Homotopy analysis method for solving a class of nonlinear mixed Volterra-Fredholm integro-differential equations of fractional order

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Abstract: In this paper, we describe the solution approaches based on Homotopy Analysis Method for the follwing Nonlinear Mixed Volterra-Fredholm integro-differential equation of fractional order

$$\begin{aligned} {}^{C}D^{\alpha}u(t) &= \varphi(t)+\lambda\int_0^t\int_0^T k(x,s)F\left(u(s)\right)dxds, \\ u^{(i)}(0) &= c_i, \quad i=0,...,n-1, \end{aligned} \tag{1}$$

where $t\in\Omega=[0;T]$, $k:\Omega\times\Omega\longrightarrow\mathbb{R}$, $\varphi:\Omega\longrightarrow\mathbb{R}$, are known functions, $F:C(\Omega,\mathbb{R})\longrightarrow\mathbb{R}$ is nonlinear function, c_i $(i=0,...,n-1)$ and λ are constants, ${}^{C}D^{\alpha}$ is the Caputo derivative of order α with $n-1<\alpha\leq n$. In addition some examples are used to illustrate the accuracy and validity of this approach.

Keywords: Homotopy Analysis Method; Caputo fractional derivative; Volterra-Fredholm integro-differential equation.

AMS 2010 Mathematics Subject Classifcation: 34A08, 26A33.

1 Introduction

To be completed.

The reader is advised to read the references [1-8].

2 Preliminaries

Definition 1 *Let $\alpha \in \mathbb{R}^+$ and $f \in L^1[a,b]$.The Riemann-Liouville fractional integral of ordre α for a function f is defined as*

$$\left(J^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{2}$$

with Γ is Gamma Euler function defined as

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$$

where $t \in [a,b]$

Definition 2 *Let $f \in L^1[a,b]$ and $\alpha \in \mathbb{R}^+$ where $n-1 < \alpha \leq n$, The Riemann-Liouville fractional derivative of ordre α.for a function f is defined as*

$$^{RL}D_t^{\alpha} f(t) = D^n J_a^{n-\alpha} f(t), \tag{3}$$

with $D^n = \frac{d^n}{dt^n}$.

Definition 3 *The Caputo fractional derivative of ordre $\alpha \in \mathbb{R}^+$.for a function f is defined as*

$$^{C}D_a^{\alpha} f(t) = J^{n-\alpha} \left(\frac{d^n}{dt^n} f(t) \right), \tag{4}$$

where $\in L^1[a,b], n-1 < \alpha \leq n,\ n \in \mathbb{N}^*$.

Remark 4 *Let $\alpha > 0$ and $\beta > 0$, for all $f \in L^1[a,b]$, we have the follewing properties:*

$$J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t) = J^{\alpha+\beta} f(t) \tag{5}$$

$$^{C}D_a^{\alpha} \left[J_a^{\alpha} f(t)\right] = f(t) \tag{6}$$

$$J_a^{\alpha} \left[{}^{C}D_a^{\alpha} f(t)\right] = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!} \tag{7}$$

$$J^{\beta} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\beta+\mu+1)} t^{\beta+\mu}, \quad \mu > -1 \tag{8}$$

3 Basic idea of Homotopy Analysis Method

Now we construct the zero-order deformation equation

$$(1-q)\pounds[\phi(t;q) - u_0(t)] = q\hbar H(t) N[\phi(t;q)], \tag{9}$$

subject to the following initial conditions

$$u_0(t) = \phi(t;0), \tag{10}$$

where $\pounds$ is the linear operator, $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $u_0(t)$ is an initial guess of the solution $u(t)$ and $\phi(t; q)$ is an unknown function on the dependent variables t and q.

