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Nonholonomic Frames for Finsler Space with Deformed Special (α, β) Metric

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Abstract: The purpose of present paper to determine the Finsler spaces due to deformation of special Finsler (α, β) metrics. Consequently, we obtain the nonholonomic frame with the help of Riemannian metric $\alpha^2 = a_{ij}(x)y^iy^j$, one form metric $\beta = b_i(x)y^i$ and Douglas metric $L(\alpha, \beta) = \left(\alpha + \frac{\beta^2}{\alpha}\right)$ such as forms

$$\begin{aligned} \text{I. } L(\alpha,\beta) &= \alpha\beta, \\ \text{II. } L(\alpha,\beta) &= \left(\alpha + \frac{\beta^2}{\alpha}\right)\beta = \alpha\beta + \frac{\beta^3}{\alpha}. \end{aligned}$$

The first metric of the above deformation is obtained by the product of Riemannian metric and one form and second one is the product of Douglas metric and 1-form metric.

Key Words: Finsler space, (α, β)-metrics, Riemannian metric, one form metric, Douglas metric, GL-metric, nonholonomic Finsler frame.

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§1. Introduction

P.R. Holland [1], [2] studies about the nonholonomic frame on space time which was based on the consideration of a charged particle moving in an external electromagnetic field in the year 1982. In the year 1987, R.S. Ingarden [3] was the first person, to point out that the Lorentz force law can be written in above case as geodesic equation on a Finsler space called Randers space. Further in 1995, R.G. Beil [5], [6] have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries.

In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M. In this case we have considered the fundamental tensor field might be the deformation of two different special Finsler spaces from the (α, β)- metrics. First we consider a nonholonomic frame for a Finsler space with (α, β)- metrics such as first product of Riemannian metric and 1-form metric and second is the product of Douglas metric and 1-form metric. Further we obtain

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a corresponding frame for each of these two Finsler deformations. This is an extension work of Ioan Bucataru and Radu Miron [10], Tripathi [14, 16] and Narasimhamurthy [15].

§2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with(α,β)-metrics [11].

Definition 2.1 *A Finsler space $F^n = \{M, F(x,y)\}$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by*

$$F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}, \tag{2.1}$$

where $\alpha^2(x,y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

The first Finsler spaces with (α,β)–metrics were introduced by the physicist G. Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested a more general case by considering, $a_{ij}(x)$ the components of a Riemannian metric on the base manifold M, $a\,(x,y) > 0$ and $b\,(x,y) \geq 0$ Two functions on TM and $B\,(x,y) = B_i\,(x,y)\,(dx^i)$ a vertical 1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x,y)B_j(x,y).$$

Now a days the above generalized Lagrange metric is known as the Beil metric. The metric tensor g_{ij} is also known as a Beil deformation of the Riemannian metric a_{ij}. It has been studied and applied by R. Miron and R.K. Tavakol in General Relativity for $a(x,y) = \exp(2\sigma(x,y))$ and b = 0. The case a(x, y) = 1 with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [6]. Further Bucataru [12] considered the class of Lagrange spaces with (α,β)-metric and obtained some new and interesting results in the year 2002.

A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland. If we do not ask for the function L to be homogeneous of order two with respect to the (α,β) variables, then we have a Lagrange space with (α,β)-metric. Next we defined some different Finsler space with (α,β)-metrics.

Further consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space(M,F). Taking into account the homogeneity of α and F we have the following formulae:

$$\begin{gathered} p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i}; \\ l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^i}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i, \\ l^i = \frac{1}{L}p^i; l^il_i = p^ip_i = 1; l^ip_i = \frac{\alpha}{L}; p^il_i = \frac{L}{\alpha}; \end{gathered} \tag{2.2}$$

$$b_iP^i = \frac{\beta}{\alpha}; b_il^i = \frac{\beta}{L}$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_iP_j + P_ib_j - \frac{\beta}{\alpha}p_ip_j = \frac{L}{\alpha}(a_{ij} - p_ip_j) + l_il_j. \tag{2.3}$$

