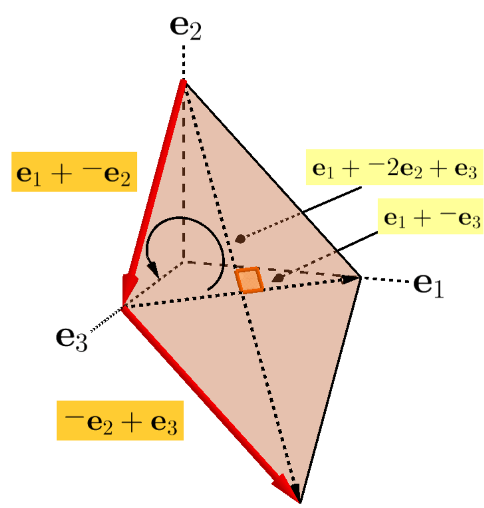
Making Sense of Adding Bivectors

July 12, 2018

James Smith
nitac14b@yahoo.com
https://mx.linkedin.com/in/james-smith-1b195047



"The two red vectors revealed as ${}^-\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{e}_1 + {}^-\mathbf{e}_2$."

Abstract

As a demonstration of the coherence of Geometric Algebra's (GA's) geometric and algebraic concepts of bivectors, we add three geometric bivectors according to the procedure described by Hestenes and Macdonald, then use bivector identities to determine, from the result, two bivectors whose outer product is equal to the initial sum. In this way, we show that the procedure that GA's inventors defined for adding geometric bivectors is precisely that which is needed to give results that coincide with those obtained by calculating outer products of vectors that are expressed in terms of a 3D basis. We explain that that accomplishment is no coincidence: it is a consequence of the attributes that GA's designers assigned (or didn't) to bivectors.

Contents

1 Introduction

Many doubts about 3D Geometric Algebra (GA3) can be avoided by recognizing that it is an attempt to express spatial relations in the form of variables that can be manipulated via well-defined operations. The developers of GA3 identified, as sufficient for that purposes, the set of real numbers plus the objects that we call vectors, bivectors, and trivectors. Those objects and their properties, along with the operations that can be effected upon them, are described in sources such as [1] and [2].

We are interested here in the GA operation known as "addition of bivectors". As a definition of that operation, [1] and [2] give a geometric procedure that may well strike students as arbitrary, with no obvious motivation or justification. However, we will demonstrate that that definition is one of the things that makes GA3 "fit together". We will make that demonstration by effecting the sum $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$ (Fig. 1) according to the procedure given by [1] (p. 25) and [2] (p. 74), then commenting upon the result. We'll assume that the reader has some familiarity with basics of GA3, but we'll begin by reviewing key properties of bivectors.

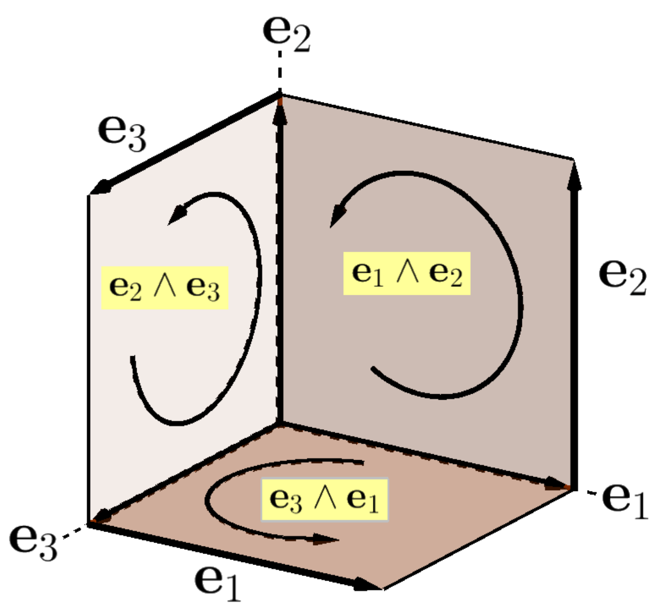


Figure 1: The three bivectors that we'll "add" according to the procedure by which that operation is defined. We've drawn oriented arcs to show each bivector's "sense". The vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the basis vectors of GA3. Because they are mutually perpendicular, the outer product $\mathbf{e}_i \wedge \mathbf{e}_j$ of any two of them (that is, such that $i \neq j$) is equal to their geometric product $\mathbf{e}_i\mathbf{e}_j$.

1.1 Review of Bivectors and How to Add Them

Let's begin by clearing-up a potential source of confusion: the term "bivector" can refer either to the outer product of two vectors (for example, $\mathbf{a} \wedge \mathbf{b}$) or to an oriented portion of a plane. Sometimes, a single bivector is considered in both ways within a single sentence, as in "The next step requires us to reshape bivector $\mathbf{a} \wedge \mathbf{b}$ into the product $\mathbf{c} \wedge \mathbf{d}$." The conceptual leaps in that sentence include (1) imaging the product $\mathbf{a} \wedge \mathbf{b}$ as a rectangle that measures $\|\mathbf{a}\|$ by $\|\mathbf{b}\|$, then (2) reshaping that rectangle into one with dimensions $\|\mathbf{c}\|$ by $\|\mathbf{d}\|$, and finally (3) interpreting the new rectangle as the product $\mathbf{c} \wedge \mathbf{d}$.

To help avoid confusions, we might wish at times to refer to the oriented segment of a plane as a geometric bivector, and to the outer product of two vectors as an algebraic bivector.

Some terminology that might prove useful: geometric bivectors and algebraic bivectors.

1.2 Bivectors' Properties

GA's inventors assigned to bivectors only such attributes as seemed most likely to facilitate formulation and solution of problems that have geometric content. The attribute "location" was not necessary, nor was "shape". Thus, geometric bivectors are defined as equal to each other if they have the same area, are parallel to each other, and have the same sense (Fig. 2).

