The Function $f(x)$ = C and the Continuum Hypothesis

Abstract

Part 1 examines whether or not an analysis of the behavior of the function f(x) = C, where C is any constant, on the interval (a, b) where a and b are real numbers and a < b, will provide a method of proving the truth or falsity of the Continuum Hypothesis (CH). The argument will be presented in three theorems and one corollary. The first theorem proves, by construction, the countability of the domain d of f(x) = C on the interval (a, b) where a and b are real numbers. The second theorem proves, by substitution, that the set of natural numbers ℕ has the same cardinality as the subset ***S*** of real numbers on the given interval. The corollary extends the proof of theorem 2 to show that ℕ and ℝ are of the same cardinality. The third theorem proves, by logical inference, that the CH is true.

Part 2 is a demonstration of how the set of natural numbers ℕ can be put into a one to one correspondence with the power set of natural numbers, P(ℕ). From this I will derive the bijective function f : ℕ → P(ℕ). Lastly, I'll propose a conjecture asserting that f(x) = C can be employed to construct a one to one correspondence between the natural numbers and any infinite set that can be cast as the domain of the function.

Appendix A extends the methodology for creating a bijection between infinite sets to the function $f(x) = x^2$ using random real numbers from the domain of the function as input to $\overset{n}{\underset{i=1}{\mathrm{E}}}(f)\,\mathrm{d_i}$ in order to show how the constructed array would appear in practical application.

Introduction

Georg Cantor's creation of hierarchal infinities in the late 19^{th} century, whereby infinite sets come in a variety of sizes despite being infinite, goes solidly against the grain of intuition. How can two sets, both with an endless supply of members, not be the same size? That is the question this work will attempt to answer.

The proofs offered in this paper rely on construction and mathematical induction via the mathematical operation of evaluating the results of the function $f(x)$ = C for:

1. the domain defined by the real numbers between a and b and
2. the domain defined by the power set of the natural numbers.

In any interval on the real number line there are infinitely many real numbers. The real numbers in the interval contain no gaps, that is they form a continuum. If the interval (a, b) where a and b are real numbers defines the domain of a function then that domain must contain all the numbers between a and b, no exceptions. The question then is, does the definition of the domain constitute an implicit list of all the elements of the domain? This question relates to Cantor's proof that the real numbers are uncountable; that they cannot be put into a one to one correspondence with the natural numbers.

Point, Line

Primitive 1: A **point** is that which has no part.
Primitive 2: A **line** is breadthless length.[1]

A point has no dimension, yet any two of them define the endpoints of a line which in turn contains infinitely many points in between. Points are indistinguishable until, that is, we start to label them. Generally, points are labeled with numbers and so become unique things in the mathematical universe. The numbers used to label points on a line are called the real numbers and are denoted by $\mathbb{R}$.

Though distinct, points on a line form a smooth, continuous length. And when points on a line are labeled with numbers there is no "next number" on the line, for no matter how small an interval one defines there are always an infinite number of points between the endpoints of the interval. The points on a line are referred to as a continuum.

Set

Georg Cantor gave the following definition of a set at the beginning of his Beiträge zur Begründung der transfiniten Mengenlehre:[2]

Definition 1: "A set is a gathering together into a whole of *definite, distinct objects* [emphasis added] of our perception [Anschauung] or of our thought—which are called elements of the set."

The element or member of a set is a "definite, distinct object"; wholly its own thing. Cantor defined a set as a collection of these definite, distinct objects whereby each element of the set exists in a manner totally unconnected from all other elements of the set. Now the points on a line cannot form a set because points in and of themselves have no distinguishing characteristics to tell one from another. However, labeling each point on a line with a real number differentiates each point from all other points on the line. We can then gather together all the labeled points on the line into a set of definite, distinct objects and so form the set of real numbers. We call the set of real numbers $\mathbb{R}$ and if r is a real number then

$$\mathbb{R}, \left\{\forall(\mathrm{r}) \mid \mathrm{r} \text{ is real}\right\}$$

is read, the set of real numbers consists of all numbers r such that r is a real number.

