Abstract

We derive various weighted summation identities, including binomial and double binomial identities, for Tribonacci numbers. Our results contain some previously known results as special cases.

1 Introduction

For \( m \geq 3 \), the Tribonacci numbers are defined by

\[
T_m = T_{m-1} + T_{m-2} + T_{m-3}, \quad T_0 = 0, \quad T_1 = T_2 = 1.
\]  

(1.1)

By writing \( T_{m-1} = T_{m-2} + T_{m-3} + T_{m-4} \) and eliminating \( T_{m-2} \) and \( T_{m-3} \) between this recurrence relation and the recurrence relation (1.1), a useful alternative recurrence relation is obtained for \( m \geq 4 \):

\[
T_m = 2T_{m-1} - T_{m-4}, \quad T_0 = 0, \quad T_1 = T_2 = 1, \quad T_3 = 2.
\]  

(1.2)

Extension of the definition of \( T_m \) to negative subscripts is provided by writing the recurrence relation (1.2) as

\[
T_{-m} = 2T_{m+3} - T_{m+4}.
\]  

(1.3)

Anantakitpaisal and Kuhapatanakul \([2]\) proved that

\[
T_{-m} = T_{m-1}^2 - T_{m-2}T_m.
\]  

(1.4)

The following identity (Feng \([3]\), equation (3.3); Shah \([7]\), (ii)) is readily established by the principle of mathematical induction:

\[
T_{m+r} = T_rT_{m-2} + (T_{r-1} + T_r)T_{m-1} + T_{r+1}T_m.
\]  

(1.5)

Irmak and Alp \([5]\) derived the following identity for Tribonacci numbers with indices in arithmetic progression:

\[
T_{lm+r} = \lambda_1(t)T_{l(m-1)+r} + \lambda_2(t)T_{l(m-2)+r} + \lambda_3(t)T_{l(m-3)+r},
\]  

(1.6)
where,
\[ \lambda_1(t) = \alpha^t + \beta^t + \gamma^t, \quad \lambda_2(t) = -\alpha \beta^t - (\alpha \gamma)^t - (\beta \gamma)^t, \quad \lambda_3(t) = (\alpha \beta \gamma)^t, \]
where \( \alpha, \beta \) and \( \gamma \) are the roots of the characteristic polynomial of the Tribonacci sequence \( x^3 - x^2 - x - 1 \). Thus,
\[
\alpha = \frac{1}{3} \left( 1 + 3\sqrt[3]{19 + 3\sqrt{33}} + 3\sqrt[3]{19 - 3\sqrt{33}} \right),
\]
\[
\beta = \frac{1}{3} \left( 1 + \omega \sqrt[3]{19 + 3\sqrt{33}} + \omega^2 \sqrt[3]{19 - 3\sqrt{33}} \right)
\]
and
\[
\gamma = \frac{1}{3} \left( 1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega \sqrt[3]{19 - 3\sqrt{33}} \right),
\]
where \( \omega = \exp(2i\pi/3) \) is a primitive cube root of unity. Note that \( \lambda_1(t), \lambda_2(t) \) and \( \lambda_3(t) \) are integers for any positive integer \( t \); in particular, \( \lambda_1(1) = 1 = \lambda_2(1) = \lambda_3(1) \).

2 Weighted sums

Lemma 1 ([1], Lemma 2). Let \( \{X_m\} \) be any arbitrary sequence, where \( X_m, m \in \mathbb{Z} \), satisfies a second order recurrence relation \( X_m = f_1 X_{m-a} + f_2 X_{m-b} \), where \( f_1 \) and \( f_2 \) are arbitrary non-vanishing complex functions, not dependent on \( m \), and \( a \) and \( b \) are integers. Then,
\[
f_2 \sum_{j=0}^{k} \frac{X_{m-ka-b+aj}}{f_1^j} = \frac{X_m}{f_1^k} - f_1 X_{m-(k+1)a}, \tag{2.1}
\]
\[
f_1 \sum_{j=0}^{k} \frac{X_{m-kb-a+bj}}{f_2^j} = \frac{X_m}{f_2^k} - f_2 X_{m-(k+1)b} \tag{2.2}
\]
and
\[
\sum_{j=0}^{k} \frac{X_{m-(b-a)k+a+(b-a)j}}{(-f_2/f_1)^j} = \frac{f_1 X_m}{(-f_2/f_1)^k} + f_2 X_{m-(k+1)(b-a)} \tag{2.3}
\]
for \( k \) a non-negative integer.