Zeroth-order deformation equation

When the parameter q increases from 0 to 1, then the homotopy solution $\phi(t; q)$ varies from $u_0(t)$ to solution $u(t)$ of the original equation (1). Using the parameter q. The function $\phi(t; q)$ can be expanded in Taylor series as follows:

$$\phi(t; q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) q^m, \tag{11}$$

where

$$u_m(t) = \frac{1}{m!} \left. \frac{\partial^m \phi(t; q)}{\partial q^m} \right|_{q=0}. \tag{12}$$

Assuming that the auxiliary parameter $\hbar$ is properly selected so that the above series is convergent when $q = 1$, then the solution $u(t)$ can be given by

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t). \tag{13}$$

Hight-order deformation equation

Define the vectore:

$$\overrightarrow{u}_n = \{u_0(t), u_1(t), u_2(t), ..., u_n(t)\}. \tag{14}$$

Differentiating the zero-order deformation eauqtion (9) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the socalled mth-order deformation equation:

$$\pounds[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t) \Re_m(\overrightarrow{u}_{m-1}(t)), \tag{15}$$

where

$$\Re_m(\overrightarrow{u}_{m-1}(t)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(t; q)]}{\partial q^{m-1}} \right|_{q=0}, \tag{16}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{17}$$

4 Main results

We Consider the Nonlinear Mixed Volterra-Fredholm integro-differential equation of fractional order.

$$\begin{cases} {}^C D^\alpha u(t) = \varphi(t) + \lambda \int_0^t \left(\int_0^T k(x, s) F(u(s)) ds \right) dx, \\ u^{(i)}(0) = c_i, \; i = 0, 1, ..., n-1, \end{cases} \tag{18}$$

where ${}^{C}D^{\alpha}$ is the Caputo derivative of order α,with $n-1<\alpha\leqslant n$, $\Omega=[0;T]$, $T>0$, $k:\Omega\times\Omega\longrightarrow\mathbb{R}$, $\varphi:\Omega\longrightarrow\mathbb{R}$, are known functions, $F:C(\Omega,\mathbb{R})\longrightarrow\mathbb{R}$ is nonlinear function, c_i $(i=0,...,n-1)$, and λ are constants.

Note that the high-order deformation Eq.(9) is governing by the linear operator $\pounds$ and the term $\Re_m(\overrightarrow{u}_{m-1}(t))$, can be expressed simply by (15) for any nonlinear operator N. We are now ready to construct a series solution corresponding to the integro-differential equation (18). For this purpose, let

$$N[\phi(t;q)]={}^{C}D^{\alpha}\phi(t;q)-\varphi(t)-\lambda\int_0^t\int_0^T k(x,s)F(\phi(s;q))dxds. \tag{19}$$

The corresponding m^{th}-order deformation Eq. (19) reads:

$$\pounds[u_m(t)-\chi_m u_{m-1}(t)]=\hbar H(t)\Re_m(\overrightarrow{u}_{m-1}(t)). \tag{20}$$

One has:

$$\begin{aligned}\Re_m(\overrightarrow{u}_{m-1}(t)) &= -(1-\chi_m)\varphi(t)+\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}}\;{}^{C}D^{\alpha}\phi(t,q)\right]_{q=0}\\ &\quad -\frac{\lambda}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}}\int_0^t\int_0^T k(x,s)F(\phi(s,q))dxds\right]_{q=0}\end{aligned} \tag{21}$$

It is worth to present a simple iterative scheme for $u_m(t)$. To this end, the linear operator $\pounds$ is chosen to be $\pounds=\frac{d^{\eta}}{dt^{\eta}}$, as an initial guess $u_0(t)$ is taken, a nonzero auxiliary function $H(t)=1$ are taken. This is substituted into (20) to give the recurrence relation:

$$u_m(t)-\chi_m u_{m-1}(t)=\hbar J^{\eta}\Re_m(\overrightarrow{u}_{m-1}(t)), \tag{22}$$

where $\alpha\leqslant\eta\leqslant n$, and

$$u_m^{(i)}(0)=b_k,\quad i=0,...,n-1. \tag{23}$$

By Eq. (21) and Eq.(22), we obtain

$$\begin{aligned}u_m(t) &= \chi_m u_{m-1}(t)+\hbar J^{\eta-\alpha}u_{m-1}(t)-(1-\chi_m)\hbar J^{\eta}\varphi(t)\\ &\quad -\hbar\sum_{k=0}^{n-1}\frac{b_k}{k!}\frac{\Gamma(k+1)}{\Gamma(\eta-\alpha+k+1)}t^{\eta-\alpha+k}\\ &\quad -\frac{\lambda\hbar}{(m-1)!}J^{\eta}\int_0^t\left[\int_0^T k(x,s)\frac{\partial^{m-1}}{\partial q^{m-1}}F\left(\sum_{m=0}^{+\infty}u_m(s)q^m\right)ds\right]_{q=0}dx,\end{aligned} \tag{24}$$

