On The Non-Real Nature of $x \cdot 0$ ($x \in \mathbb{R}_{\neq 0}$)

The Set of Null Imaginary Numbers $\mathbb{I}_\emptyset$

Part I: Fundamentals

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Abstract

In this letter we discuss the inconsistencies of $0/0 \cdot x = y$, $x, y \in \mathbb{R}_{\neq 0}$ from the perspective of the zero property multiplication (ZPM) $x \cdot 0 = y \cdot 0$ on $\mathbb{R}$. We axiomatize $x \cdot 0$ as a number $i_0(x)$ that has a real part $\Re(i_0(x)) = 0$ but indeed is not real. From this we define the set of null imaginary numbers $\mathbb{I}_\emptyset$ as $\{i_0(x) | \forall x \in \mathbb{R}_{\neq 0}\} \cup \{0\}$. We present the elementary algebra on $\mathbb{I}_\emptyset$ based on the definitions of uniqueness (i.e., if $x \neq y \Leftrightarrow i_0(x) \neq i_0(y)$) and the null division (i.e., $i_0(x)/0 = x \neq 0$). Also, *under the condition of existence of* $\mathbb{I}_\emptyset$, we show that $0/0 = i_0(0)/i_0(0) = 1$ does not cause the logic trivialism of the real mathematic.

1 Introduction

1.1 Elementary Inconsistencies of $0/0 \in \mathbb{R}$

Division by zero is forbidden for real numbers because we do not know how to prevent the inconsistencies that result. To briefly recall some of those inconsistencies, let us define the inequalities 1, 2 and 3 for the constants $x, y \in \mathbb{R}_{\neq 0}$.

$$x \neq 0 \tag{1}$$
$$y \neq 0 \tag{2}$$
$$x \neq y \tag{3}$$

The **zero property of multiplication** (ZPM) tells us the Eq. 4 is valid.

$$x \times 0 = y \times 0 = 0 \tag{4}$$

From Eq. 4 one may wonder whether the equality 5 holds. If $0/0 = 1$ then $x = y$, which contradicts the initial statement 3. If $0/0 = 0$ then $y = 0$ which contradicts the initial statement 2.

$$0/0 \times x \stackrel{?}{=} y \tag{5}$$

1.2 What Really Does Cause the Inconsistencies of $0/0$?

ZPM defines zero as the absorbing element for multiplication on $\mathbb{R}$. The inconsistencies we have just described in section 1.1 strongly suggests that *there may be a dimension in which ZPM* ***fails*** *to absorb two distinct numbers in exactly the same way.* In fact, the true nature of $x \cdot 0$ cannot be entirely explained in $\mathbb{R}$ (even in $\mathbb{C}$) unless one **forbids** division by zero. Because of this, **it seems to be more appropriate to axiomatize $k \cdot 0$ as a number that has a null real part but, indeed, is not real**. Thus, to account the differences in the 'absorptions' $x \cdot 0$ and $y \cdot 0$ ($x \neq y \in \mathbb{R}_{\neq 0}$), we propose the function $i_0(x)$, $x \in \mathbb{R}_{\neq 0}$, we refer to as the **null imaginary number** of x.

2 Notation, Definitions and Properties

In this section we present definitions and properties of the null imaginary numbers $\mathbb{I}_\emptyset$. We always denote both x and y as distinct non-zero real numbers unless differently stated. Similarly, **all usual properties of $\mathbb{R}$ does hold for $\mathbb{I}_\emptyset$ unless differently stated**.

2.1 Fundamentals

The process of multiplying $x \in \mathbb{R}_{\neq 0}$ by zero we refer to as *imagination* (Def. 1).

Definition 1 (Imagination) *We define a null imaginary number $i_0(x) \in \mathbb{I}_\emptyset, x \in \mathbb{R}_{\neq 0}$ as*

$$x \cdot 0 = i_0(x) \tag{6}$$

Imagination on x yields a *null imaginary number* (or just *n-imaginary*) $i_0(x)$ which is member of the null imaginary set of numbers $\mathbb{I}_\emptyset$ (Def. 2).

Definition 2 (The Set $\mathbb{I}_\emptyset$ of Null Imaginary Numbers)

$$\mathbb{I}_\emptyset = \{i_0(x) | \forall x \in \mathbb{R}_{\neq 0}\} \cup \{0\}. \tag{7}$$

No two real numbers lead to the same n-imaginary number on $\mathbb{I}_\emptyset$ (Def. 3). We extend this definition for the n-imaginary numbers too (Def. 4).