Theorem 2.1([10]) *For a Finsler space (M,F) consider the metric with the entries:*

$$Y^i_j = \sqrt{\frac{\alpha}{L}}(\delta^i_j - l^il_j + \sqrt{\frac{\alpha}{L}}p^ip_j) \tag{2.4}$$

defined on TM. Then $Y_j = Y^i_j(\frac{\partial}{\partial y^i}), \quad j \in 1,2,3,\cdots,n$ *is a nonholonomic frame.*

Theorem 2.2([7]) *With respect to frame the holonomic components of the Finsler metric tensor* $a_{\alpha\beta}$ *is the Randers metric* g_{ij}*, i.e,*

$$g_{ij} = Y^\alpha_i Y^\beta_j a_{\alpha\beta}. \tag{2.5}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j, \beta^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)–metric $F^2(x,y) = L\{\alpha(x,y),\beta(x,y)\}$ we have the Finsler invariants [13].

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial\alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial\beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}; \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial\alpha}), \tag{2.6}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with (α,β)-metric we have

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{2.7}$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by [13]

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_iy_j. \tag{2.8}$$

From (2.8) we can see that $g_{ij}$is the result of two Finsler deformations

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j),$$

$$II. \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho^2_{-1})b_ib_j. \tag{2.9}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (2.9) is according to the theorem (7.9.1) in [10], given by

$$X^i_j = \sqrt{\rho}\delta^i_j - \frac{1}{\beta^2}\{\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}}\}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{2.10}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{2.11}$$

Again the frame that corresponds to the II_{nd} deformation (2.9) given by

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}\right)}\}b^i b_j, \tag{2.12}$$

where $C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \tag{2.13}$$

Theorem 2.3([10]) *Let $F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\}$ be the metric function of a Finsler space with (α,β) metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) and (2.12) respectively.

§3. Nonholonomic Frames for Finsler Space with Deformed (α, β) Metric

In this section we consider two cases of nonholonomic Finlser frames with special (α, β)-metrics, such an I^{st} Finsler frame product of Riemannian metric, one form metric and II^{nd} Finsler frame product of Douglas metric and 1-form metric.

3.1 Nonholonomic Frame for $L = \alpha\beta$

In the first case, for a Finsler space with the fundamental function $L = \alpha\beta$ the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= \frac{\beta}{2\alpha}, \quad \rho_0 = 0, \\ \rho_{-1} &= \frac{1}{2\alpha}, \quad \rho_{-2} = \neg\frac{\beta}{2\alpha^3}, \\ B^2 &= \frac{1}{4\alpha^4}(\alpha^2 b^2 - \beta^2). \end{aligned} \tag{3.1}$$

Using (3.1) in (2.10) we have

$$X_j^i = \sqrt{\frac{\beta}{2\alpha}}\delta_j^i - \frac{1}{4\alpha^2\beta^2}\left[\sqrt{\frac{\beta}{2\alpha}} + \sqrt{\frac{\beta - 4\alpha^4\beta}{2\alpha}}\right](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j). \tag{3.2}$$

Again using (3.1) in (2.12) we have

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{2\beta C^2}{\alpha}}\}b^i b_j, \tag{3.3}$$

where $C^2 = \frac{\beta}{2\alpha}b^2 - \frac{1}{2\alpha^3\beta}(\alpha^2 b^2 - \beta^2)^2$.

Theorem 3.1 *Let $L = \alpha\beta$ be the metric function of a Finsler space with (α, β) metric for which the condition (2.7) is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is nonholonomic Finsler Frame with X^i_k and Y^k_j are given by (3.2) and (3.3) respectively.