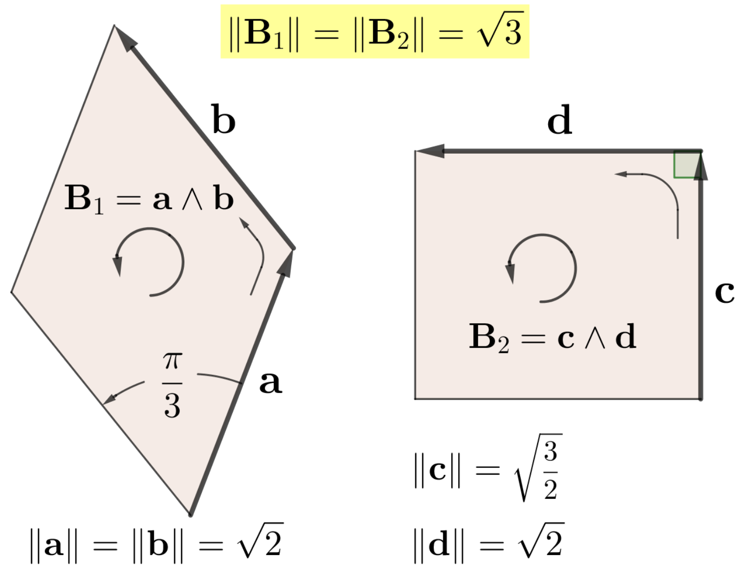


Figure 2: An algebraic bivector does not have unique factorizations. An equivalent statement is that the corresponding geometric bivector does not possess the quality of "shape". Thus, if $\mathbf{B}_1$ and $\mathbf{B}_2$ lie in parallel planes, then they are equal according to the postulates of GA3 because they have the same area and sense.

We'll wish, sometimes, to express those criteria according to observations made by Macdonald ([2]). For example (p. 98),

> [The algebraic bivector $(2\mathbf{e}_1) \wedge (3\mathbf{e}_2)$] represents an area of 6 in the plane with basis $\{2\mathbf{e}_1, 3\mathbf{e}_2\}$. The area has neither a shape nor a position in the plane. Thus the [algebraic] bivector $(6\mathbf{e}_1) \wedge (\mathbf{e}_2)$] represents the same area in the same plane, even though it represents a different rectangle.

See also Fig. 2

From the preceding passage, we can deduce two criteria for the equality of any pair of algebraic bivectors: $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{c} \wedge \mathbf{d}$:

1. $\mathbf{c}$ and $\mathbf{d}$ must be linear combinations of $\mathbf{a}$ and $\mathbf{b}$. This criterion ensures that the corresponding geometric bivectors are parallel.

2. $\|\mathbf{a} \wedge \mathbf{b}\| = \|\mathbf{c} \wedge \mathbf{d}\|$ This criterion ensures that the areas of the corresponding geometric bivectors are equal.

The dual of a multivector $\mathcal{M}$ in GA3 is $\mathcal{M} I_3^{-1}$, where $I_3^{-1} = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1$.

What, then, of the bivectors' sense? From Macdonald's discussion (p. 106) of duals of multivectors, we can deduce that two bivectors have the same sense if and only if their duals are equal. That is, if $(\mathbf{a} \wedge \mathbf{b})\, I_3^{-1} = (\mathbf{c} \wedge \mathbf{d})\, I_3^{-1}$. After further thought, we can see that that direction of a bivector's dual (which is a vector) depends upon the bivector's norm as well as the bivector's plane. Therefore,

> Two bivectors $\mathbf{B}_1$ and $\mathbf{B}_2$ are equal if and only if their duals are equal.

Although that statement will prove useful to us later, the equality of bivectors' areas and of their directions of rotation will often prove useful in themselves.

These criteria for equality would take much less convenient forms if GA's inventors had included shape and location among the attributes of bivectors. An important consequence of these criteria is that given any geometric bivector $\mathbf{G}$ and any vector $\mathbf{p}$ that is parallel to it, we can write the corresponding algebraic bivector $\mathbf{G}$ as the outer product of $\mathbf{p}$ and some vector $\mathbf{s}$ that is perpendicular to $\mathbf{p}$. How can we identify $\mathbf{s}$? We begin by recognizing that because $\mathbf{p} \perp \mathbf{s}$, $\mathbf{G} = \mathbf{p}\mathbf{s}$. Thus,

$$\mathbf{s} = \mathbf{p}^{-1}\mathbf{G}.$$

1.3 The Geometric Procedure for Adding Bivectors

Fig. 3 is an adaptation of the diagram used by [1] (p. 25) and [2] (p. 74) to define and illustrate the procedure. To help us understand how to apply that procedure, we'll arrive at Fig. 3 starting from a situation in which $\mathbf{M}_1$ and $\mathbf{M}_2$ have arbitrary shapes (Fig. 4).

Our first step is to identify a vector ($\mathbf{w}$) that is common to both bivectors (Fig. 5).

Next, we find the vector $\mathbf{u}$ such that $\mathbf{u}$ is perpendicular to $\mathbf{w}$, and $\mathbf{u}\wedge\mathbf{w} = \mathbf{M}_1$. This step enables to draw $\mathbf{M}_1$ as $\mathbf{u}\wedge\mathbf{w}$, which is equal to $\mathbf{u}\mathbf{w}$ because $\mathbf{u} \perp \mathbf{w}$ (Fig. 6). Similarly, we find the vector $\mathbf{v}$, perpendicular to $\mathbf{w}$, such that $\mathbf{v} \wedge \mathbf{w} = \mathbf{M}_2$, so that we can draw $\mathbf{M}_2$ as $\mathbf{v} \wedge \mathbf{w}$, which is equal to $\mathbf{v}\mathbf{w}$.

Here, we are already switching between thinking of $\mathbf{M}_1$ as a geometric bivector and as an algebraic one.

Having drawn both bivectors according to the above-mentioned factorizations, we position $\mathbf{u}$ and $\mathbf{v}$ as required to sum those vectors, while leaving the orientations of $\mathbf{M}_1$ and $\mathbf{M}_2$ unchanged (Fig. 7)

Now, we construct the vector $\mathbf{u} + \mathbf{v}$ (Fig. 8).

To finish, we draw the rectangle corresponding to the factorization $(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w}$, which is equal to $(\mathbf{u} + \mathbf{v})\,\mathbf{w}$ because $(\mathbf{u} + \mathbf{v}) \perp \mathbf{w}$. Considered as an algebraic bivector, the "sum" $\mathbf{M}_1 + \mathbf{M}_2$ is that product. As a geometric bivector, $\mathbf{M}_1 + \mathbf{M}_2$ is the corresponding section of the plane (Fig. 9). We should emphasize that the product $(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w}$ is just one factorization of the bivector defined as the "sum" of $\mathbf{M}_1$ and $\mathbf{M}_2$.