which yields

$$\begin{aligned} u_m(t) &= \chi_m u_{m-1}(t) - (1-\chi_m)\,\hbar\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\varphi(s)ds \\ &+\hbar\int_0^t \frac{(t-s)^{\eta-\alpha-1}}{\Gamma(\eta-\alpha)}u_{m-1}(s)ds - \hbar\sum_{k=0}^{n-1}\frac{b_k\Gamma(k+1)}{k!\Gamma(\eta-\alpha+k+1)}t^{\eta-\alpha+k} \\ &-\frac{\lambda\hbar}{(m-1)!}\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} \\ &\times\left(\int_0^s\Big[\int_0^T k(x,y)\frac{\partial^{m-1}}{\partial q^{m-1}}F\Big(\sum_{m=0}^{+\infty}u_m(y)q^m\Big)dy\Big]_{q=0}dx\right)ds. \end{aligned} \tag{25}$$

Special case: if F is linear function. Choose $F(u(s)) = u(s)$, we get

$$\begin{aligned} u_m(t) &= \chi_m u_{m-1}(t) + \hbar\int_0^t \frac{(t-s)^{\eta-\alpha-1}}{\Gamma(\eta-\alpha)}u_{m-1}(s)ds \\ &-\hbar\sum_{k=0}^{n-1}\frac{b_k\Gamma(k+1)}{k!\Gamma(\eta-\alpha+k+1)}t^{\eta-\alpha+k} - (1-\chi_m)\,\hbar\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\varphi(s)ds \\ &-\frac{\lambda\hbar}{(m-1)!}\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\left(\int_0^s\Big[\int_0^T k(x,y)u_{m-1}(y)dy\Big]dx\right)ds. \end{aligned} \tag{26}$$

5 Applications

We consider the follwing problem

$$\begin{cases} {}^C D^\alpha u(t) = 2t + \lambda\displaystyle\int_0^t\left(\int_0^1 (s-x)\left[(u(s))^2 - u(s)\right]ds\right)dx, \\ u^{(i)}(0) = 0, \quad i = 0,1,...,n-1, \end{cases} \tag{27}$$

where $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}^*$, $t \in [0,1]$, $\lambda \in \mathbb{R}$.
Choose $\pounds = {}^C D^\eta$, with $\alpha \leqslant \eta \leqslant n$, we obtain

$${}^C D^\eta\left[u_m(t) - \chi_m u_{m-1}(t)\right] = \hbar H(t)\Re_m(\overrightarrow{u}_{m-1}(t)), \tag{28}$$

and

$$\begin{aligned} \Re_m(\overrightarrow{u}_{m-1}(t)) &= {}^C D^\alpha u_{m-1}(t) - (1-\chi_m)(2t) \\ &-\lambda\sum_{k=0}^{m-1}\int_0^t\left(\int_0^1 (s-x)u_k(s)u_{m-1-k}(s)ds\right)dx \\ &+\lambda\int_0^t\left(\int_0^1 (s-x)u_{m-1}(s)ds\right)dx, \end{aligned} \tag{29}$$

subject to the initial conditions

$$u_m^{(i)}(0) = 0,\ i = 0, 1, ..., n-1. \tag{30}$$

5.1 Convergence theorem

Theorem 5 *Let the serie* $\sum_{m=0}^{+\infty} u_m(t)$ *is converge where* $u_m \in C(\Omega, \mathbb{R})$ *is produced by high-order deformation (28) and the serie* $\sum_{m=0}^{+\infty} D^\alpha u_m(t)$ *also converge. Then* $\sum_{m=0}^{+\infty} u_m(t)$ *is the exact solution of the problem (27)*