3.2 Nonholonomic Frame for $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$

In the second case, for a Finsler space with the fundamental function $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$ the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= \frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}, \quad \rho_0 = \frac{3\beta}{\alpha}, \\ \rho_{-1} &= \frac{\alpha^2 - 3\beta^2}{2\alpha^3}, \quad \rho_{-2} = \frac{3\beta^3 - \alpha^2\beta}{2\alpha^5}, \\ B^2 &= \frac{(\alpha^2 - 3\beta)^2(\alpha^2 b^2 - \beta^2)}{4\alpha^8}. \end{aligned} \tag{3.4}$$

Using (3.4) in (2.10) we have

$$\begin{aligned} X^i_j &= \sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}}\delta^i_j - \frac{(\alpha^2 - 3\beta^2)^2}{4\alpha^6\beta^2}[\sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}} \\ &+ \sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3} + \frac{2\alpha^5\beta}{(3\beta^2 - \alpha^2)}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j). \end{aligned} \tag{3.5}$$

Again using (3.4) in (2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{\alpha\beta C^2}{\alpha^2 - 3\beta^2}}\}b^i b_j, \tag{3.6}$$

where $C^2 = \frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}b^2 + \frac{(3\beta^2 - \alpha^2)}{2\alpha^3\beta}(\alpha^2 b^2 - \beta^2)^2$.

Theorem 3.2 *Let $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$ be the metric function of a Finsler space with (α,β) metric for which the condition (2.7) is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is nonholonomic Finsler Frame with X^i_k and Y^k_j are given by (3.5) and (3.6) respectively.

§4. Conclusions

Nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli and Bucataru ([7], [8]) has been determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with (α,β)-metric, it appears a natural question: does how many Finsler space with(α,β)-metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with (α, β)-metrics.

In this work, we consider the Douglas metric, Riemannian metric and 1-form metric we determine the nonholonomic Finsler frames. But, in Finsler geometry, there are many(α, β)-metrics in future work we can determine the frames for them also.

References

[1] Holland P.R., Electromagnetism, particles and anholonomy, *Physics Letters*, **91**(6) (1982), 275-278.

[2] Holland P.R., Anholonomic deformations in the ether: a significance for the electrodynamic potentials, In: Hiley B.J., Peat F.D. (eds.), *Quantum Implications*, Routledge and Kegan Paul, London and New York, 295-311 (1987).

[3] Ingarden R.S., On physical interpretations of Finsler and Kawaguchi spaces, *Tensor N.S.*, **46**,(1987), 354-360.

[4] Randers G., On asymmetric metric in the four space of general relativity, *Phys. Rev.*, **59**,(1941), 195-199.

[5] Beil R.G., Comparison of unified field theories, *Tensor N.S.*, **56**,(1995), 175-183.

[6] Beil R.G., Finsler and Kaluza-Klein gauge theories, *Intern. J. Theor. Phys.*, 32, 6(1993), 1021-1031.

[7] Miron R. and Anastasiei M.,*The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ., FTPH, No.59, (1994).

[8] Antonelli P.L., Bucataru I., Finsler connections in anholonomic geometry of a Kropina space, —it Nonlinear Studies, **8** (1), (2001), 171-184.

[9] Hrimiuc D., Shimada H., On the L-duality between Lagrange and Hamilton manifolds, *Nonlinear World*, **3**, (1996), 613-641.

[10] Ioan Bucataru, Radu Miron, *Finsler-Lagrange Geometry: Applications to Dynamical Systems* CEEX ET 3174/2005-2007, and CEEX M III 12595/2007 (2007).

[11] Matsumoto M., Theory of Finsler spaces with (α, β)-metric, *Rep. Math. Phys.*, **31**, (1992), 43-83.

[12] Bucataru I., Nonholonomic frames on Finsler geometry, *Balkan Journal of Geometry and its Applications*, **7** (1), (2002), 13-27.

[13] Matsumoto M., *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaishesha Press, Otsu, Japan, 1986.

[14] Tripathi Brijesh Kumar, Pandey K. B. and Tiwari R.B., Nonholonomic frame for Finsler space with (α, β)-metric, *International J. Math. Combin.*, Vol.**1**, (2016), 109-115.

[15] Narasimhamurthy S.K., Mallikarjun Y. Kumar, Kavyashree A. R., Nonholonomic frames for Finsler Space with special (α, β)-metric, *International Journal of Scientific and Research Publications*, **4**(1), (2014), 1-7.

[16] Tripathi Brijesh Kumar and Chaubey V. K., Nonholonomic frames for Finsler space with generalized Kropina metric, *International Journal of Pure and Applied Mathematics*, Vol. 108(4), (2016) 921-928, doi:10.12732/ijpam.v108i4.17.