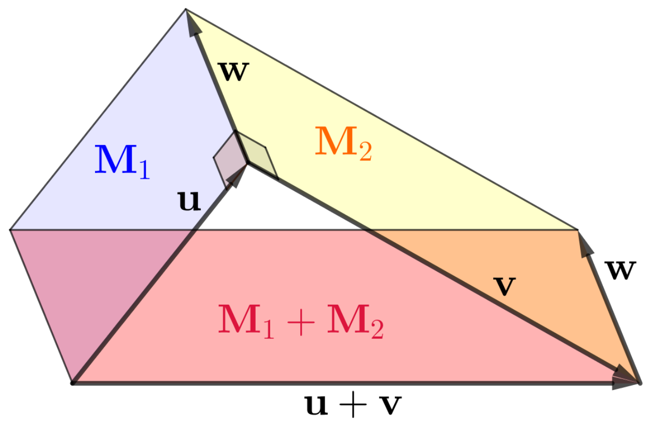


Figure 3: An adaptation of the illustration used by [1] and [2] in explaining the addition of bivectors. $\mathbf{M_1}$ and $\mathbf{M}_2$ are the bivectors that are to be added. Vector $\mathbf{w}$ is common to both. The vector $\mathbf{u}$ is perpendicular to $\mathbf{w}$, and is such that $\mathbf{u} \wedge \mathbf{w} = \mathbf{M}_1$. Similarly, $\mathbf{v}$ is perpendicular to $\mathbf{w}$, and is such that $\mathbf{v} \wedge \mathbf{w} = \mathbf{M}_2$. The result $(\mathbf{M}_1 + \mathbf{M}_2)$ of the procedure is outer product of the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{w}$.

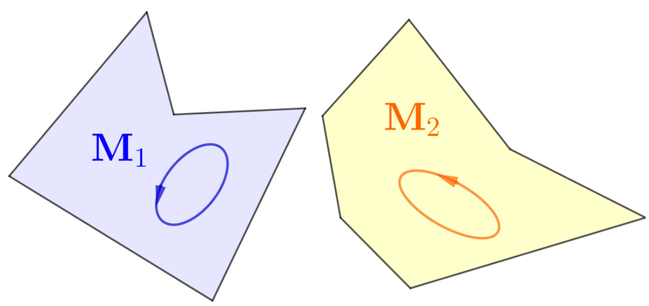


Figure 4: The starting point from which we will arrive at the arrangement shown in Fig. 3. Bivectors $\mathbf{M}_1$ and $\mathbf{M}_2$ are shown with arbitrary shapes. The circles with arrows shown the sense of the two bivectors; the circles look like ellipses in this diagram because of the viewing angle.

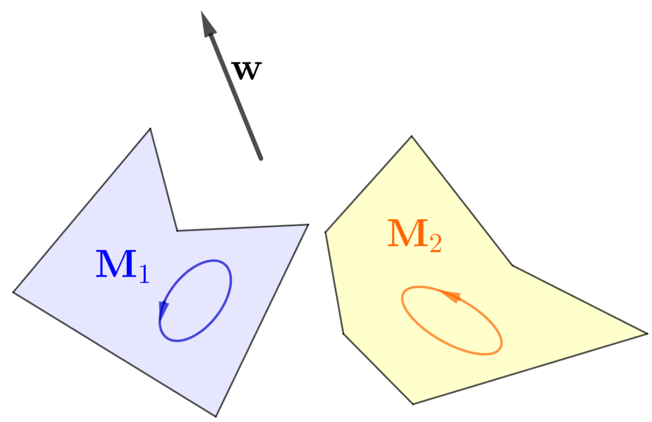


Figure 5: Bivectors $\mathbf{M}_1$ and $\mathbf{M}_2$ are still shown with arbitrary shapes, but we have now identified and drawn a vector $\mathbf{w}$ that is common to both bivectors.

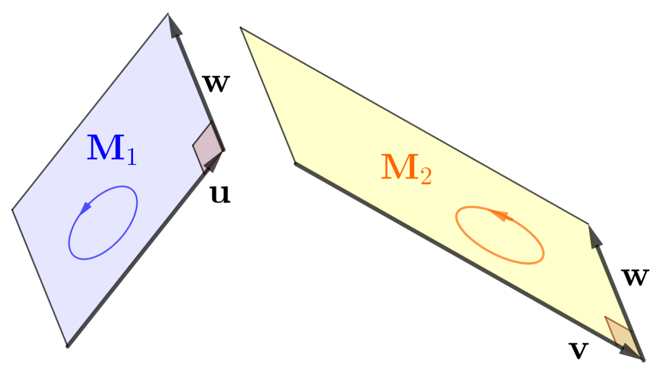


Figure 6: Bivector $\mathbf{M}_1$ drawn according to the factorization $\mathbf{u} \wedge \mathbf{w}$, and bivector $\mathbf{M}_2$ drawn according to the factorization $\mathbf{v} \wedge \mathbf{w}$.

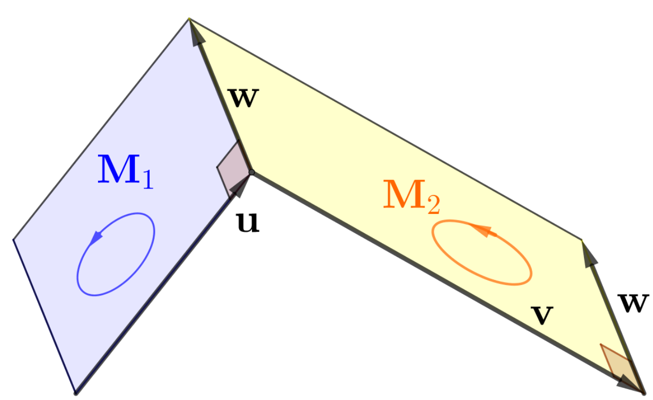


Figure 7: After moving $\mathbf{M}_2$ as needed to bring $\mathbf{u}$ and $\mathbf{v}$ into the position needed to find the vector sum $\mathbf{u} + \mathbf{v}$. The two bivectors are joined along their common vector $\mathbf{w}$.

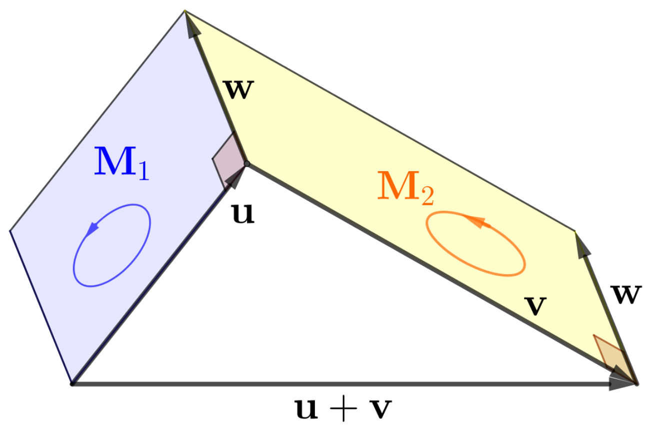


Figure 8: Showing the vector sum $\mathbf{u} + \mathbf{v}$.