Theorem 1. The following identities hold for any integers \( m \) and \( k \):
\[
\sum_{j=0}^{k} 2^{-j} T_{m-k-4+j} = 2T_{m-k-1} - 2^{-k} T_m, \tag{2.4}
\]
\[
2 \sum_{j=0}^{k} (-1)^j T_{m-4k-4+j} = (-1)^k T_m + T_{m-4k-4} \tag{2.5}
\]
and
\[
\sum_{j=0}^{k} 2^{j} T_{m-3k+1+3j} = 2^{k+1} T_m - T_{m-3k-3} \tag{2.6}
\]
Proof. From the recurrence relation (1.2), make the identifications $f_1 = 2$, $f_2 = -1$, $a = 1$ and $b = 4$ and use these in Lemma 1 with $X = T$.

Particular instances of identities (2.4)–(2.6) are the following identities:

$$\sum_{j=0}^{k} 2^{-j}T_j = 4 - 2^{-k}T_{k+4},$$

(2.7)

giving,

$$\sum_{j=0}^{\infty} 2^{-j}T_j = 4,$$

(2.8)

and

$$2 \sum_{j=0}^{k} (-1)^jT_{4j} = (-1)^kT_{4k+1} - 1$$

(2.9)

and

$$\sum_{j=0}^{k} 2^jT_{3j} = 2^{k+1}T_{3k-1}.$$  

(2.10)

Lemma 2 (Partial sum of an $n^{th}$ order sequence). Let \{X_j\} be any arbitrary sequence, where $X_j$, $j \in \mathbb{Z}$, satisfies a $n^{th}$ order recurrence relation $X_j = f_1X_{j-c_1} + f_2X_{j-c_2} + \cdots + f_nX_{j-c_n}$, where $f_1$, $f_2$, $\ldots$, $f_n$ are arbitrary non-vanishing complex functions, not dependent on $j$, and $c_1$, $c_2$, $\ldots$, $c_n$ are fixed integers. Then, the following summation identity holds for arbitrary $x$ and non-negative integer $k$:

$$\sum_{j=0}^{k} x^jX_j = \sum_{m=1}^{n} \left\{ x^{c_m} f_m \left( \sum_{j=1}^{c_m} x^{-j}X_j - \sum_{j=k-c_m+1}^{k} x^jX_j \right) \right\}.$$  

Proof. Recurrence relation:

$X_j = \sum_{m=1}^{n} f_mX_{j-c_m}.$

We multiply both sides by $x^j$ and sum over $j$ to obtain

$$\sum_{j=0}^{k} x^jX_j = \sum_{m=1}^{n} \left( f_m \sum_{j=0}^{k} x^jX_{j-c_m} \right) = \sum_{m=1}^{n} \left( x^{c_m} f_m \sum_{j=-c_m}^{k} x^jX_j \right),$$

after shifting the summation index $j$. Splitting the inner sum, we can write

$$\sum_{j=0}^{k} x^jX_j = \sum_{m=1}^{n} x^{c_m} f_m \left( \sum_{j=-c_m}^{-1} x^jX_j + \sum_{j=0}^{k} x^jX_j + \sum_{j=k+1}^{k-c_m} x^jX_j \right).$$

Since

$$\sum_{j=-c_m}^{-1} x^jX_j \equiv \sum_{j=1}^{c_m} x^{-j}X_j \quad \text{and} \quad \sum_{j=k+1}^{k-c_m+1} x^jX_j \equiv \sum_{j=k-c_m}^{k} x^jX_j,$$
the preceding identity can be written
\[
\sum_{j=0}^{k} x^j X_j = \sum_{m=1}^{n} x^m f_m \left( \sum_{j=1}^{c_m} x^{-j} X_{-j} + \sum_{j=0}^{k} x^j X_j - \sum_{j=k-c_m+1}^{k} x^j X_j \right).
\]
Thus, we have
\[
S = \sum_{m=1}^{n} x^m f_m \left( \sum_{j=1}^{c_m} x^{-j} X_{-j} + \sum_{j=0}^{k} x^j X_j \right),
\]
where
\[
S = S_k(x) = \sum_{j=0}^{k} x^j X_j.
\]
Removing brackets, we have
\[
S = \sum_{m=1}^{n} x^m f_m \left( \sum_{j=1}^{c_m} x^{-j} X_{-j} - \sum_{j=k-c_m+1}^{k} x^j X_j \right) + S \sum_{m=1}^{n} x^m f_m,
\]
from which the result follows by grouping the \( S \) terms.

