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Motion Planning in Certain Lexicographic Product Graphs

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Abstract: Let G be an undirected graph with n vertices in which a robot is placed at a vertex say v, and a hole at vertex u and in all other $(n-2)$ vertices are obstacles. We refer to this assignment of robot and obstacles as a configuration C_u^v of G. Suppose we have a one player game in which the robot or obstacle can be slide to an adjacent vertex if it is empty i.e. if it has a hole. The goal is to take the robot to a particular destination vertex using minimum number of moves. In this article, we give the minimum number of moves required for the motion planning problem in Lexicographic products of some graphs. In addition, we proved the necessary and sufficient condition for the connectivity of the lexicographic product of two graphs.

Key Words: Motion in a graph, lexicographic product graphs, k-factor of graphs.

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§1. Introduction

Given a graph G, with a robot placed at one of it's vertices and movable obstacles at some other vertices. Assuming that we are allowed to slide the robot and obstacles to an adjacent vertex if it is empty. Let $u, v \in V(G)$, and suppose that the robot is at v and the hole at u and obstacles at other vertices we refer to this as a configuration C_u^v. Now we use $u \overset{r}{\leftarrow} v$ and $u \overset{o}{\leftarrow} v$ to denote respectively, the robot move and the obstacle move from vertex v to an adjacent vertex u where $u, v \in E(G)$. A simple move is referred to as moving an obstacle or the robot to an adjacent empty vertex while a graph G is k-reachable if there exists a k-configuration such that the robot can reach any vertex of the graph in a finite number of simple moves. The objective is to find a minimum sequence of moves that takes the robot from (source) vertex p to a (destination) vertex t.

For two vertices $u, v \in V(G)$, let $d_G(u, v)$ denotes the distance between u and v in G. Most of the distances used in this article are in G so we use $d(u, v)$ instead of $d_G(u, v)$ to represent the distance between the vertices u and v in G. We denote the complete, complement of a complete, cycle, path graph and complete graph minus one factor on n vertices by K_n, $\overline{K}_n$, C_n, P_n and $K_n - I$ respectively.

The motion planning problem in graph was proposed by Papadimitriou et al [9] where it was shown that with arbitrary number of holes, the decision version of such problem is NP-

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complete and that the problem is complex even when it is restricted to planar graphs. They also gave time algorithm for trees. The result in [9] was improve in [2]. Robot motion planning on graphs (RMPG) is a graph with a robot placed at one of its vertices and movable obstacle at some of the other vertices while generalization of RMPG problem is the Multiple robot motion planning in graph (MRMPG) whereby we have k different robots with respective destinations. Ellips and Azadeh [5] studied MRMPG on trees and introduced the concept of minimal solvable trees. Auletta et al [1] also studied the feasibility of MRMPG problem on trees and gave an algorithm that, on input of two arrangements of k robots on a tree of order n, decides in time $O(n)$ whether the two arrangements are reachable from one another. Parberry [8] worked on grid of order n^2 with multiple robots while Deb and Kapoor [4,3] generalized and apply the technique used in [8] to calculate the minimum number of moves for the motion planning problem for the cartesian product of two given graphs.

The MRMPG problem of grid graph of order n^2 with n^2-1 robots is known as (n^2-1)-puzzle. The objective of (n^2-1)-puzzle is to verify whether two given configuration of the grid graph of order n^2 are reachable from each other and if they are reachable then to provide a sequence of minimum number of moves that takes one configuration to the other. The (n^2-1)-puzzle have been studied extensively in [7, 8, 10, 11].

Our work was motivated by Deb and Kapoor [3] whereby they gave minimum sequence of moves required for the motion planning problem in Cartesian product of two graphs having girth 6 or more. They also proved that the path traced by the robot coincides with a shortest path in case of Cartesian product graphs of graphs. In this paper, we consider the case of lexicographic product graphs. Here we give the minimum number of moves required for the motion planning problem in the Lexicographic product of two graphs say G and H, where G and H are specified in each of our cases.

1.1 Lexicographic Product of Graphs

Definition 1.1 *The lexicographic product $G \circ H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ in which (u_i, v_j) and (u_p, v_q) are adjacent if one of the following condition holds:*

(i) $\{u_i, u_p\} \in E(G)$;

(ii) $u_i = u_p$ *and* $\{v_j, v_q\} \in E(H)$.

The graphs G and H are known as the factors of $G \circ H$. Now onwards G and H are simple graphs with $V(G) = \{1, 2, 3, \cdots, m\}$ unless otherwise stated.