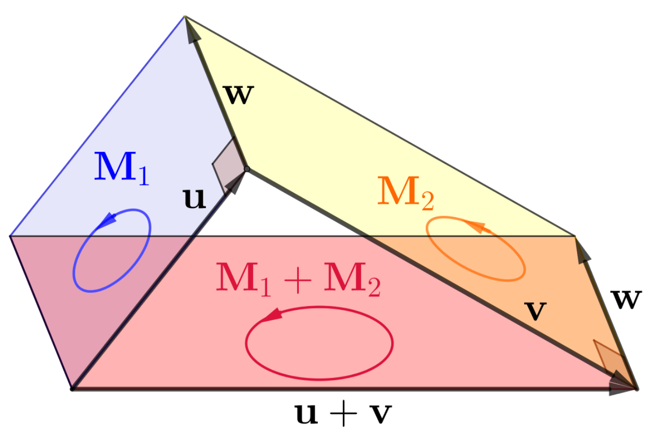


Figure 9: Final step: Draw the rectangle for the outer product $(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w}$. That product is one factorization of the bivector that is defined as the "sum" $\mathbf{M}_1 + \mathbf{M}_2$. We've now arrived at the diagram shown in Fig. 3.

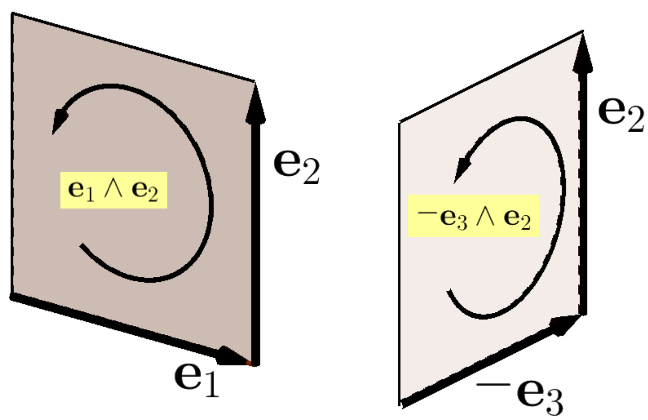


Figure 10: The result of the initial steps in adding geometric bivectors $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_3$. We identified $\mathbf{e}_2$ as the vector common to both, and therefore drew $\mathbf{e}_2\mathbf{e}_3$ as $-\mathbf{e}_3\mathbf{e}_2$.

2 Adding Our Three Unit Bivectors

As mentioned in the Introduction, we wish now to use the bivector-addition procedure to find the single bivector that is equal to the sum $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$. Because the addition of bivectors is both commutative and associative, we could do that addition in any order. Nevertheless, we'll to it in the order in which it is written. That is, as $(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3) + \mathbf{e}_3\mathbf{e}_1$. As we did when reviewing the procedure in the previous section, we will alternate between thinking of bivectors as geometric ones and algebraic ones.

2.1 Adding $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_3$

To use the procedure that we saw in the previous section, we first need to identify some vector $\mathbf{w}$ that is common to both $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_3$. Clearly, $\mathbf{e}_2$ is one such vector. We also need to identify the vector $\mathbf{u}$ such that $\mathbf{u} \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$, with $\mathbf{u} \perp \mathbf{e}_2$. That vector is $\mathbf{e}_1$. Next, we need to find the vector $\mathbf{v}$ such that $\mathbf{v} \wedge \mathbf{e}_2 = \mathbf{e}_2\mathbf{e}_3$, with $\mathbf{v} \perp \mathbf{e}_2$. That vector is $-\mathbf{e}_3$ (Fig. 10).

Note that we could use any scalar multiple of $\mathbf{e}_2$ as our $\mathbf{w}$.

Now, we move the two bivectors into the position needed for adding the vectors $\mathbf{e}_1$ and $-\mathbf{e}_3$ (Fig. 11).

After constructing the vector $\mathbf{e}_1 + -\mathbf{e}_3$, we finish by constructing the rectangle corresponding to the product $(\mathbf{e}_1 + -\mathbf{e}_3) \wedge \mathbf{e}_2$, which is also $(\mathbf{e}_1 + -\mathbf{e}_3)\,\mathbf{e}_2$ because $\mathbf{e}_1 + -\mathbf{e}_3$ and $\mathbf{e}_2$ are perpendicular (Fig. 12).

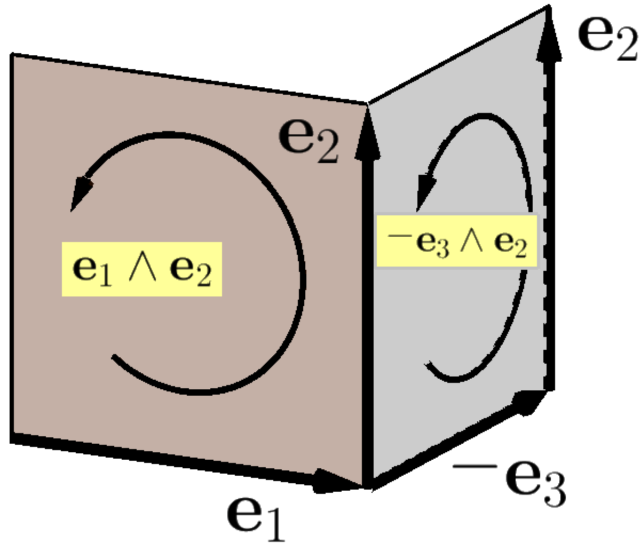


Figure 11: After moving $\mathbf{e}_1\mathbf{e}_2$ and $-\mathbf{e}_3\mathbf{e}_2$ into the position needed for adding the vectors $\mathbf{e}_1$ and $-\mathbf{e}_3$.

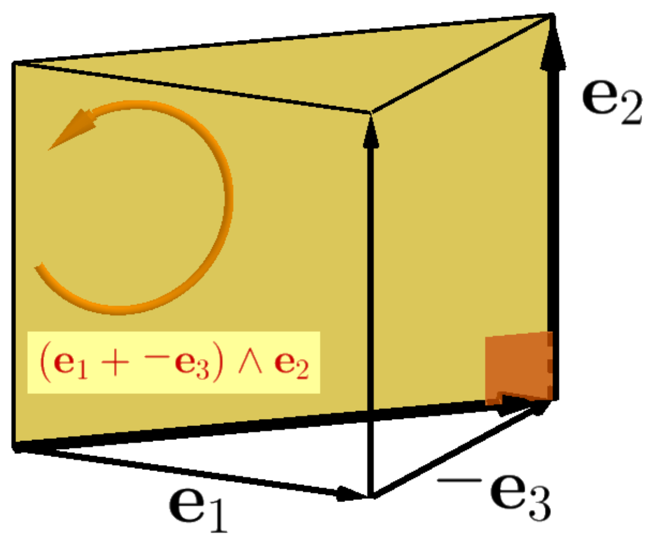


Figure 12: After constructing the vector $\mathbf{e}_1 + -\mathbf{e}_3$, we finish by constructing the rectangle corresponding to the product $(\mathbf{e}_1 + -\mathbf{e}_3) \wedge \mathbf{e}_2$, which is also $(\mathbf{e}_1 + -\mathbf{e}_3)\,\mathbf{e}_2$ because $\mathbf{e}_1 + -\mathbf{e}_3$ and $\mathbf{e}_2$ are perpendicular.