**Lemma 3** (Generating function). Under the conditions of Lemma 2, if additionally \( x^k X_k \) vanishes in the limit as \( k \) approaches infinity, then
\[
S_\infty(x) = \sum_{j=0}^{\infty} x^j X_j = \frac{\sum_{m=1}^{n} \left( x^m f_m \sum_{j=1}^{c_m} x^{-j} X_{-j} \right)}{1 - \sum_{m=1}^{n} x^m f_m},
\]
so that \( S_\infty(x) \) is a generating function for the sequence \( \{X_j\} \).

**Theorem 2** (Sum of Tribonacci numbers with indices in arithmetic progression). For arbitrary \( x \), any integers \( t \) and \( r \) and any non-negative integer \( k \), the following identity holds:
\[
(1 - \lambda_1(t)x - \lambda_2(t)x^2 - \lambda_3(t)x^3) \sum_{j=0}^{k} x^j T_{tj+r} = T_r + (x\lambda_2(t) + x^2\lambda_3(t))T_{r-t}
\]
\[
+ x\lambda_3(t)T_{r-2t} + x^{k+1}T_{(k+1)t+r} - x^{k+2}\lambda_2(t) + x\lambda_3(t))T_{kt+r}
\]
\[
- x^{k+2}\lambda_3(t)T_{(k-1)t+r},
\]
where,
\[
\lambda_1(t) = \alpha^t + \beta^t + \gamma^t, \quad \lambda_2(t) = -(\alpha\beta)^t - (\alpha\gamma)^t - (\beta\gamma)^t, \quad \lambda_3(t) = (\alpha\beta\gamma)^t,
\]
where \( \alpha \), \( \beta \) and \( \gamma \) are the roots of the characteristic polynomial of the Tribonacci sequence \( x^3 - x^2 - x - 1 \).

**Proof.** Write identity (1.6) as \( X_j = f_1 X_{j-1} + f_2 X_{j-2} + f_3 X_{j-3} \) and identify the sequence \( \{X_j\} = \{T_{tj+r}\} \) and the constants \( c_1 = 1, c_2 = 2, c_3 = 3 \) and the functions \( f_1 = \lambda_1(t), f_2 = \lambda_2(t), f_3 = \lambda_3(t) \), and use these in Lemma 2.

\[\square\]
Corollary 3 (Generating function of the Tribonacci numbers with indices in arithmetic progression). For any integers $t$ and $r$, any non-negative integer $k$ and arbitrary $x$ for which $x^kT_k$ vanishes as $k$ approaches infinity, the following identity holds:

$$
\sum_{j=0}^{\infty} x^jT_{kj+r} = \frac{T_r + (x\alpha + x^2\beta)T_{r-t} + x\gamma T_{r-2t}}{1 - x\alpha - x^2\beta - x^3\gamma},
$$

where,

$$
\lambda_1 = \alpha^t + \beta^t + \gamma^t, \quad \lambda_2 = -(\alpha\beta)^t - (\alpha\gamma)^t - (\beta\gamma)^t, \quad \lambda_3 = (\alpha\beta\gamma)^t,
$$

where $\alpha$, $\beta$ and $\gamma$ are the roots of the characteristic polynomial of the Tribonacci sequence $x^3 - x^2 - x - 1$.

Many instances of Theorem 2 may be explored. In particular, we have

$$(\lambda_1(t) + \lambda_2(t) + \lambda_3(t) - 1) \sum_{j=0}^{k} T_{kj+r} = -T_r - (\lambda_2(t) + \lambda_3(t))T_{r-t}$$

$$- \lambda_3(t)T_{r-2t} + T_{(k+1)t+r}$$

$$+ (\lambda_2(t) + \lambda_3(t))T_{kt+r} + \lambda_3(t)T_{(k-1)t+r}, \tag{2.11}$$

which at $r = 0$ gives

$$(\lambda_1(t) + \lambda_2(t) + \lambda_3(t) - 1) \sum_{j=0}^{k} T_{ij} = -\lambda_2(t) + \lambda_3(t)(T_{k-1}^2 - T_{k-2}T_k)$$

$$- \lambda_3(t)(T_{2k-1}^2 - T_{2k-2}T_{2k}) + T_{(k+1)t}$$

$$+ (\lambda_2(t) + \lambda_3(t))T_{kt} + \lambda_3(t)T_{(k-1)t}, \tag{2.12}$$

and

$$(1 + \lambda_1(t) - \lambda_2(t) + \lambda_3(t)) \sum_{j=0}^{k} (-1)^jT_{kj+r} = T_r + (\lambda_3(t) - \lambda_2(t))T_{r-t}$$

