Existence of solutions for fractional Langevin equations with boundary conditions on an infinite interval

Zaid Laadjal
Departement of Mathematics and Computer Sciences,
ICOSI Laboratory, University of Khenchela, (40000), Algeria
E-mail: *zaid.laadjal@yahoo.com*

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Abstract

Abstract: In this paper, we investigate the existence and uniqueness of solutions for the following fractional Langevin equations with boundary conditions

$$\begin{cases} D^{\alpha}\left(D^{\beta}+\lambda\right)u\left(t\right)=f\left(t,u\left(t\right)\right), \quad t\in(0,+\infty) \\ u(0)=D^{\beta}u\left(0\right)=0, \\ \lim\limits_{t\to+\infty}D^{\alpha-1}u\left(t\right)=\lim\limits_{t\to+\infty}D^{\alpha+\beta-1}u\left(t\right)=au(\xi), \end{cases} \tag{1}$$

where $1<\alpha\leqslant 2$ and $0<\beta\leqslant 1$, such that $1<\alpha+\beta\leqslant 2$, with $a,b\in\mathbb{R}$, $\xi\in\mathbb{R}^{+}$ and D^{α}, D^{β} are the Riemman-Liouville fractional derivative. Some new results are obtained by applying standard fixed point theorems.

Keywords: fractional Langevin equation; Riemman-Liouville fractional derivative; Infinite interval; fixed point theorem.

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1 introduction

To be completed.

2 Preliminaries

Definition 1 *[2]The Riemann-Liouville fractional integral of ordre $\alpha\in\mathbb{R}^{+}$.for a function $f\in L^{1}[a,b]$ is defined as*

$$\left(I^{\alpha}f\right)\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{t}\left(t-s\right)^{\alpha-1}f\left(s\right)ds \tag{2}$$

where Γ is Gamma Euler function.defined as

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$$

Definition 2 *[2]Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}^*$ where $n-1 < \alpha < n$, The Riemann-Liouville fractional derivative of ordre α.for a function $f \in L^1[a,b]$ is defined as*

$$D_a^\alpha f(t) = D^n I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, d\tau, \tag{3}$$

with $D^n = \frac{d^n}{dt^n}$.

Properties

Let $\delta > 0$ and $\beta > 0$, for all $f \in L^1[a,b]$, we have

$$I^\delta I^\beta f(t) = I^\beta I^\delta f(t) = I^{\delta+\beta} f(t) \tag{4}$$

$$I^\alpha t^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)} t^{\alpha+\eta}, \quad \eta > -1. \tag{5}$$

If $\beta > \delta > 0$ we have

$$D^\delta I^\beta f(t) = I^{\beta-\delta} f(t) \tag{6}$$

Lemma 3 *[2]Let $\alpha \in \mathbb{R}^+$ where $n-1 < \alpha \leqslant n$, wiht $n \in \mathbb{N}^*$. Then the differential equation $D^\alpha u(t) = 0$, has this general solution*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}, \tag{7}$$

where $c_i \in \mathbb{R}$, with $i = 0, 1, 2, ..., n$.

Lemma 4 *[2]Let $\alpha > 0$. Then*

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, with $i = 0, 1, 2, ..., n$, and $n-1 < \alpha \leqslant n$.

3 Main results

Lemma 5 *Let $1 < \alpha \leqslant 2$ and $0 < \beta \leqslant 1$, where $1 < \alpha + \beta \leqslant 2$, and let $h(t) \in C(\mathbb{R}^+, \mathbb{R})$. The following problem*

$$\begin{cases} D^\alpha \left(D^\beta + \lambda\right) u(t) = h(t), \qquad t \in (0, +\infty) \\ u(0) = D^\beta u(0) = 0, \\ \lim\limits_{t\to+\infty} D^{\alpha-1} u(t) = \lim\limits_{t\to+\infty} D^{\alpha+\beta-1} u(t) = au(\xi), \end{cases} \tag{8}$$

has equivalent to the fractional integral equation

$$\begin{aligned} u(t) &= -\lambda\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} u(s)\,ds + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} h(s)ds \\ &\quad + \frac{\mu}{a(1+\lambda)} t^{\alpha+\beta-1}\int_0^{+\infty} h(s)ds + \frac{\mu\lambda t^{\alpha+\beta-1}}{\Gamma(\beta)}\int_0^{\xi} (\xi-s)^{\beta-1} u(s)\,ds \\ &\quad - \frac{\mu t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^{\xi} (\xi-s)^{\alpha+\beta-1} h(s)ds, \end{aligned} \tag{9}$$

where

$$\mu = \frac{a(1+\lambda).}{a(1+\lambda)\xi^{\alpha+\beta-1} - \Gamma(\alpha+\beta)}. \tag{10}$$