Proof. We have $\sum_{m=0}^{+\infty} u_m(t)$ converge, then $\lim_{m\to+\infty} u_m(t) = 0$. And

$$\sum_{m=1}^{n} [u_m(t) - \chi_m u_{m-1}(t)] = u_n(t), \tag{31}$$

thus

$$\lim_{n\to+\infty} \sum_{m=1}^{n} [u_m(t) - \chi_m u_{m-1}(t)] = \lim_{n\to+\infty} u_n(t) = 0, \tag{32}$$

we obtain

$$\begin{aligned} {}^C D^\eta \sum_{m=1}^{+\infty} [u_m(t) - \chi_m u_{m-1}(t)] &= \sum_{m=1}^{+\infty} {}^C D^\eta [u_m(t) - \chi_m u_{m-1}(t)] \\ &= \hbar H(t) \sum_{m=1}^{+\infty} \Re_m(\overrightarrow{u}_{m-1}(t)) = 0. \end{aligned} \tag{33}$$

By $\hbar \neq 0$ and $H(t) \neq 0$, we get

$$\sum_{m=1}^{+\infty} \Re_m(\overrightarrow{u}_{m-1}(t)) = 0. \tag{34}$$

Using (29), we have

$$\begin{aligned}
\sum_{m=1}^{+\infty} \Re_m(\overrightarrow{u}_{m-1}(t)) &= \sum_{m=1}^{+\infty} D^{\alpha} u_{m-1}(t) - \sum_{m=1}^{+\infty} (1-\chi_m)(2t) \\
&\quad -\lambda \sum_{m=1}^{+\infty}\sum_{k=0}^{m-1} \int_0^t \left(\int_0^1 (s-x) u_k(s) u_{m-1-k}(s) ds \right) dx \\
&\quad +\lambda \sum_{m=1}^{+\infty} \int_0^t \left(\int_0^1 (s-x) u_{m-1}(s) ds \right) dx \\
&= \sum_{m=1}^{+\infty} D^{\alpha} u_{m-1}(t) - 2t \\
&\quad -\lambda \int_0^t \left(\int_0^1 (s-x) \sum_{m=1}^{+\infty}\sum_{k=0}^{m-1} u_k(s) u_{m-1-k}(s) ds \right) dx \\
&\quad +\lambda \int_0^t \left(\int_0^1 (s-x) \sum_{m=1}^{+\infty} u_{m-1}(s) ds \right) dx \\
&= \sum_{m=0}^{+\infty} D^{\alpha} u_m(t) - 2t \\
&\quad -\lambda \int_0^t \left(\int_0^1 (s-x) \sum_{m=0}^{+\infty} u_m(s) \sum_{k=0}^{+\infty} u_k(s) ds \right) dx \\
&\quad +\lambda \int_0^t \left(\int_0^1 (s-x) \sum_{m=0}^{+\infty} u_m(s) ds \right) dx \\
&= {}^{C}D^{\alpha} \sum_{m=0}^{+\infty} u_m(t) - 2t \\
&\quad -\lambda \int_0^t \left(\int_0^1 (s-x) \left(\sum_{m=0}^{+\infty} u_m(s) \right)^2 ds \right) dx \\
&\quad +\lambda \int_0^t \left(\int_0^1 (s-x) \sum_{m=0}^{+\infty} u_m(s) ds \right) dx \\
&= {}^{C}D^{\alpha} S(t) - 2t - \lambda \int_0^t \left(\int_0^1 (s-x) \left[S^2(s) - S(s) \right] ds \right) dx,
\end{aligned}$$

where $S(t) = \sum_{m=0}^{+\infty} u_m(t)$. By Eq. (34) we have

$$ {}^{C}D^{\alpha} S(t) - 2t - \lambda \int_0^t \left(\int_0^1 (s-x) \left[S^2(s) - S(s) \right] ds \right) dx = 0. \tag{35}$$