$$- \lambda_3(t)T_{r-2t} + (-1)^kT_{(k+1)t+r}$$

$$+ (-1)^k(\lambda_3(t) - \lambda_2(t))T_{kt+r}$$

$$- (-1)^k\lambda_3(t)T_{(k-1)t+r}, \tag{2.13}$$

which at $r = 0$ gives

$$(1 + \lambda_1(t) - \lambda_2(t) + \lambda_3(t)) \sum_{j=0}^{k} (-1)^jT_{ij} = (\lambda_3(t) - \lambda_2(t))(T_{k-1}^2 - T_{k-2}T_k)$$

$$- \lambda_3(t)(T_{2k-1}^2 - T_{2k-2}T_{2k}) + (-1)^kT_{(k+1)t}$$

$$+ (-1)^k(\lambda_3(t) - \lambda_2(t))T_{kt}$$

$$- (-1)^k\lambda_3(t)T_{(k-1)t}. \tag{2.14}$$

Many previously known results are particular instances of the identities (2.11) and (2.13). For example, Theorem 5 of [6] is obtained from identity (2.12) by setting $t = 4$. Sums of
Tribonacci numbers with indices in arithmetic progression are also discussed in references [4, 5, 6] and references therein, using various techniques.

Weighted sums of the form \( \sum_{j=0}^{k} j^p T_{j+r} \), where \( p \) is a non-negative integer, may be evaluated by setting \( x = e^y \) in the identity of Theorem 2, differentiating both sides \( p \) times with respect to \( y \) and then setting \( y = 0 \). The simplest examples in this category are the following:

\[
2 \sum_{j=0}^{k} jT_{j+r} = -T_{r-2} + 3T_{r+1} + (k-1)T_{k+r-1} + (2k-1)T_{k+r} + (k-2)T_{k+r+1}
\]  
\[
2 \sum_{j=0}^{k} j^2T_{j+r} = -3T_{r-1} - 5T_{r} - 6T_{r+1} + (k^2 - 2k + 3)T_{k+r-1} + (2k^2 - 2k + 5)T_{k+r} + (k^2 - 4k + 6)T_{k+r+1},
\]

with the particular cases

\[
2 \sum_{j=0}^{k} jT_j = 2 + (k-1)T_{k-1} + (2k-1)T_k + (k-2)T_{k+1}
\]  
\[
2 \sum_{j=0}^{k} j^2T_j = -6 + (k^2 - 2k + 3)T_{k+r-1} + (2k^2 - 2k + 5)T_k + (k^2 - 4k + 6)T_{k+1}.
\]

3 Weighted binomial sums

Lemma 4 ([1], Lemma 3). Let \( \{X_m\} \) be any arbitrary sequence. Let \( X_m, m \in \mathbb{Z} \), satisfy a second order recurrence relation \( X_m = f_1X_{m-a} + f_2X_{m-b} \), where \( f_1 \) and \( f_2 \) are non-vanishing complex functions, not dependent on \( m \), and \( a \) and \( b \) are integers. Then,

\[
\sum_{j=0}^{k} \binom{k}{j} \left( \frac{f_1}{f_2} \right)^j X_{m-bk+(b-a)j} = \frac{X_m}{f_2^k},
\]  
\[
\sum_{j=0}^{k} \binom{k}{j} \frac{X_{m+(a-b)k+bj}}{(-f_2)^j} = \left( -\frac{f_1}{f_2} \right)^k X_m
\]  
\[
\text{and}
\sum_{j=0}^{k} \binom{k}{j} \frac{X_{m+(b-a)k+aj}}{(-f_1)^j} = \left( -\frac{f_2}{f_1} \right)^k X_m,
\]

for \( k \) a non-negative integer.
Theorem 4. The following identities hold for any integer $m$ and any non-negative integer $k$:

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^j T_{m-4k+3j} = (-1)^k T_m, \tag{3.4}
\]

\[
\sum_{j=0}^{k} \binom{k}{j} T_{m-3k+4j} = 2^k T_m \tag{3.5}
\]

and

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^{-j} T_{m+3k+j} = 2^{-k} T_m. \tag{3.6}
\]

**Proof.** Identify $X = T$ in Lemma 4 and use the $f_1$, $f_2$, $a$ and $b$ values found in the proof of Theorem 1. \qed

Particular cases of (3.4), (3.5) and (3.6) are the following identities:

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^j T_{3j} = (-1)^k T_{4k}, \tag{3.7}
\]

\[
\sum_{j=0}^{k} \binom{k}{j} T_{4j} = 2^k T_{3k} \tag{3.8}
\]

and

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^{-j} T_j = 2^{-k}(T_{3k-1}^2 - T_{3k-2}T_{3k}). \tag{3.9}
\]