Definition 3 (Uniqueness of Imaginary Numbers)

$$x \neq y \quad \Leftrightarrow \quad i_0(x) \neq i_0(y),\ x, y \in \mathbb{R} \tag{8}$$

Definition 4 (Null imaginary Power) *Let* $i_0(x) \in \mathbb{I}_\emptyset$, $x \in \mathbb{R}$ *and* $n \in \mathbb{N}^*$. *Then:*

$$x \cdot \prod_{i=1}^{n} 0 = i_0(x)^n \tag{9}$$

$$i_0(x) = i_0(x)^n \quad \Leftrightarrow \quad n = 1 \tag{10}$$

Despite the uniqueness on $\mathbb{I}_\emptyset$, all n-imarginary numbers preserve and share the same null real part (Def. 5).

Definition 5 (The Real Part $\Re(i_0(x))$ of a Null Imaginary Number $i_0(x)$)

$$\Re(i_0(x)) = 0, \forall x \in \mathbb{R} \tag{11}$$

In other words, Defs. 3 and 5 tell us that Eqs. 12 and 13 do hold, respectively.

$$x \cdot 0 \neq y \cdot 0 \tag{12}$$

$$\Re(x \cdot 0) = \Re(y \cdot 0) = 0 \tag{13}$$

The reverse process of imagination (i.e., dividing a null imaginary number by zero) we refer to as *realization* (Def. 6).

Definition 6 (Realization)

$$\frac{i_0(x)}{0} = x \tag{14}$$

We define $i_0(0) = 0$ as the shared origin of $\mathbb{I}_\emptyset$ and $\mathbb{R}$ (Def. 7).

Definition 7 (Dual Nature of Zero)

$$i_0(x) = x \quad \Leftrightarrow \quad x = 0 \tag{15}$$

in other words,

$$i_0(0) = 0 \tag{16}$$

$$i_0(x) \neq x, \forall x \neq 0 \in \mathbb{R} \tag{17}$$

$$\mathbb{I}_\emptyset \cap \mathbb{R} = \{0\} \tag{18}$$

2.2 Real and Null Imaginary Division

Another form of *imagination* for a real number $x \in \mathbb{R}_{\neq 0}$ is division by zero, (Property 1).

Property 1 (*Imagination* by Division) *Let* $x \in \mathbb{R}_{\neq 0}$. *Then,* $x/0$ *is an n-imaginary number.*

Proof

$$\frac{x}{0} = \frac{1}{0 \cdot x^{-1}}$$

$$\frac{x}{0} = \frac{1}{i_0(x^{-1})} \tag{19}$$

$\square$

In fact, considering the property 2, (i.e., $y \cdot i_0(x) = i_0(xy)$), from Eq. 19 one gets the equality 20:

$$\begin{aligned} x \cdot i_0(x^{-1}) &= 0 \cdot 1 \\ i_0(x \cdot x^{-1}) &= i_0(1) \end{aligned} \tag{20}$$

Property 2 (Multiplication $\mathbb{R} \times \mathbb{I}_\emptyset$)

$$y \cdot i_0(x) = i_0(yx) \tag{21}$$

Proof

$$\begin{aligned} y \cdot i_0(x) &= y \cdot 0 \cdot x \\ &= y \cdot x \cdot 0 \\ &= yx \cdot 0 \\ &= i_0(yx) \end{aligned}$$

$\square$

2.3 Meaning of $0/0$ Based on $\mathbb{I}_\emptyset$

Let us consider now the uniqueness on $\mathbb{I}_\emptyset$ (i.e., $x \cdot 0 \neq y \cdot 0$, definition 3) to Def. 0/0 (Eq. 22). Based on Def. 1 (Eq. 6), one gets Eq. 23. Next, Eq. 25 results according to Def. 6 (Eq. 14). Therefore, by defining the set $\mathbb{I}_\emptyset$, we define $0/0 = 1$.

Property 3 *(The Set $\mathbb{I}_\emptyset$ defines $0/0 = 1$)*
Proof

$$\frac{0}{0} = \frac{x}{1} \tag{22}$$

$$\frac{i_0(1)}{0} = x, \text{ (Def. 1)} \tag{23}$$

$$1 = x, \text{ (Def. 6)} \tag{24}$$

Therefore, assuming the existence of $\mathbb{I}_\emptyset$,

$$\frac{0}{0} = 1 \tag{25}$$

$\square$

2.4 Elementary Algebra on $\mathbb{I}_\emptyset$

Property 4 (Sum) *Let $x, y \in \mathbb{R}$. Then*

$$i_0(x) + i_0(y) = i_0(x + y) \tag{26}$$

Proof:

$$\begin{aligned} i_0(x) + i_0(y) &= x \cdot 0 + y \cdot 0 \\ &= 0 \cdot (x + y) \\ &= i_0(x + y) \end{aligned}$$