Suppose we are dealing with p-copies of a graph G and we are denoting these p-copies of G by G^i, where $i = \{1, 2, 3, \cdots, p\}$. Then for each vertex $u \in V(G)$ we denote the corresponding vertex in the i^{th} copy G^i by u^i. The girth of a graph G, denoted by $g(G)$ is the length of the shortest cycle contained in graph G.

Example 1.2 Let $G = P_2$ and $H = C_4$. The graph $G \circ H$ is shown in Figure 1 below.

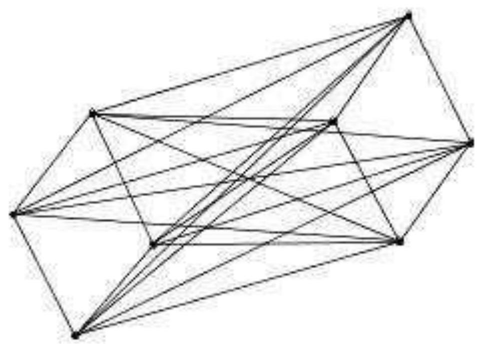

Figure 1 $P_2 \circ C_4$

Remark 1.3 The Lexicographic product $G \circ H$, of graphs G and H, is the graph obtained by replacing each vertex of G by a copy of H and every edge of G by the complete bipartite graph $K_{[H],[H]}$.

1.2 Connectivity of Lexicographic Product of Two Graphs

Here we aim at proving a corollary which Deb and Kapoor [3] mentioned as concerning the condition for which the lexicographic product $G \circ H$ is connected.

Proposition 1.4 (See [6]) *Suppose that u^i and v^j are two vertices in $G \circ H$. Then*

$$d_{(G \circ H)}(u^i, v^j) = \begin{cases} d_H(u,v), & if\ i = j \ and\ d_G(i) = 0 \\ \min\{d_H(u,v), 2\}, & if\ i = j \ and\ d_G(i) \neq 0 \\ d_G(i,j), & if\ i \neq j \end{cases}$$

Theorem 1.5 *Let G and H be two non-trivial graphs. Then $G \circ H$ is connected if and only if G is connected.*

Proof Assume that $G \circ H$ is connected. We only need to show that graph G is connected. Given that $u, v \in V(H)$ and $i, j \in V(G)$. Let u^i be an arbitrary vertex in $G \circ H$. Since $G \circ H$ is connected it means that the vertex u^i has an edge with at least a vertex in $G \circ H$ (especially in one of the factors in the product graph), let this vertex be v^j. Next, by definition of the lexicographic product graph, for u^i and v^j to have an edge in $G \circ H$ then $\{i, j\} \in E(G)$. Which implies that there is a path between i and j in G. But u^i is an arbitrary vertex in $G \circ H$ we conclude that there is a path between each pair of vertices in graph G. Therefore G is connected. Conversely, suppose that graph G is connected. It suffices us to show that $G \circ H$ is connected. Since G is connected it implies that for all $i, j \in V(G)$ where i and j are distinct $d(i, j) \neq 0$. We shall prove this by contradiction. Assume that the graph $G \circ H$ is disconnected. If $G \circ H$ is disconnected it means that there exist an arbitrary pair of vertices

(u^i, v^j) in $G \circ H$ such that $d_{G\circ H}(u^i, v^j) = 0$ for all $i, j \in V(G)$ and $u, v \in V(H)$. Since i and j are distinct then by Proposition 1.4 we have that $d_{G\circ H}(u^i, v^j) = d_G(i, j)$. But $d_G(i, j) \neq 0$ therefore $d_{G\circ H}(u^i, v^j) \neq 0$ a contradiction. Since the pair (u^i, v^j) is arbitrary we conclude that the product graph $G \circ H$ is connected. This completes the proof. $\square$

§2. Robot Moves in Lexicographic Product of a Graph and Complement of Complete Graph

Definition 2.1 *An edge u^p, v^q in $G \circ H$ is said to be a G-edge if $u = v$ and $\{p, q\} \in E(G)$.*

Definition 2.2 *Given two graphs G and H. For any $u^p, v^q \in V(G \circ H)$,we call the distance between u and v in H to be the H-distance between u^p and v^q in $G \circ H$. We use $d_G(u^p, v^q)$ and $d_H(u^p, v^q)$ to denote the G-distance and H-distance between u^p, v^q in $G \circ H$, respectively.*

Now when there is no confusion about the graph in question G, we use $d(u, v)$ instead of $d_G(u, v)$ to represent the distance between u and v in G.