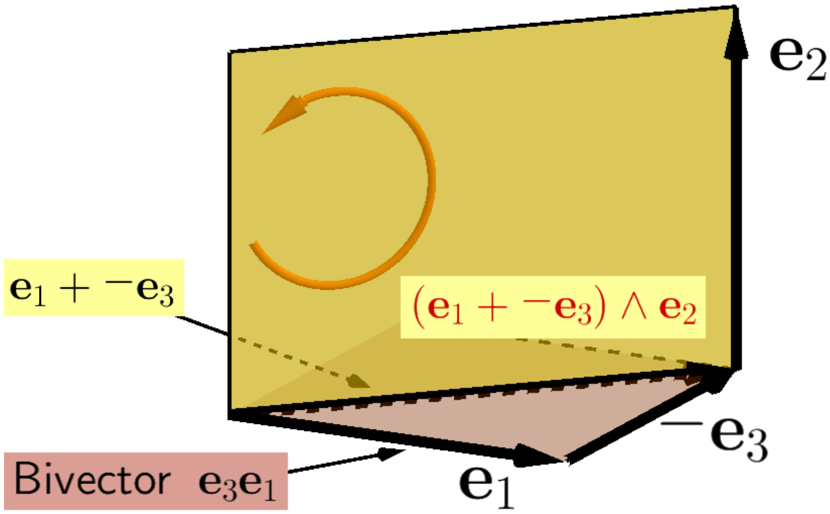


Figure 13: As the vector $\mathbf{w}$ that is common to $(\mathbf{e}_1 + {}^{-}\mathbf{e}_3) \wedge \mathbf{e}_2$ and $\mathbf{e}_3 \wedge \mathbf{e}_1$, we will use $\mathbf{e}_1 + {}^{-}\mathbf{e}_3$.

2.2 Adding $(\mathbf{e}_1 + {}^{-}\mathbf{e}_3) \wedge \mathbf{e}_2$ and $\mathbf{e}_3 \wedge \mathbf{e}_1$

Again, we begin by identifying some vector $\mathbf{w}$ that is common to the two bivectors that we wish to add. We'll use $\mathbf{e}_1 + {}^{-}\mathbf{e}_3$ (Fig 13). Therefore, we'll need to rewrite $(\mathbf{e}_1 + {}^{-}\mathbf{e}_3) \wedge \mathbf{e}_2$ as the outer product of some vector $\mathbf{u}$ with $\mathbf{e}_1 + {}^{-}\mathbf{e}_3$, where $\mathbf{u}$ is perpendicular to $\mathbf{e}_1 + {}^{-}\mathbf{e}_3$:

The vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ in this section are not the same as in the previous.

$$\mathbf{u} \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_3) = (\mathbf{e}_1 + {}^{-}\mathbf{e}_3) \wedge \mathbf{e}_2.$$

By inspection, we can see that $\mathbf{u}$ is ${}^{-}\mathbf{e}_2$.

Using the vector $\mathbf{e}_1 + {}^{-}\mathbf{e}_3$ as our $\mathbf{w}$ will also require us to "reshape" the bivector $\mathbf{e}_3\mathbf{e}_1$. Put more correctly, we need to draw that geometric bivector as the rectangle corresponding to the outer product of some vector $\mathbf{v}$ with $\mathbf{e}_1 + {}^{-}\mathbf{e}_3$, where $\mathbf{v}$ and $\mathbf{e}_1 + {}^{-}\mathbf{e}_3$ are perpendicular:

$$\begin{aligned}
\mathbf{v} \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_3) &= \mathbf{e}_3\mathbf{e}_1 \\
\mathbf{v}\,(\mathbf{e}_1 + {}^{-}\mathbf{e}_3) &= \mathbf{e}_3\mathbf{e}_1 \\
\mathbf{v} &= \mathbf{e}_3\mathbf{e}_1 \left[\mathbf{e}_1 + {}^{-}\mathbf{e}_3\right]^{-1} \\
\mathbf{v} &= \mathbf{e}_3\mathbf{e}_1 \left[\frac{\mathbf{e}_1 + {}^{-}\mathbf{e}_3}{2}\right] \\
\mathbf{v} &= \frac{\mathbf{e}_1 + \mathbf{e}_3}{2}.
\end{aligned}$$

With the bivector $(\mathbf{e}_1 + {}^{-}\mathbf{e}_3) \wedge \mathbf{e}_2$ rewritten as $\mathbf{e}_2 \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_3)$, and $\mathbf{e}_3\mathbf{e}_1$ "reshaped" as $\left[\frac{\mathbf{e}_1 + \mathbf{e}_3}{2}\right] \wedge \left[\mathbf{e}_1 + {}^{-}\mathbf{e}_3\right]$ (Fig. 14), we finish by constructing the vector ${}^{-}\mathbf{e}_2 + \frac{\mathbf{e}_1 + \mathbf{e}_3}{2} = \frac{\mathbf{e}_1 + {}^{-}\mathbf{e}_2 + \mathbf{e}_3}{2}$, and finally the bivector $\left[\frac{\mathbf{e}_1 + {}^{-}\mathbf{e}_2 + \mathbf{e}_3}{2}\right] \wedge \left[\mathbf{e}_1 + {}^{-}\mathbf{e}_3\right]$ (Fig. 15).