Continuous and Discrete

We now have an interesting situation to consider. On the one hand the real numbers, when viewed as a line form a continuum containing no gaps, only an infinite regression of numbers between any two numbers no matter how close together they are. On the other hand, when viewed as a set, the

[1] "Elements Book 1" Euclid
[2] "Eine Menge, ist die Zusammenfassung bestimmter, wohlunterschiedener Objekte unserer Anschauung oder unseres Denkens – welche Elemente der Menge genannt werden – zu einem Ganzen." "Archived copy". Archived from the original on 2011-06-10. Retrieved 2011-04-22.

real numbers are distinctly individual entities with essentially no relation to one another other than being real numbers.

The real number line is ordered. If we select a convenient starting point say 0, then moving right away from 0 all numbers get larger while moving left away from 0 all numbers get smaller. The real number line has no endpoints.

The set of real numbers need not be ordered in any way. Unlike the real number line, the set of real numbers has a first element although the identity of that element may be any real number whatsoever. The set of real numbers contains all possible real numbers as individual elements. It is a discrete entity.

The question then is, what exactly are the real numbers? Are they a continuum or a discrete set? Or might they be both and what they appear to be is dependent on the context in which they are used?

Function

Definition 2: A **function** is an equation for which any x, from the domain of the function, that can be plugged into the equation will yield exactly one y out of the equation.

Notation 1: Let y be written $f(x)$.
Notation 2: Let the set of x values be denoted by $\{x\}$ and called values of the domain.
Notation 3: Let the set of $f(x)$ values by denoted by $\{f(x)\}$ and be called values of the range.
Notation 4: Let the domain of a function be denoted by d.
Notation 5: Let the range of the function be denoted by r.

Equivalency 1: $d = \{x\}$.
Equivalency 2: $r = \{f(x)\}$

A single independent variable function is an object that takes an input value and via a rule creates an output value consisting of an ordered pair. If we have a function $f(x) = x^2$ with a domain of (1, 2, 3), evaluating the function over the domain will yield the set of ordered pairs $\{(1, 1), (2, 4), (3, 9)\}$. If we plot the resultant set on a graph we will have 3 points, one at (1, 1), one at (2, 4) and one at (3, 9). A finite set cast as the domain of a function of one independent variable will always produce a plot of points on a graph.

If we take the same function and define the domain as the closed interval [1, 3]; after evaluating the function over the entire domain we will have an infinite set of ordered pairs. The plot of the set on a graph will yield a smooth curve with no breaks between the points (1, 1) and (3, 9), going through (2, 4). The process of evaluating the function over its entire domain and plotting the resultant set of ordered pairs has transformed a discrete set into a continuum.

Definition 3: A **function index** is a count of the number of times the function has been evaluated for values of x, the index value is denoted by n.

Notation 6: Let the index of the function be denoted by i.

Notation 7: Let the set of n values be denoted by {n} and be called the values of the index.

Equivalency 3: i = {n}

Notation 8: Let the expression

$$\overset{n}{\underset{i=1}{\ddot{E}}}(f)\, d_i$$

be taken to mean "evaluate the function *f* over the domain d where i determines the number of domain elements to evaluate the function for, where i = (1, 2 …)." **Note:** For infinite domains n = ∞. Call the expression the evaluate function operator.

Below is an example of how the evaluate function operator can be used.

