Arithmetic of Analysis
(Supremum and Infimum)

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Abstract:*I can still remember my expression and feeling when we were asked to show that* $sup(A+B) = sup(A) + sup(B)$. *It was an herculean task because the concept was too difficult to grasp with the use of approximation property until I discovered an easy route. In a bid to restrict my papers to just few pages, I will focus more on examples than theorems.*

Introduction

Let A ⊆ ℝ be any non-empty subset of ℝ, then A is said to be bounded from above in case there exists a real number L ∈ ℝ such that x ≤ L, ∀ x ∈ A.
Remark: An upper bound for a set A may not be unique. If L is an upper bound for A and there exists c ∈ ℝ such that L ≤ c, then c is also an upper bound for A.

Let A ⊆ ℝ be any non-empty subset of ℝ, then A is said to be bounded from below in case there exists a real number W ∈ ℝ such that x ≤ W, ∀ x ∈ A.
Remark: A lower bound for a set A may not be unique. If W is a lower bound for A and there exists d ∈ ℝ such that d ≤ W, then d is also a lower bound for A.

A non-subset of ℝ may have both lower and upper bound. Such subsets are special in the sense that they can be contained in some close interval. These subsets have name

Definition: A non-empty subset A ⊂ ℝ is said to be bounded if A has both upper and lower bound (A is bounded from below and above).
Remark: A set is said to be unbounded if it is not bounded.

Definition: Let A be a non-empty subset of ℝ such that A has an upper bound. L ∈ ℝ is called the supremum of A if the following conditions are satisfied;

1. L is an upper bound
2. If z is any other upper bound. Then L ≤ z

Notation: Sup(A) denotes supremum of A
Remark: The second condition above means a supremum is the least element of all the upper bounds. Hence, the name least upper bound is sometime used to mean supremum and we write lub(A) for sup(A).

The following propositions will be stated only

1. Let A be non-empty subset of ℝ such that the infimum (supremum) exists, then it is unique.
2. Let A ⊆ ℝ such that inf(A) and sup(A) exists, then inf(A) ≤ sup(A).

Examples

1. Show that sup(A+B) = sup(A) + sup(B)
   Solution: Let
$$A = [x : x \in A]$$
$$B = [y : y \in B]$$
$$A + B = [x + y : \forall\ x \in A \ and \ \forall\ y \in B]$$
$$x + y \leq sup(A + B)$$

fix y;

$$x \leq sup(A+B) - y$$

Since x is arbitrary in A, it means that,

$$sup(A) \leq sup(A+B) - y$$

Similarly;

$$sup(A) \leq sup(A+B) - y$$
$$y \leq sup(A+B) - sup(A)$$

Since y is arbitrary in B, it means that,

$$sup(B) \leq sup(A+B) - sup(A)$$

$$sup(A) + sup(B) \leq sup(A+B) \tag{1}$$

Considering the other part; since

$$x \leq sup(A)$$
$$y \leq sup(B)$$

then;

$$x + y \leq sup(A) + sup(B)$$

since x+y is arbitrary in A+B, then

$$sup(A+B) \leq sup(A) + sup(B) \tag{2}$$

Combining both equations

$$sup(A) + sup(B) = sup(A+B)$$

Let $\alpha \in$ R. Let A and B be non empty subsets of R such that inf and sup in the following statements all exist. Prove that

2. $sup(\alpha + A) = \alpha + sup(A)$
3. $inf(\alpha + A) = \alpha + inf(A)$
4. $sup(AB) = sup(A)\ sup(B)$
5. $inf(AB) = inf(A)\ inf(B)$
6. If $\alpha > 0, sup(\alpha A) = \alpha\ sup(A)$
7. If $\alpha > 0, inf(\alpha A) = \alpha\ inf(A)$
8. If $\alpha < 0, sup(\alpha A) = \alpha\ inf(A)$
9. If $\alpha < 0, inf(\alpha A) = \alpha\ sup(A)$
10. $sup(\frac{1}{A}) = \frac{1}{inf(A)}$, where $0 \notin A$

Solution (8): Using the same set defined above, then

$$\alpha A = [\alpha x : \forall\ x \in A],\ where\ \alpha < 0$$

$$\alpha x \leq sup(\alpha A)$$
$$\frac{sup(\alpha A)}{\alpha} \leq x$$

Since x is arbitrary in A, then

$$\frac{sup(\alpha A)}{\alpha} \leq inf(A)$$

$$\alpha \ \inf(A) \leq sup(\alpha A) \tag{3}$$

Similarly;

$$inf(A) \leq x$$

$$\alpha x \leq \alpha inf(A)$$

Since αx is arbitrary in αA, then

$$sup(\alpha A) \leq \alpha inf(A) \tag{4}$$

Combining both,

$$sup(\alpha A) = \alpha inf(A)$$