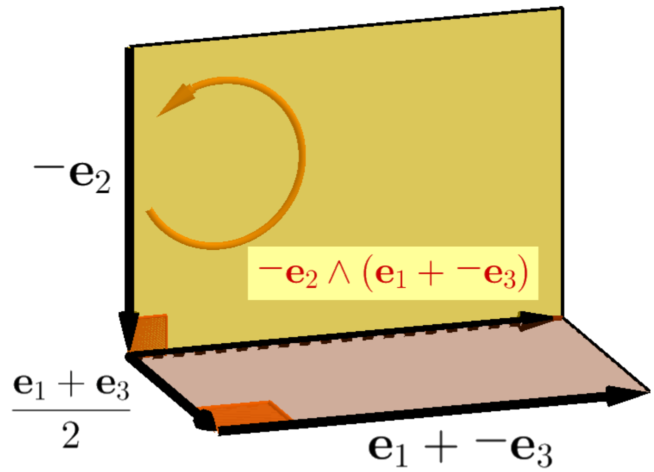


Figure 14: After "rewriting" the bivector $(\mathbf{e}_1 + {}^{-}\mathbf{e}_3) \wedge \mathbf{e}_2$ as $\mathbf{e}_2 \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_3)$, and "reshaping" $\mathbf{e}_3\mathbf{e}_1$ as $\left[\frac{\mathbf{e}_1 + \mathbf{e}_3}{2}\right] \wedge [\mathbf{e}_1 + {}^{-}\mathbf{e}_3]$.

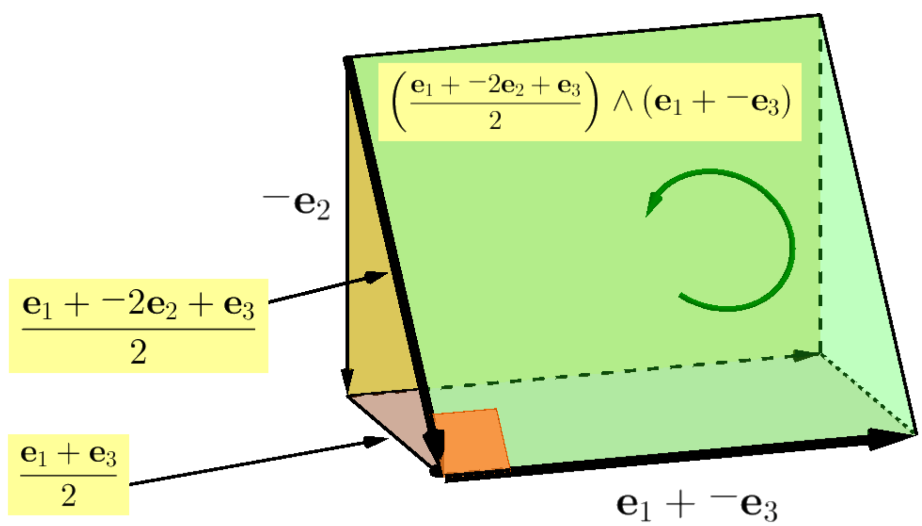


Figure 15: After adding the vectors ${}^{-}\mathbf{e}_2$ and $\frac{\mathbf{e}_1 + \mathbf{e}_3}{2}$ to give $\frac{\mathbf{e}_1 + {}^{-}\mathbf{e}_2 + \mathbf{e}_3}{2}$, then drawing the bivector $\left[\frac{\mathbf{e}_1 + {}^{-}2\mathbf{e}_2 + \mathbf{e}_3}{2}\right] \wedge [\mathbf{e}_1 + {}^{-}\mathbf{e}_3]$, which is the final result.

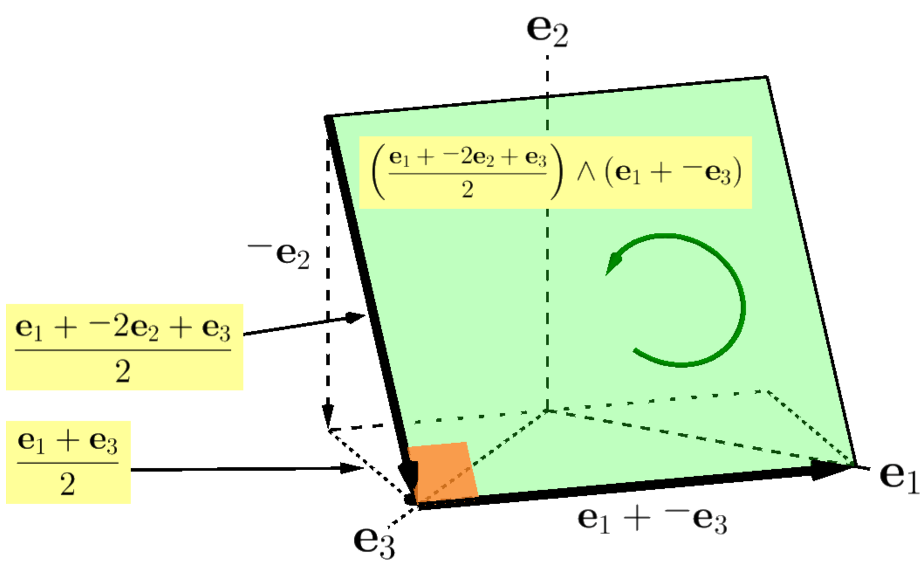


Figure 16: Our result for the sum $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$, along with our frame of reference.

3 Examining the Result

Let's present the bivector that we identified as the sum $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$ along with the frame of reference. 16. Now, compare that result to the rhombic bivector shown in Fig. 17. Both bivectors in contain the vectors $\left[\frac{\mathbf{e}_1 + {}^{-}2\mathbf{e}_2 + \mathbf{e}_3}{2}\right]$ and $[\mathbf{e}_1 + {}^{-}\mathbf{e}_3]$. They also have the same magnitude (area) and sense (rotation). Therefore, according to the postulates of GA, the two bivectors are equal.

Examining the rhombic bivector, we see that it is the plane segment for the outer product of the two red vectors shown in Fig. 18. Further examination shows that those vectors are ${}^{-}\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{e}_1 + {}^{-}\mathbf{e}_2$. Thus, the rhombic bivector is the plane segment for the product $({}^{-}\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_2)$. We have now demonstrated that the algebraic bivector corresponding to the geometric bivector drawn in Fig. 16, which we obtained by effecting the sum $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$, is equal to the product $({}^{-}\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_2)$. As another way of demonstrating that $({}^{-}\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_2)$ and $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$ are the same algebraic bivector, let's calculate their respective duals.

Yes, we can also demonstrate that equality by simply expanding $({}^{-}\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_2)$. We'll do that in a moment.

We'll begin with the dual of $({}^{-}\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_2)$. For reasons that we will explain during the Discussion, we will calculate that dual in a way that does not require us to expand the product.