In view of the above definition, we now have this proposition.

Proposition 2.3 *Given two graphs G and H. Let $\{i, j\}, \{j, k\} \in E(G)$ and $u, v \in V(H)$. Then $d_{G\circ H-u^i}(v^i, u^j) = d_{G\circ H}(v^i, u^j)$.*

Proof To prove this, notice that $d_{G\circ H-u^i}(v^i, u^j) = 1$ which is same as $d_{G\circ H}(v^i, u^j)$.

Each vertex set of copy H^i is adjacent to all other vertices in copy H^j for all $\{i, j\} \in E(G)$ in Lexicographic product graphs. $\square$

The following two results was given in [4].

Lemma 2.4([4]) *Given two graphs G and H. Let $\{i, j\}, \{j, k\} \in E(G)$ such that $\{i, k\} \notin E(G)$ and $u, v, w \in V(H)$. Consider the configuration $C_{u^i}^{v^j}$ of $G \circ H$. Then we require at least three moves to move the robot from v^j to w^k.*

Lemma 2.5([4]) *Given two graphs G and H. Let $u, v, w \in V(H)$ such that $\{u, v\}, \{v, w\} \in E(H)$ and $\{u, w\} \notin E(H)$. For some $i \in V(G)$, consider the configuration $C_{u^i}^{v^i}$ of $G \circ H$. Then minimum of three moves are required to move the robot from v^i to w^i.*

Proposition 2.6 *Let G be a graph and H a complement of a complete graph on n vertices. Let $\{i, j\}, \{j, k\} \in E(G)$ and $u, v \in V(H)$. Then starting from the configuration $C_{v^i}^{u^i}$ we need at least two moves to move the robot to u^j.*

Proof To move the robot from u^i to u^j before it, the hole is required to move from v^i to u^j. This takes just a move since $d_{G\circ H-u^i}(v^i, u^j) = 1$. Then the move $u^j \xleftarrow{r} u^i$ takes the robot to u^j. Hence the result follows. $\square$

Corollary 2.7 *Let G be any graph and H a complement of a complete graph on n vertices. Let $\{i, j\}, \{j, k\} \in E(G)$ and $u, v \in V(H)$, where u and v are distinct. Then starting from the configuration $C_{v^i}^{v^j}$ we need at least three moves to move the robot to v^k.*

Proof Observe that $\{v^i, v^j\}, \{v^j, v^k\} \in E(G \circ H)$. For the robot to move to v^k before it, the hole must move from v^i to v^k. This takes $d_{G\circ H}(v^i, v^k) = 2$. Then the move $v^k \overset{r}{\leftarrow} v^j$ moves the robot from v^j to v^k. Hence the result follows. □

Definition 2.8 *A robot move in $G \circ H$ is called a G-move if the edge along which the move take place is a G-edge.*

Definition 2.9 *Let T be a sequence of moves that take the robot from u^p to v^q in $G \circ H$. An H-move (respectively G-move) in T of the robot is said to be a secondary H-move (respectively G-move) if it is preceded by an H-move (respectively G-move). An H-move (respectively G-move) in T of the robot is said to be a primary H-move (respectively G-move) if it is preceded by a G-move (respectively H-move). Also the edge corresponding to a primary G-move (respectively H-move) in T is said to be a primary G-edge (respectively H-edge). In view of the above definitions we have the following remark.*

Remark 2.10 Given graph G and H a complement of a complete graph on n vertices.

(i) In view of Proposition 2.6, to perform the first move of the robot we require at least 2 moves;

(ii) In view of Corollary 2.7, to perform each secondary move of the robot we require at least 3 moves.

Theorem 2.11 *Let G be a graph and H a complement of a complete graph on n vertices. Let $i, j, k \in V(G)$ and $u, v, w \in V(H)$. Then each robot and obstacle moves in a minimum sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{u^j}^{w^j}$ in $G \circ H$ is a G-move. Also such a minimum sequence involves exactly a number of G-moves of the robot and $3a$ moves in total, where $a = d(i, j) \geq 1$.*

Proof Since $\{u, v\}, \{v, w\} \notin E(H)$ it means $d_H(u^i, v^i) = d_H(v^i, w^i) = 0$. The first part of this lemma follows. Now, let T be a sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{u^j}^{w^j}$ in $G \circ H$. First, let z be the number of robot moves in T. By Proposition 2.6, we need at least two moves to accomplish the first move of the robot. Observe that each remaining $z - 1$ robot moves in T is a secondary G-move. So by Remark 2.10, we need minimum of $3(z - 1)$ moves to accomplish the $z - 1$ secondary G-moves (since the first robot move places the hole at a preceding copy of the robot). Now, if $w^j \overset{r}{\leftarrow} w^a$ is the z^{th} robot move in T, it will leave the graph $G \circ H$ with the configuration $C_{w^a}^{w^j}$. Since $d_{G\circ H}(w^a, u^j) = 1$, so we need minimum of one more move to take the hole from w^a to u^j. Hence T involves minimum $3z$ moves. Notice that, the expression $3z$ takes the minimum value when z is minimum.