Proof. We applied the operartor I^α on $D^\alpha\left(D^\beta+\lambda\right)u(t) = h(t)$, we get

$$\left(D^\beta+\lambda\right)u(t) = I^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{11}$$

where $c_1, c_2 \in \mathbb{R}$,
by the boundary condition $u(0) = 0$, and $D^\beta u(0) = 0$, we have $c_2 = 0$, thus

$$D^\beta u(t) = -\lambda u(t) + I^\alpha h(t) + c_1 t^{\alpha-1}, \tag{12}$$

applied the operartor I^β

$$u(t) = -\lambda I^\beta u(t) + I^{\alpha+\beta} h(t) + c_1 I^\beta t^{\alpha-1} + c_3 t^{\beta-1}, \tag{13}$$

where $c_3 \in \mathbb{R}$
by the boundary condition $u(0) = 0$ we have $c_3 = 0$, therefor

$$u(t) = -\lambda I^\beta u(t) + I^{\alpha+\beta} h(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}. \tag{14}$$

Applied the operator $D^{\alpha+\beta-1}$

$$\begin{aligned} D^{\alpha+\beta-1}u(t) &= -\lambda D^{\alpha+\beta-1} I^\beta u(t) + D^{\alpha+\beta-1} I^{\alpha+\beta} h(t) \\ &\quad + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} D^{\alpha+\beta-1} t^{\alpha+\beta-1}, \end{aligned} \tag{15}$$

which yields

$$D^{\alpha+\beta-1}u(t) = -\lambda D^{\alpha+\beta-1} I^\beta u(t) + Ih(t) + c_1\Gamma(\alpha). \tag{16}$$

We have

$$\begin{aligned} D^{\alpha+\beta-1} I^{\beta} u(t) &= \frac{d}{dt} I^{1-(\alpha+\beta-1)} I^{\beta} u(t) \\ &= \frac{d}{dt} I^{2-\alpha} u(t) \\ &= \frac{d}{dt} I^{1-(\alpha-1)} u(t) \\ &= D^{\alpha-1} u(t), \end{aligned} \tag{17}$$

substituting (17) into (16), we obtain

$$D^{\alpha+\beta-1} u(t) = -\lambda D^{\alpha-1} u(t) + Ih(t) + c_1 \Gamma(\alpha), \tag{18}$$

thus

$$\lim_{t\to+\infty} D^{\alpha+\beta-1} u(t) = -\lambda \lim_{t\to+\infty} D^{\alpha-1} u(t) + \lim_{t\to+\infty} Ih(t) + c_1 \Gamma(\alpha). \tag{19}$$

By (19), (14) and the boundary conditions $\lim_{t\to+\infty} D^{\alpha-1} u(t) = \lim_{t\to+\infty} D^{\alpha+\beta-1} u(t) = au(\xi)$, we obtain

$$c_1 = \frac{\mu\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \left(\frac{1}{a(1+\lambda)} \lim_{t\to+\infty} Ih(t) + \lambda \left(I^{\beta} u\right)(\xi) - \left(I^{\alpha+\beta} h\right)(\xi) \right), \tag{20}$$

where μ defined as in (10),
substituting (20) into (14), we obtain

$$\begin{aligned} u(t) &= -\lambda I^{\beta} u(t) + I^{\alpha+\beta} h(t) + \mu t^{\alpha+\beta-1} \\ &\quad \times \left(\frac{1}{a(1+\lambda)} \lim_{t\to+\infty} Ih(t) + \lambda \left(I^{\beta} u\right)(\xi) - \left(I^{\alpha+\beta} h\right)(\xi) \right). \end{aligned} \tag{21}$$

Therefor

$$\begin{aligned} u(t) &= -\lambda \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s)\, ds + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds \\ &\quad + \frac{\mu}{a(1+\lambda)} t^{\alpha+\beta-1} \int_0^{+\infty} h(s) ds + \frac{\mu\lambda t^{\alpha+\beta-1}}{\Gamma(\beta)} \int_0^{\xi} (\xi-s)^{\beta-1} u(s)\, ds \\ &\quad - \frac{\mu t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^{\xi} (\xi-s)^{\alpha+\beta-1} h(s) ds \end{aligned} \tag{22}$$

The proof is complete ■

Consider the space defined by

$$X = \left\{ u \in C(\mathbb{R}^+, \mathbb{R}),\ \sup_{t\geqslant 0} \frac{u(t)}{1+t^{\alpha+\beta-1}} \text{ is bounded on } \mathbb{R}^+ \right\} \tag{23}$$

and with the norm

$$\|u\|_X = \sup_{t\geqslant 0} \frac{|u(t)|}{1+t^{\alpha+\beta-1}}. \tag{24}$$

Lemma 6 *[1] The space* $(X, \|.\|_X)$ *is Banach space.*

We define the operator $P : X \to X$ by

$$\begin{aligned} Pu\,(t) &= -\lambda\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\,u\,(s)\,ds + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\,f(s,u(s))ds \\ &\quad + \frac{\mu}{a\,(1+\lambda)}t^{\alpha+\beta-1}\int_0^{+\infty} f(s,u(s))ds + \frac{\mu\lambda t^{\alpha+\beta-1}}{\Gamma(\beta)}\int_0^{\xi} (\xi-s)^{\beta-1}\,u\,(s)\,ds \\ &\quad - \frac{\mu t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^{\xi} (\xi-s)^{\alpha+\beta-1}\,f(s,u(s))ds \end{aligned} \tag{25}$$

where μ defined as in (10).
To be completed.

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[3] To be completed.