In GA3, the dual of a bivector is a vector. Therefore, we can write that

$$[({}^{-}\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_2)]\,\mathcal{I}_3^{-1} = \langle [({}^{-}\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_2)]\,\mathcal{I}_3^{-1} \rangle_1.$$

Next, per [2] (p. 101), we write $({}^{-}\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^{-}\mathbf{e}_2)$ as $\langle ({}^{-}\mathbf{e}_2 + \mathbf{e}_3)\,(\mathbf{e}_1 + {}^{-}\mathbf{e}_2) \rangle_2$,

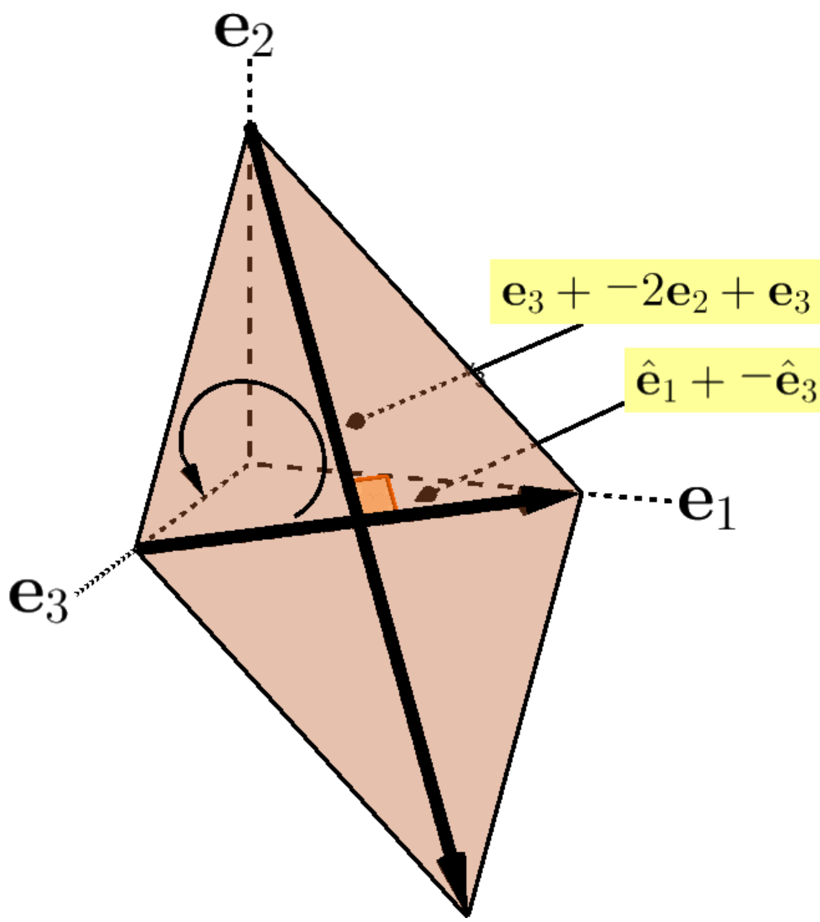


Figure 17: A bivector for comparison with our result for the sum $\mathbf{e}_1\mathbf{e}_2+\mathbf{e}_2\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_1$ (Fig. 16). The arrow on the end of the arc indicates this bivector's sense of rotation. This bivector, which is in the shape of a rhombus, has the same magnitude (area) and sense (rotation) as that shown in Fig. 16. Like that bivector, it also contains the vectors $\left[\dfrac{\mathbf{e}_1 + {}^{-}2\mathbf{e}_2 + \mathbf{e}_3}{2}\right]$ and $[\mathbf{e}_1 + {}^{-}\mathbf{e}_3]$. Therefore, according to the postulates of GA, the bivector shown in this Figure and that in Fig. 16 are equal.

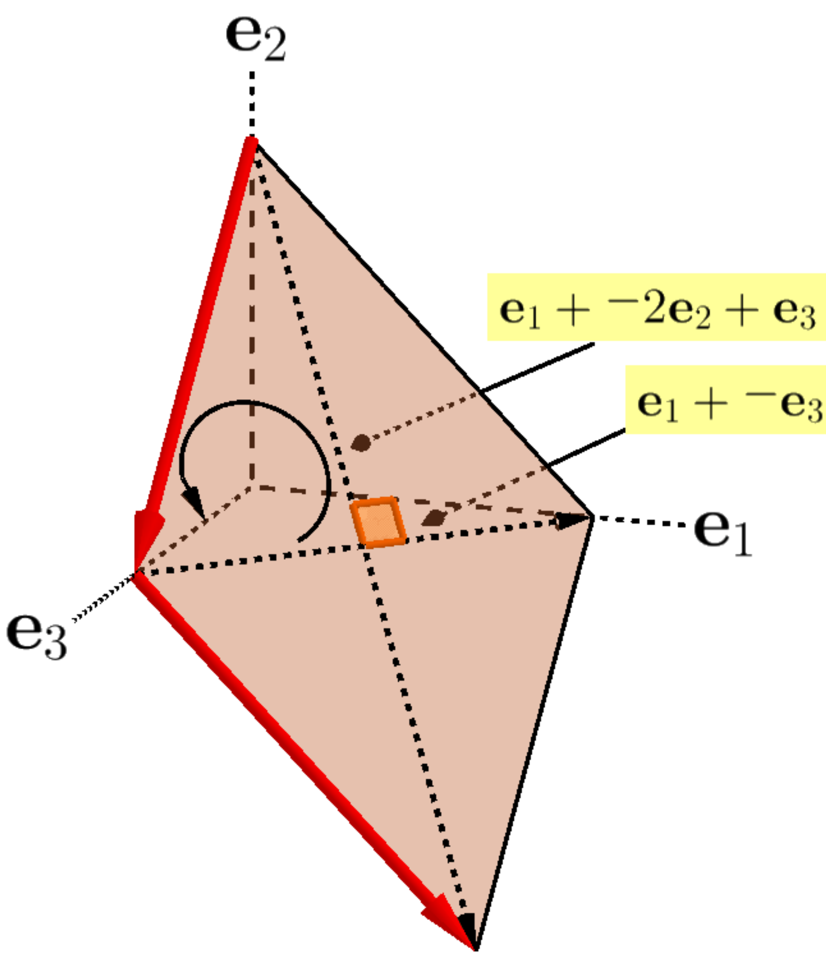


Figure 18: Demonstration that the bivector shown in Fig. 17 is that which corresponds to the exterior product of the two red vectors.