Next, let $d(i, j) = a$ and $[i = i_0, i_1, i_2, ..., i_a = j]$ be a path of length a connecting i and j in G. Then $[v^i = v^{i_0}, v^{i_1}, v^{i_2}, \cdots, v^{i_a} = w^j]$ is a path of length a in $G \circ H$ joining v^i to w^j. So the sequence of moves $u^i \overset{o^*}{\longleftarrow} v^{i_1} \overset{r}{\leftarrow} v^{i_0} \overset{o^*}{\longleftarrow} v^{i_2} \overset{r}{\leftarrow} v^{i_1} \overset{o^*}{\longleftarrow} v^{i_3} \overset{r}{\leftarrow} v^{i_2} \overset{o^*}{\longleftarrow} v^{i_4} \cdots v^i_{a-2} \overset{o^*}{\longleftarrow} v^{i_a} \overset{r}{\leftarrow} v^i_{a-1} \overset{o^*}{\longleftarrow} u^j$ takes the robot from v^i to w^j along this path. Also it involves exactly a number of G-moves of the robot. Furthermore, from the given sequence above obstacle moves involves exactly $2 + 2(a - 1)$ moves. Therefore a minimum sequence of moves T that takes the configuration $C_{u^i}^{v^i}$ to $C_{u^j}^{w^j}$ involves exactly $3a$ moves. □

Theorem 2.12 *Let G be a graph and H a complement of a complete graph on n vertices. Let $\{i, j\} \in E(G)$ and $u, v, w \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{w^j}^{w^i}$ in $G \circ H$ involves exactly 5 moves.*

Proof Combining Proposition 2.6 and Corollary 2.7 gives the result. □

The above Lemma gives the minimum number of moves required to take the robot from a given factor to itself and to another factor $G \circ H$. The proof of this lemma is immediate from Theorems 2.11 and 2.12.

Lemma 2.13 *Consider the graph $G \circ H$. Let $u, v \in V(H)$ with the initial configuration $C_{u^i}^{v^i}$, where G is any graph and H a complement of a complete graph on n vertices. Then*

(i) to move the robot from H^i to H^i we require at least 5 G-moves;

(ii) to move the robot from H^i to H^j we require at least $1 + 2(a-1) + a$ G-moves. Where $a = d(i, j) \geq 1$.

Corollary 2.14 *Let G be a path and H a complement of a complete graph on n vertices. Let $i, j, k \in V(G)$ and $u, v, w \in V(H)$. Then each robot and obstacle moves in a minimum sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{w^j}^{u^j}$ in $G \circ H$ is a G-move. Also such a minimum sequence involves exactly $3a$ moves in total, where $a = d(i, j) \geq 1$.*

Proof The proof of this corollary is immediate from Lemma 2.13. □

Corollary 2.15 *Let G be a path and H a complement of a complete graph on n vertices. Let $i, j \in V(G)$ and $u, v, w \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{w^j}^{w^i}$ in $G \circ H$ involves exactly 5 moves.*

proof The proof of this corollary is immediate from Lemma 2.13. □

Note that for such a product (Corollaries 2.14 and 2.15) there is no shortest path. Next, we consider the case when graph G is a complete graph. We would do this by stating the lemma below without proof.

Lemma 2.16 *Let G be complete and H a complement of a complete graph on n vertices. Let $i, j \in V(G)$ and $u, v, w \in V(H)$. Then starting from configuration $C_{u^i}^{v^i}$,*

(i) we require at least 5 moves to move the robot to w^i;

(ii) we require a minimum of 2 moves to move the robot to w^k.

§3. Robot Moves in $K_n - I \circ H$

In this section we investigate a case where graph G is a complete graph minus a 1-factor and H a complete graph or it's complement.