Using Eq. (30), the inicial condition

$$S(0)=\sum_{m=0}^{+\infty}u_m(0)=0. \tag{36}$$

Therefore $\sum_{m=0}^{+\infty}u_m(t)$ is the exact solution of the Eq. (27).
The proof is complete. $\blacksquare$

a) If choose to the initial condition

$$u_0(t)=0, \tag{37}$$

then, we obtain

$$u_1(t)=-\frac{2\hbar}{\Gamma(\eta+2)}t^{\eta+1}, \tag{38}$$

and

$$\begin{aligned} u_2(t) &= +\frac{2\lambda\hbar^2}{\left[\Gamma(\eta+3)\right]^2}t^{\eta+2}-\left(\frac{2\lambda\hbar^2}{(\eta+3)\left[\Gamma(\eta+2)\right]^2}+\frac{2\hbar}{\Gamma(\eta+2)}\right)t^{\eta+1} \\ &\quad -\frac{2\hbar^2}{\Gamma(2\eta-\alpha+2)}t^{2\eta-\alpha+1}, \end{aligned} \tag{39}$$

$$\vdots$$

which yields

$$\begin{aligned} u(t) &\simeq u_0(t)+u_1(t)+u_2(t) \\ &\simeq \frac{2\lambda\hbar^2}{\left[\Gamma(\eta+3)\right]^2}t^{\eta+2}-\left(\frac{2\lambda\hbar^2}{(\eta+3)\left[\Gamma(\eta+2)\right]^2}+\frac{4\hbar}{\Gamma(\eta+2)}\right)t^{\eta+1} \\ &\quad -\frac{2\hbar^2}{\Gamma(2\eta-\alpha+2)}t^{2\eta-\alpha+1} \end{aligned} \tag{40}$$

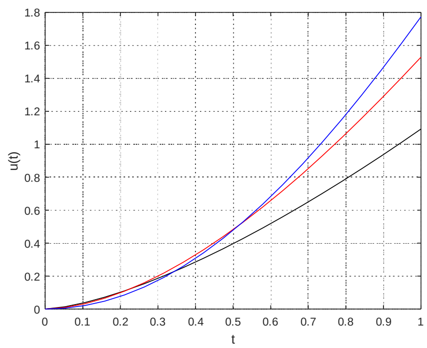

FIG 1: $\lambda = 1$, $\alpha = 0.5$ black line:$\hbar = -0.4$, $\eta = 0.5$, red line:$\hbar = -0.7$, $\eta = 0.75$, blue line:$\hbar = -1, \eta = 1$.

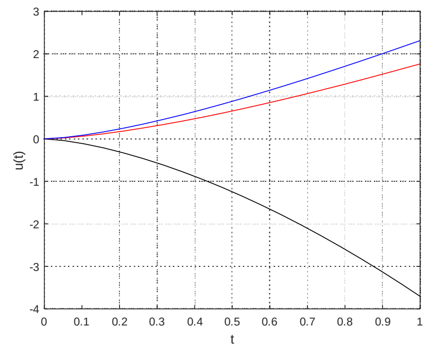

FIG 2: $\lambda = 1$, $\eta = \alpha = 0.5$ black line:$\hbar = 1$, red line:$\hbar = -0.7$, blue line:$\hbar = -1$.

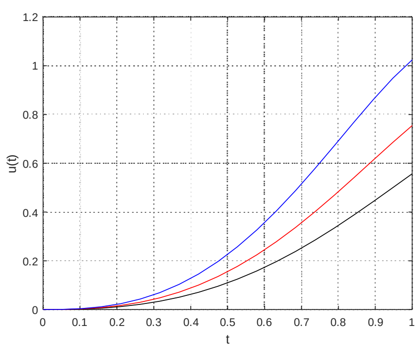

FIG 3: $\lambda = 3$, $\eta = \alpha = 1.5$ black line:$\hbar = -0.5$, red line:$\hbar = -0.7$, blue line:$\hbar = -1$.

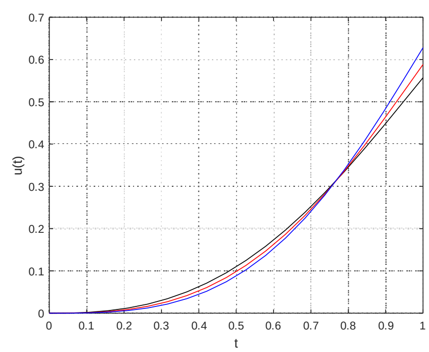

FIG 4: $\lambda = 3$, $\alpha = 1.5$ black line:$\hbar = -0.5$, $\eta = 1.5$, red line:$\hbar = -0.7$, $\eta = 1.75$, blue line:$\hbar = -1, \eta = 2$.

To be completed.

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[9] To be completed.