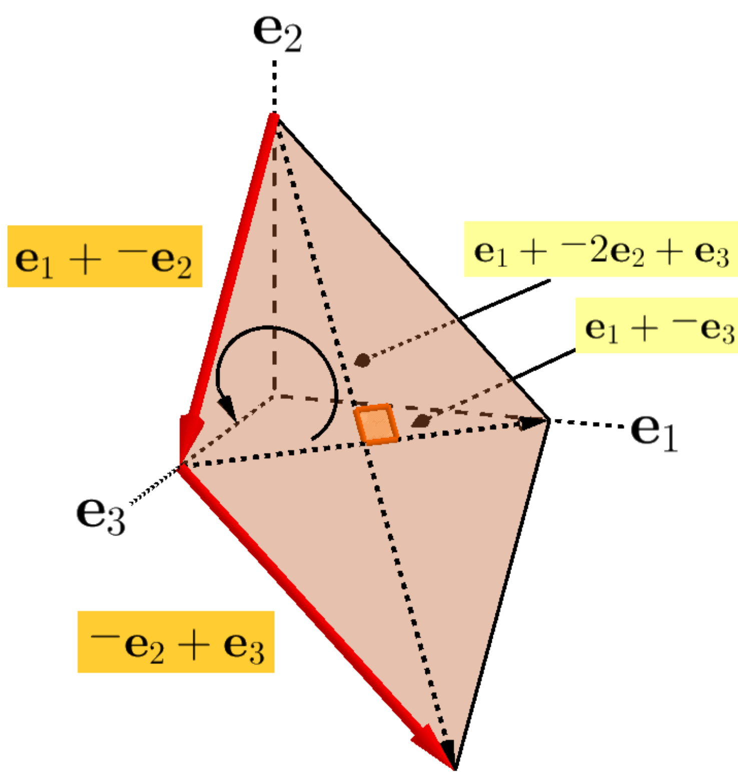


Figure 19: The two red vectors from 18 revealed as ${}^-\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{e}_1 + {}^-\mathbf{e}_2$. Therefore, this rhombic bivector is the geometric bivector for the product $({}^-\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + {}^-\mathbf{e}_2)$.

which is $(-\mathbf{e}_2 + \mathbf{e}_3)\,(\mathbf{e}_1 + -\mathbf{e}_2) - (-\mathbf{e}_2 + \mathbf{e}_3)\cdot(\mathbf{e}_1 + -\mathbf{e}_2)$:

$$
\begin{aligned}
[(-\mathbf{e}_2 + \mathbf{e}_3)\wedge(\mathbf{e}_1 + -\mathbf{e}_2)]\,\mathcal{I}_3^{-1} &= \langle[(-\mathbf{e}_2 + \mathbf{e}_3)\,(\mathbf{e}_1 + -\mathbf{e}_2) - (-\mathbf{e}_2 + \mathbf{e}_3)\cdot(\mathbf{e}_1 + -\mathbf{e}_2)]\,\mathcal{I}_3^{-1}\rangle_1 \\
&= \langle(-\mathbf{e}_2 + \mathbf{e}_3)\,(\mathbf{e}_1 + -\mathbf{e}_2)\,\mathcal{I}_3^{-1}\rangle_1 - \underbrace{\langle(-\mathbf{e}_2 + \mathbf{e}_3)\cdot(\mathbf{e}_1 + -\mathbf{e}_2)\,\mathcal{I}_3^{-1}\rangle_1}_{=0} \\
&= \langle(-\mathbf{e}_2 + \mathbf{e}_3)\,(\mathbf{e}_1 + -\mathbf{e}_2)\,(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)\rangle_1 \\
&= \langle(-\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_1 + -\mathbf{e}_3\mathbf{e}_2)\,(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)\rangle_1 \\
&= \langle(-\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_1 + -\mathbf{e}_3\mathbf{e}_2)\,(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)\rangle_1 \\
&= \langle\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\rangle_1 + \langle-\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\rangle_1 + \langle-\mathbf{e}_3\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\rangle_1 + \langle\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\rangle_1 \\
&= \mathbf{e}_3 + 0 + \mathbf{e}_2 + \mathbf{e}_1 \\
&= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3.
\end{aligned}
$$

Finding the dual of $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$ is more straightforward:

$$(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1)\,(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3.$$

Because their duals are equal, so are $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$ and $(-\mathbf{e}_2 + \mathbf{e}_3)\wedge(\mathbf{e}_1 + -\mathbf{e}_2)$. That equality can be further confirmed by simply expanding $(-\mathbf{e}_2 + \mathbf{e}_3)\wedge(\mathbf{e}_1 + -\mathbf{e}_2)$.

4 Discussion

Let's begin by reviewing what we've done thus far. We added the geometric bivectors $\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_3$, and $\mathbf{e}_3\mathbf{e}_1$ according to the procedure described by [1] and [2]. That procedure required us to "reshape" and "relocate" those bivectors—steps that GA's inventors made permissible by excluding shape and location as attributes of geometric bivectors. Using said procedure, we found that the result of $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$ is the geometric bivector that corresponds to the algebraic bivector $\left[\dfrac{\mathbf{e}_1 + -2\mathbf{e}_2 + \mathbf{e}_3}{2}\right]\wedge[\mathbf{e}_1 + -\mathbf{e}_3]$.

By reshaping that geometric bivector, we found that it is equal to the geometric bivector $(-\mathbf{e}_2 + \mathbf{e}_3)\wedge(\mathbf{e}_1 + -\mathbf{e}_2)$. Then, we confirmed that $(-\mathbf{e}_2 + \mathbf{e}_3)\wedge(\mathbf{e}_1 + -\mathbf{e}_2)$ is equal to the sum $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$ by showing that the dual of $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$ is equal to that of $(-\mathbf{e}_2 + \mathbf{e}_3)\wedge(\mathbf{e}_1 + -\mathbf{e}_2)$. Importantly, we calculated the dual for $(-\mathbf{e}_2 + \mathbf{e}_3)\wedge(\mathbf{e}_1 + -\mathbf{e}_2)$ in a way that did not require us to expand that outer product. Only after we established the equality of the two bivectors via their duals did we expand the product $(-\mathbf{e}_2 + \mathbf{e}_3)\wedge(\mathbf{e}_1 + -\mathbf{e}_2)$ to show that it is indeed equal to the algebraic bivector $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$.

The significance of what we have done is not that we added the geometric bivectors $\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_3$, and $\mathbf{e}_3\mathbf{e}_1$ and ended up with the algebraic bivector $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1$. Instead, the significance is that the procedure for adding geometric bivectors is precisely that which is needed to give results that coincide with those obtained by calculating outer products of vectors that are expressed in terms of a 3D basis. That accomplishment is no coincidence: it is a consequence of the attributes that GA's designers assigned (or didn't) to bivectors.

References

[1] D. Hestenes, 1999, *New Foundations for Classical Mechanics*, (Second Edition), Kluwer Academic Publishers (Dordrecht/Boston/London).

[2] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).