1. Let *f* be the function *f*(x) = C.

2. Substituting *f*(x) = C for *f* above we have

$$\overset{\infty}{\underset{i=1}{\ddot{E}}}\left(f(x) = C\right) d_i$$

3. Let r be an element of $\mathbb{R}$ so that

$$\mathbb{R}, \{r \in \mathbb{R} \mid r \text{ is real}\}$$

4. Let the domain d of *f*(x) = C be the set of real numbers r in $\mathbb{R}$ such that a < r < b

$$d = (\forall r) \{r \in \mathbb{R}\}, a < r < b$$

5. Let the 3-tuple of the index i and the input value r_i and the range value C_i be written

$$(i, r_i, C_i)$$

and be the output value of one iteration of

$$\overset{\infty}{\underset{i=1}{\ddot{E}}}\left(f(x) = C\right) d_i$$

6. Let the set of all 3-tuples thus created be called the function's indexed set, denoted by

$$S_i = \{(i_1, r_1, C_1), (i_2, r_2, C_2) \ldots (i_n, r_n, C_n) \ldots\}$$

7. Create an array from each 3-tuple in the function's indexed set, $\mathbf{S_i}$ as is demonstrated below.

$$
\begin{array}{c|cccccc}
\mathrm{i} & \mathrm{i}_1 & \mathrm{i}_2 & \mathrm{i}_3 & \ldots & \mathrm{i}_n & \ldots \\
 & \updownarrow & \updownarrow & \updownarrow & \ldots & \updownarrow & \ldots \\
\mathrm{x} & \mathrm{r}_1 & \mathrm{r}_2 & \mathrm{r}_3 & \ldots & \mathrm{r}_n & \ldots \\
f(\mathrm{x}) & \mathrm{C}_1 & \mathrm{C}_2 & \mathrm{C}_3 & \ldots & \mathrm{C}_n & \ldots
\end{array}
$$

Part 1 – Proof of the Continuum Hypothesis

Theorems 1 – 3 and Corollary 1

Given:

1. The set of natural numbers ℕ. Let n stand for an element of ℕ so that

$$\mathbb{N}, \{n \in \mathbb{N} \mid 1 \leq n\}$$

2. The subset ***S*** of the set of real numbers ℝ. Let r stand for an element of ***S*** so that

$$S, \{r \in S \mid a < r < b\}$$

where a and b are real numbers.

3. The function *f*(x) = C on the real interval (a, b)

$$f(x) = \text{C where C is constant and } a < x < b$$

4. The domain d of *f*(x) = C is a subset ***S*** of the real numbers between a and b

$$S, \{r \in S \mid a < r < b\}$$

5. The index i is the set of natural numbers

$$\mathbb{N}, \{n \in \mathbb{N} \mid 1 \leq n\}$$

Theorem 1: The domain d of *f*(x) = C is countable over the interval (a, b) where,

$$d = (\forall r) \{r \in \mathbb{R}\}, a < r < b$$

Proof by Construction: Invoking the evaluate function operator on *f*(x) = C we have

$$\overset{\infty}{\underset{i=1}{\mathrm{E}}} (f(x) = C)\, d_i \text{ over } (a, b) \text{ where } i = (1, 2 \ldots)$$

which forms the basis for the indexed set of the function

$$S_i = \{(i_1, r_1, C_1), (i_2, r_2, C_2) \ldots (i_n, r_n, C_n) \ldots\}$$

The 3-tuples that constitute $\mathbf{S_i}$ can be formatted as an array showing the values of i, x and *f*(x) respectively, depicted below.

i	1	2	3	…	n	…
	$\updownarrow$	$\updownarrow$	$\updownarrow$	…	$\updownarrow$	…
x	r_1	r_2	r_3	…	r_n	…
f(x)	C_1	C_2	C_3	…	C_n	…

There are an infinite number of elements in the domain and since the domain, contains all real numbers between a and b, there will not be any missing real numbers in the array. Conversely there will be an infinite number of index of values of i to match with elements of the domain. There is exactly one i for every r and exactly one r for every i in the array. This shows that there exists a one to one correspondence between i and d and this completes the proof. The correspondence can be expressed as the bijective function,

$$f: \mathrm{i} \rightarrow \mathrm{d}$$

Having shown that $f: \mathrm{i} \rightarrow \mathrm{d}$ exists we can assert that the cardinal numbers of the sets comprising i and d are equal and that the sets are the same size.

Theorem 2: The set ℕ of natural numbers

$$\mathbb{N}, \left\{ \mathrm{n} \in \mathbb{N} \mid 1 \leq \mathrm{n} \right\}$$

and the subset ***S*** of real numbers

$$\boldsymbol{S}, \left\{ \mathrm{r} \in \boldsymbol{S} \mid \mathrm{a} < \mathrm{r} < \mathrm{b} \right\}$$

have the same cardinality and as a result, are the same size.