Lemma 3.1 *Let G be a complete graph minus a 1-factor and H a complete graph (or it's complement). Let $i, j \in V(G)$ but $\{i, j\} \notin E(G)$ and $u, v \in V(H)$. Then each robot move in a* *minimum sequence of moves that takes* $C_{v^i}^{u^i}$ *to* $C_{v^j}^{u^j}$ *is a* G*-move. Also such a minimum sequence involves exactly* 6 *moves.*

Proof let T be a sequence of moves that takes $C_{v^i}^{u^i}$ to $C_{v^j}^{u^j}$. By Proposition 2.6, two moves is required to move the robot to an adjacent vertex. Next by Corollary 2.7, four additional moves is required to take the robot and hole to their required destination, while the remaining move is the last move of the hole. So, the sequence T of moves $v^i \xleftarrow{o^*} u^k \xleftarrow{r} u^i \xleftarrow{o^*} v^k \xleftarrow{o^*} u^j \xleftarrow{r} u^k \xleftarrow{o^*} v^j$ takes the robot and the hole to the required destination, and each move in this sequence is a G-move. Also T involves exactly six moves. □

Lemma 3.2 *Let* G *be a complete graph minus a* 1*-factor and* H *a complete graph (or it's complement). Let* $i, j \in V(G)$ *but* $\{i, j\} \notin E(G)$ *and* $u, v \in V(H)$*. Then each robot move in a minimum sequence of moves that takes* $C_{v^j}^{u^i}$ *to* $C_{v^i}^{u^j}$ *is a* G*-move. Also such a minimum sequence involves exactly* 6 *moves.*

Proof The proof can be drawn in the same line as that of Lemma 3.1. □

Lemma 3.3 *Let* G *be a complete graph minus a* 1*-factor and* H *a complement of a complete graph with* n *vertices. Let* $i, j \in V(G)$ *but* $\{i, j\} \notin E(G)$ *and* $u, v \in V(H)$*. Then each robot move in a minimum sequence of moves that takes* $C_{v^j}^{u^i}$ *to* $C_{v^j}^{v^i}$ *is a* G*-move. Also such a minimum sequence involves exactly* 6 *moves.*

Proof Let T be a sequence of moves that takes $C_{v^j}^{u^i}$ to $C_{v^j}^{v^i}$. By Proposition 2.6, we need at least 2 moves to accomplish the first G-move of the robot. Notice that the last move of the robot is also a G-move. Now if $v^i \xleftarrow{r} v^m$ is the last move of the robot, it will leave the graph $G \circ H$ with the configuration $C_{v^m}^{v^i}$ and this would require 3 moves. Since $d_{G \circ H}(v^m, v^j) = 1$ so we need minimum of one more move to take the hole from v^m to v^j. Therefore T involves exactly 6 moves. □

Lemma 3.4 *Let* G *be a complete graph minus a* 1*-factor and* H *a complete graph. Let* $i, j \in V(G)$ *but* $\{i, j\} \notin E(G)$ *and* $u, v \in V(H)$*. Then starting from the configuration* $C_{u^j}^{u^i}$ *of* $G \circ H$*, we require at least* 3 *moves to move the robot to* v^i*.*

Proof For the robot to move to v^i before it, the hole must be moved from u^j to v^i. This takes 2 moves since $d_{G \circ H}(u^j, v^i) = 2$. Then the move $v^i \xleftarrow{r} u^i$ takes the robot from u^i to v^i. Hence the result follows □

In view of Lemmas 3.1, 3.2 and 3.3 we have the following theorem.

Theorem 3.5 *Consider the graph* $G \circ H$*. Let* G *be a complete graph minus a* 1*-factor and* H *a graph. Let* $i, j \in V(G)$ *where* $\{i, j\} \notin E(G)$ *and* $u, v \in V(H)$*. According as the hole is either at the* i^{th} *or* j^{th} *copy of* $G \circ H$*. Then to move the robot from*

(*i*) H^i *to* H^j *we require* 5 *moves if* H *is either* K_n *or* $\overline{K_n}$*;*

(*ii*) H^i *to* H^i *we require* 5 *moves if* H *is* $\overline{K_n}$*.*

Finally, we have the corollary below which is as a result of Lemma 3.4.

Corollary 3.6 *Consider the graph $G \circ H$. Let G be a complete graph minus a 1-factor and H a complete graph. Let $i, j \in V(G)$ but $\{i, j\} \notin E(G)$ and $u, v \in V(H)$. Then to move the robot from H^i to H^i we require 3 moves according as the hole is either at the j^{th} or i^{th} copy of $G \circ H$ respectively.*

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