\,(\forall k)\,(3k + r \leq j)$$

Let's take the equalities 3.11 on page 42, 3.12 and 3.13 on this page 43 and write

$$\mathcal{A}_0 - \mathcal{A}_1 \equiv \sum_{j=0}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,0} - \sum_{j=1}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,1} \quad (p)$$
$$\equiv \mathrm{a}_{0,0} + \sum_{j=1}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,0} - \sum_{j=1}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,1} \quad (p)$$

$$\mathcal{A}_0 - \mathcal{A}_2 \equiv \sum_{j=0}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,0} - \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,2} \quad (p)$$
$$\equiv \mathrm{a}_{0,0} + \mathrm{a}_{1,0}\frac{y}{z} + \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,0} - \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,2} \quad (p)$$

$$\mathcal{A}_1 - \mathcal{A}_2 \equiv \sum_{j=1}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,1} - \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,2} \quad (p)$$
$$\equiv \mathrm{a}_{1,1} + \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,1} - \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \mathrm{a}_{j,2} \quad (p)$$

Let us write

$$\mathrm{a}_{j,0} - \mathrm{a}_{j,1} = \alpha_{j,(0,1)} \Longleftrightarrow \mathrm{a}_{j,0} = \mathrm{a}_{j,1} + \alpha_{j,(0,1)} \tag{3.14}$$
$$\mathrm{a}_{j,0} - \mathrm{a}_{j,2} = \alpha_{j,(0,2)} \Longleftrightarrow \mathrm{a}_{j,0} = \mathrm{a}_{j,2} + \alpha_{j,(0,2)}$$
$$\mathrm{a}_{j,1} - \mathrm{a}_{j,2} = \alpha_{j,(1,2)} \Longleftrightarrow \mathrm{a}_{j,1} = \mathrm{a}_{j,2} + \alpha_{j,(1,2)}$$

Note that these last three equalities lead to

$$\alpha_{j,(0,2)} = \alpha_{j,(0,1)} + \alpha_{j,(1,2)} \tag{3.15}$$

Then

$$\mathcal{A}_0 - \mathcal{A}_1 \equiv \mathrm{a}_{0,0} + \sum_{j=1}^{n-1} \left(\frac{y}{z}\right)^j \alpha_{j,(0,1)} \quad (p)$$

$$\equiv \mathrm{a}_{0,0} + \alpha_{1,(0,1)} \frac{y}{z} + \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \alpha_{j,(0,1)} \quad (p) \qquad (3.16)$$

$$\mathcal{A}_0 - \mathcal{A}_2 \equiv \mathrm{a}_{0,0} + \mathrm{a}_{1,0} \frac{y}{z} + \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \alpha_{j,(0,2)} \quad (p) \qquad (3.17)$$

$$\mathcal{A}_1 - \mathcal{A}_2 \equiv \mathrm{a}_{1,1} + \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \alpha_{j,(1,2)} \quad (p) \qquad (3.18)$$

and we also have, by summing the équalité 3.16 and 3.18 on this page 44 (see also the équalité 3.15 on page 43)

$$\mathcal{A}_0 - \mathcal{A}_2 \equiv \mathrm{a}_{0,0} + \alpha_{1,(0,1)} \frac{y}{z} + \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \alpha_{j,(0,1)} + \mathrm{a}_{1,1} + \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \alpha_{j,(1,2)} \quad (p)$$

$$\equiv \mathrm{a}_{0,0} + \alpha_{1,(0,1)} \frac{y}{z} + \mathrm{a}_{1,1} + \sum_{j=2}^{n-1} \left(\frac{y}{z}\right)^j \alpha_{j,(0,2)} \quad (p)$$

and therefore

$$\mathrm{a}_{0,0} + \alpha_{1,(0,1)} \frac{y}{z} + \mathrm{a}_{1,1} \equiv \mathrm{a}_{0,0} + \mathrm{a}_{1,0} \frac{y}{z} \Longleftrightarrow \alpha_{1,(0,1)} \frac{y}{z} + \mathrm{a}_{1,1} \equiv \mathrm{a}_{1,0} \frac{y}{z} \quad (p) \quad (3.19)$$

but in accordance with the equalities 2.6 on page 24, 2.11 on page 26 and 2.17 on page 28

$$\mathrm{a}_{0,0} = C_0^0 = 1$$
$$\mathrm{a}_{1,0} = C_1^0 = 1$$

and similarly, in accordance with the equalities 2.7 on page 24, 2.12 on page 26 and 2.18 on page 28

$$\mathrm{a}_{1,1} = C_1^1 = 1$$

and in accordance with the relationship 3.14 on page 43

$$\alpha_{1,(0,1)} = \mathrm{a}_{1,0} - \mathrm{a}_{1,1} = 0$$

and therefore the relation 3.19 on this page 44 is written

$$1 \equiv \frac{y}{z} \Longleftrightarrow z \equiv y \quad (p)$$

We then notice that by swapping x and y, we can also write $u_3 = x^{-1}y$ and

$$\sum_{j=0}^{n-1} \left(\frac{x+y}{z}\right)^j = \sum_{j=0}^{n-1} \left(\frac{x}{z}\right)^j \left(1 + \frac{y}{x}\right)^j$$

$$\equiv \sum_{j=0}^{n-1} \left(\frac{x}{z}\right)^j \sum_{k=0}^{j} C_j^k u_3^k \quad (p)$$

$$\equiv \sum_{j=0}^{n-1} \left(\frac{x}{z}\right)^j \sum_{k=0}^{\lfloor \frac{j}{3} \rfloor} \left(C_j^{3k+0} u_3^0 + C_j^{3k+1} u_3^1 + C_j^{3k+2} u_3^2\right) \quad (p)$$

with this time

$$u_3 = x^{-1}y$$

and we deduce from this

$$1 \equiv \frac{x}{z} \Longleftrightarrow z \equiv x \quad (p)$$

But then

$$x \equiv y \Longrightarrow x^2 + xy + y^2 \equiv 3x^2 \equiv 3y^2 \equiv 0 \quad (p) \tag{3.20}$$

but we have shown (see conclusion of section 3.5.1 on page 41)

$$p > n > 3 \Longrightarrow x^2 \equiv y^2 \equiv 0 \quad (p) \tag{3.21}$$

which contradicts the initial assumptions in section 3.3 on page 38.

The conjecture is thus proved for every prime integer $n > 3$. **QED**.