Proof by Substitution: Theorem 1 proves that $f: \mathrm{i} \rightarrow \mathrm{d}$ exists and by definitions that i = ℕ and d = ***S***. Substituting ℕ for i and ***S*** for d in the bijective function

$$f: \mathrm{i} \rightarrow \mathrm{d}$$

we have

$$f: \mathbb{N} \rightarrow \boldsymbol{S}$$

which completes the proof.

Corollary 1: Cantor showed that the real numbers in the open interval (a, b) are equinumerous with ℝ. Therefore, since d of *f*(x) = C has been defined as (a, b), and d = ***S***, ℝ can be substituted for ***S*** in

$$f: \mathbb{N} \rightarrow \boldsymbol{S}$$

which yields the bijective function

$$f: \mathbb{N} \rightarrow \mathbb{R}$$

and demonstrates that the cardinal numbers of $\mathbb{N}$ and $\mathbb{R}$ are the same.

Theorem 3: The Continuum Hypothesis: there can be no infinite set with a cardinality strictly between that of the set of natural numbers $\mathbb{N}$ and the set of real numbers $\mathbb{R}$.

Proof: Corollary 1 demonstrates the bijective function,

$$f: \mathbb{N} \rightarrow \mathbb{R}$$

exists and that a one to one correspondence between $\mathbb{N}$ and $\mathbb{R}$ exists. Therefore $\mathbb{N}$ and $\mathbb{R}$ must have the same cardinal number. It is self-evident that it is impossible for any infinite set to have a cardinal number in between two infinite sets with the same cardinal number. Since $\mathbb{N}$ and $\mathbb{R}$ have the same cardinal number it is not possible for any infinite set to have a cardinal number between them. This completes the proof and confirms the truth of the hypothesis.

Part 1 Conclusion

Part 1 demonstrates that the set of natural numbers can be put into a one to one correspondence with the set of real numbers, $f: \mathbb{N} \rightarrow \mathbb{R}$. I used the function $f(x) = C$ to create an indexed array of the function's real number domain d, the constant range, C, and the index value of each iteration of the function's evaluation, i, for each member of the domain d_i.

The purpose of the exercise was to provide a constructive proof of Cantor's Continuum Hypothesis. Because the domain of $f(x) = C$ contains all real numbers, evaluating and indexing the function over the entire domain leads naturally to the bijective function $f: \mathbb{N} \rightarrow \mathbb{R}$.

In Part 2 I'll demonstrate how the set of natural numbers $\mathbb{N}$ can be put into a one to one correspondence with the power set of natural numbers, $P(\mathbb{N})$. From this I will derive the bijective function $f: \mathbb{N} \rightarrow P(\mathbb{N})$. Lastly, I'll propose a conjecture asserting that $f(x) = C$ can be employed to construct a one to one correspondence between the natural numbers and any infinite set that can be cast as the domain of the function.

Part 2 – Using $f(x) = C$ to demonstrate that $f: \mathbb{N} \rightarrow P\{\mathbb{N}\}$ exists

Theorems 4 & 5

Given:

1. The power set P of ℕ written P(ℕ). Let {p} stand for an element of P(ℕ) so that

$$P(\mathbb{N}), \{\{p\} \in P(\mathbb{N})\}$$

2. The domain d of $f(x) = C$ is the power set of the natural numbers

$$P(\mathbb{N}), \{\{p\} \in P(\mathbb{N})\}$$

that is d = **P**(ℕ).

Theorem 4: The domain d of the continuous function $f(x) = C$ is countable where d = P(ℕ), {{p} ∊ P(ℕ)}.

Proof by Construction: Invoking the evaluate function operator on $f(x) = C$ we have

$$\overset{\infty}{\underset{i=1}{\mathrm{E}}} \left(f(x) = C\right) d_i \mid i = (1, 2 \ldots)$$

which forms the basis for the indexed set of the function

$$S_i = \left\{\left(i_1, \{p\}_1, C_1\right), \left(i_2, \{p\}_2, C_2\right) \ldots \left(i_n, \{p\}_n, C_n\right) \ldots\right\}$$

The 3-tuples that constitute $\mathbf{S_i}$ can be formatted as an array showing the values of i, x and $f(x)$ respectively, depicted below.