4 Weighted double binomial sums

**Lemma 5.** Let $\{X_m\}$ be any arbitrary sequence, $X_m$ satisfying a third order recurrence relation $X_m = f_1 X_{m-a} + f_2 X_{m-b} + f_3 X_{m-c}$, where $f_1$, $f_2$ and $f_3$ are arbitrary nonvanishing functions and $a$, $b$ and $c$ are integers. Then, the following identities hold:

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_2}{f_3} \right)^j \left( \frac{f_1}{f_2} \right)^s X_{m-ck+(c-b)j+(b-a)s} = \frac{X_m}{f_3^k}, \tag{4.1}
\]

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_3}{f_2} \right)^j \left( \frac{f_1}{f_3} \right)^s X_{m-bk+(b-c)j+(c-a)s} = \frac{X_m}{f_2^k}, \tag{4.2}
\]

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_3}{f_1} \right)^j \left( \frac{f_2}{f_3} \right)^s X_{m-ak+(a-c)j+(c-b)s} = \frac{X_m}{f_1^k}, \tag{4.3}
\]

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_2}{f_3} \right)^j \left( -\frac{1}{f_2} \right)^s X_{m-(c-a)k+(c-b)j+bs} = \left( -\frac{f_1}{f_3} \right)^k X_m, \tag{4.4}
\]

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_1}{f_3} \right)^j \left( -\frac{1}{f_1} \right)^s X_{m-(c-b)k+(c-a)j+as} = \left( -\frac{f_2}{f_3} \right)^k X_m. \tag{4.5}
\]
and
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left(\frac{f_1}{f_2}\right)^j \left(-\frac{1}{f_1}\right)^s X_{m-(b-c)k+(b-a)j+s} = \left(-\frac{f_3}{f_2}\right)^k X_m. \tag{4.6}
\]

Proof. Only identity (4.1) needs to be proved as identities (4.2)-(4.6) are obtained from (4.1) by re-arranging the recurrence relation. The proof of (4.1) is by induction on \(k\), similar to the proof of Lemma 3 of [1]. \(\square\)

**Theorem 5.** The following identities hold for non-negative integer \(k\), integer \(m\) and integer \(r \not\in \{-17, -4, -1, 0\}:

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} (T_{r-1} + T_r)^{j-s} \frac{T_r^{s+1}}{T_r^2} \frac{T_m^{r+1}}{T_r^{k+1}} = \frac{T_m}{T_r^k}, \tag{4.7}
\]

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \frac{T_r^{j-s} T_r^{s+1}}{(T_{r-1} + T_r)^{j-s}} \frac{T_m^{r+1}}{T_r^{k+1}} = \frac{T_m}{(T_{r-1} + T_r)^k}, \tag{4.8}
\]

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \frac{T_r^{j-s}(T_{r-2} + T_{r-1})^s}{T_r^j} \frac{T_m^{r+1}}{T_r^{k+1}} = \frac{T_m}{T_r^k}, \tag{4.9}
\]

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-1)^s \binom{k}{j} \binom{j}{s} \frac{T_r^{j-s} T_r^{s+1}}{(T_{r-1} + T_r)^{j-s}} \frac{T_m^{r+1}}{T_r^{k+1}} = (-1)^k \left(\frac{T_{r+1}}{T_r}\right)^k T_m, \tag{4.10}
\]

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-1)^s \binom{k}{j} \binom{j}{s} \frac{T_r^{j-s} T_r^{s+1}}{(T_{r-1} + T_r)^{j-s}} \frac{T_m^{r+1}}{T_r^{k+1}} = (-1)^k \left(\frac{T_{r+1}}{T_r}\right)^k T_m. \tag{4.11}
\]

and

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-1)^s \binom{k}{j} \binom{j}{s} \frac{T_r^{j-s} T_r^{s+1}}{(T_{r-1} + T_r)^{j-s}} \frac{T_m^{r+1}}{T_r^{k+1}} = (-1)^k \left(\frac{T_{r+1}}{T_r}\right)^k T_m. \tag{4.12}
\]

Proof. Write the identity (1.5) as \(T_m = T_1 T_m - 2 + (T_{r-1} + T_r) T_m - r - 1 + T_{r+1} T_{m-r}\), identify \(f_1 = T_r, f_2 = T_{r-1} + T_r, f_3 = T_{r+1}, a = r + 2, b = r + 1, c = r\) and use these in Lemma [3] with \(X = T\). \(\square\)

**References**