□

Property 5 (Multiplication) *Let $i_0(x), i_0(y) \in \mathbb{I}_\emptyset$, $x, y \in \mathbb{R}$. Then*

$$i_0(x) \cdot i_0(y) = i_0(xy)^2 \tag{27}$$

Proof:

$$\begin{aligned} i_0(x) \cdot i_0(y) &= 0 \cdot x \cdot 0 \cdot y \\ &= x \cdot y \cdot 0 \cdot 0 \\ &= xy \cdot 0 \cdot 0 \\ &= i_0(xy) \cdot 0, \text{ (Def. 1)} \\ i_0(x) \cdot i_0(y) &= i_0(xy)^2, \text{ (Def. 4)} \end{aligned} \tag{28}$$

□

Property 6 (Division) *Let $i_0(x), i_0(y) \in \mathbb{I}_\emptyset$, $x, y \in \mathbb{R}_{\neq 0}$. Then*

$$\frac{i_0(x)}{i_0(y)} = \frac{x}{y} \tag{29}$$

Proof:

$$\begin{aligned} \frac{i_0(x)}{i_0(y)} &= \frac{0}{0} \cdot \frac{x}{y} \\ \frac{i_0(x)}{i_0(y)} &= 1 \cdot \frac{x}{y}, \text{ (Prop. 3)} \end{aligned} \tag{30}$$

□

Definition 8 (Multiplicative identity)

$$i_0(x) \cdot 1 = i_0(x) \tag{31}$$

2.5 Null Subtraction

Let us consider the specific case $x - x$, $x \in \mathbb{R}_{\neq 0}$ in face of $\mathbb{I}_\emptyset$. Considering a pure real domain, Def. 5 ensures $\Re(x - x) = 0$. However, one concerning on the null-imaginary nature of $(x - x)$, may find out it can be neither $i_0(x)$ nor $i_0(-x)$. In fact, *realization* (Def. 6) tells us that $x/0 = x \neq x - x$ as well as $(-x)/0 = -x \neq x - x$. Thus, we define $x - x \in \mathbb{I}_\emptyset$ according to the Def. 9.

Definition 9 (Real Null Subtraction)

$$\begin{aligned} x - x &= i_0(x-x) = i_0(\pm x) && (32) \\ \Re(x-x) &= \Re(i_0(x-x)) = \Re(i_0(\pm x)) = 0 && (33) \end{aligned}$$

The subtraction of an n-imaginary number by itself is defined in Def. 7 considering Def. 9.

Property 7 (Null Imaginary Subtraction) *Let* $i_0(x) \in \mathbb{I}_\emptyset$, $x \in \mathbb{R}_{\neq 0}$. *Then*

$$i_0(x) - i_0(x) = i_0(\pm x)^2 \quad (34)$$

Proof:

$$\begin{aligned} i_0(x) - i_0(x) &= x \cdot 0 - x \cdot 0 && (35) \\ &= 0 \cdot (x - x) \\ &= 0 \cdot (i_0(\pm x)), \textit{ (Def. 9)} \\ &= i_0(\pm x)^2, \textit{ (Def. 4)} && (36) \end{aligned}$$

□

3 Summary

In this letter we introduced the set of null imaginary numbers $\mathbb{I}_\emptyset$ by revisiting the zero-property multiplication on $\mathbb{R}$. We defined $\mathbb{I}_\emptyset$ such that any two null imaginary numbers $i_0(x) = x \cdot 0$ ($x \in \mathbb{R}_{\neq 0}$) and $i_0(y) = y \cdot 0$ ($x \neq y \in \mathbb{R}_{\neq 0}$) are **distinct** but present the same real part $\Re(i_0(x)) = \Re(i_0(y)) = 0$. We showed that the existence of $\mathbb{I}_\emptyset$ (along with its elementary algebra we also defined) enables $0/0 = 1$ without bringing inconsistency to $\mathbb{R}$. The meaning of $i_0(x)$ and $i_0(y)$ can translate, for instance, the amount of energy-amplitude transferred to the null imaginary dimension during the destructive interferences $f_1(t) = xcos(2\pi t) + xcos(2\pi t + \pi)$ and $f_2(t) = ycos(2\pi t) + ycos(2\pi t + \pi)$, respectively. We plan to present deeper discussion about the physical interpretation of $\mathbb{I}_\emptyset$ as well as its complex extension in future work.