$$\begin{array}{c|cccccc}
i & 1 & 2 & 3 & \ldots & n & \ldots \\
 & \updownarrow & \updownarrow & \updownarrow & \ldots & \updownarrow & \ldots \\
x & \{p\}_1 & \{p\}_2 & \{p\}_3 & \ldots & \{p\}_n & \ldots \\
f(x) & C_1 & C_2 & C_3 & \ldots & C_n & \ldots
\end{array}$$

The process of building the array of data points can go on indefinitely. There are an infinite number of elements in the domain and an infinite number of index of values of i to match with elements of the domain. There is exactly one i for every {p} and exactly one {p} for every i in the array. This shows that there exists a one to one correspondence between i and d and this completes the proof. The correspondence can be expressed as the bijective function,

$$f: i \rightarrow d$$

Having shown that $f: \mathrm{i} \rightarrow \mathrm{d}$ exists we can assert that the cardinal numbers of the sets comprising i and d are equal and that the sets are the same size.

Theorem 5: The set $\mathbb{N}$ of natural numbers

$$\mathbb{N}, \{n \in \mathbb{N} \mid 1 \leq n\}$$

and the power set $P(\mathbb{N})$ of $\mathbb{N}$

$$P(\mathbb{N}), \{\{p\} \in P(\mathbb{N})\}$$

have the same cardinality and, as a result. are the same size.

Proof by Substitution: Theorem 1 proves that $f: \mathrm{i} \rightarrow \mathrm{d}$ exists and by definitions that $\mathrm{i} = \mathbb{N}$ and $\mathrm{d} = P(\mathbb{N})$. Substituting $\mathbb{N}$ for i and $P(\mathbb{N})$ for d in the bijective function

$$f: \mathrm{i} \rightarrow \mathrm{d}$$

we have

$$f: \mathbb{N} \rightarrow P(\mathbb{N})$$

which completes the proof.

Part 2 Conclusion

Together Parts 1 & 2, having proved that

$$f: \mathbb{N} \rightarrow \mathbb{R} \quad \text{and}$$

$$f: \mathbb{N} \rightarrow P(\mathbb{N})$$

are true, calls into question the existence of a hierarchy of infinite sets as envisioned by Cantor. The construction of the arrays containing the bijections between i and d in each case is straightforward algebraic calculation.

Cantor's hierarchy of infinite sets has, at least in part, been collapsed. I will close with a conjecture I call the Domain Conjecture.

The Domain Conjecture

For an infinite set $\boldsymbol{S}$ of members $\{m_1, m_2, m_3 \ldots\}$, $f: \mathbb{N} \rightarrow \boldsymbol{S}$ is true if the domain d of $f(x) = C$ consists of the members of the set $\boldsymbol{S}$, that is $\mathrm{d} = \{m_1, m_2, m_3 \ldots\}$.

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Appendix A – Analysis of $f(x) = x^2$

In order to illustrate how the evaluate function operator is used to create bijections between sets, this example uses the function $f(x) = x^2$, $(0, 1)$.

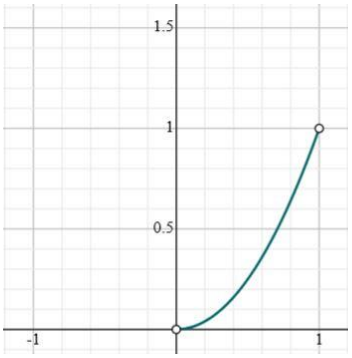
0. To evaluate a function $f(x)$, replace the function's variable (in this case x) with the given number or expression from the function's domain.

1. Let the expression

$$\overset{n}{\underset{i=1}{\mathrm{E}}}(f)\, d_i$$

be taken to mean "evaluate the function f over the domain d where i determines the number of domain elements to evaluate the function for, where i = (1, 2 …)." **Note:** For infinite domains n = ∞. Call the expression the evaluate function operator.

2. Let f be the function $f(x) = x^2$, $(0, 1)$. The function is continuous at every point in the domain which means every real number between 0 and 1 must be part of the function's domain. Otherwise the graph of the function would be a broken curve showing gaps where real numbers between 0 and 1 are missing.



3. Substituting $f(x) = x^2$ for f above we have

$$\overset{\infty}{\underset{i=1}{\mathrm{E}}}\left(f(x) = x^2\right) d_i \text{ over } (0, 1) \text{ where } i = (1, 2 \ldots)$$

4. Let r be an element of $\mathbb{R}$ so that

$$\mathbb{R}, \{r \in \mathbb{R} \mid r \text{ is real}\}$$

5. Let the domain d of $f(x) = x^2$ be the set of real numbers r in $\mathbb{R}$ such that $0 < r < 1$

$$d = (\forall r)\ \{r \in \mathbb{R}\}, 0 < r < 1$$

6. Let the 3-tuple of the index i and the input value r_i and the range value $f(x)_i$ be written

$$\left(i, r_i, f(x)_i\right)$$

and be the output value of one iteration of

$$\underset{i=1}{\overset{\infty}{\mathrm{E}}}\left(f(\mathrm{x}) = \mathrm{x}^2\right) \mathrm{d}_i \text{ over } (0, 1) \text{ where } i = (1, 2 \ldots)$$

7. Let the set of all 3-tuples thus created be called the function's indexed set, denoted by

$$\mathrm{S}_i = \left\{\left(i_1, r_1, f(\mathrm{x})_1\right), \left(i_2, r_2, f(\mathrm{x})_2\right) \ldots \left(i_n, r_n, f(\mathrm{x})_n\right) \ldots\right\}$$

8. Create an array from each 3-tuple in the function's indexed set, $\mathbf{S_i}$ as is demonstrated below.

$$\begin{array}{c|cccccc}
i & i_1 & i_2 & i_3 & \ldots & i_n & \ldots \\
 & \updownarrow & \updownarrow & \updownarrow & \ldots & \updownarrow & \ldots \\
x & r_1 & r_2 & r_3 & \ldots & r_n & \ldots \\
f(x) & f(x)_1 & f(x)_2 & f(x)_3 & \ldots & f(x)_n & \ldots
\end{array}$$

The one to one correspondence between the values in the i and x rows is readily apparent

9. Using random numbers from the function's domain we have:

d = {.25, .038, .355, … .67…}

$i_1 = 1 \quad i_2 = 2 \quad i_3 = 3 \quad i_n = n$

$x_1 = .25 \quad x_2 = .038 \quad x_3 = .355 \quad x_n = .67$

$f(x)_1 = .0625 \quad f(x)_2 = .001444 \quad f(x)_3 = .126025 \quad f(x)_n = .4489$

$\mathbf{S_i}$ = {(1, .25, .0625), (2, .038, .001444), (3, .355, .126025) … (n, .67, .4489) …}

and the array built from the set of 3-tuples:

$$\begin{array}{c|cccccc}
i & 1 & 2 & 3 & \ldots & n & \ldots \\
 & \updownarrow & \updownarrow & \updownarrow & \ldots & \updownarrow & \ldots \\
x & .25 & .038 & .355 & \ldots & .67 & \ldots \\
f(x) & .0625 & .001444 & .126025 & \ldots & .4489 & \ldots
\end{array}$$

All real numbers between 0 and 1 must be in the domain of the function by definition. That means each real number between 0 and 1 must be part of a 3-tuple in the set $\mathbf{S_i}$ and furthermore, each real number of the domain must occupy a place in the x row of the array. Each entry in the x row of the array has one and only one corresponding entry in the i row. Each i row entry has one and only one corresponding entry in the x row.

There is a constructed one to one correspondence between the function's indexed set i and the function's domain d which means

$$f: \mathrm{i} \rightarrow \mathrm{d}$$

exists and since i = $\mathbb{N}$, $\{\mathrm{n} \in \mathbb{N} \mid 1 \leq \mathrm{n}\}$ and $\mathrm{d} = (\forall \mathrm{r})\,\{\mathrm{r} \in \mathbb{R}\},\ 0 < \mathrm{r} < 1$ we can assert that by substitution.

$$f: \mathbb{N} \rightarrow \mathbb{R}.$$

I have shown the last steps in abbreviated form. The long form steps to arrive at $f: \mathbb{N} \rightarrow \mathbb{R}$ appear elsewhere in this paper.