The Function $f(x) = C$ and the Continuum Hypothesis

An Algebraic Proof of the CH

I. Introduction

The continuum hypothesis was advanced by Georg Cantor in 1878. It is a hypothesis based on the possibility that infinite sets come in different sizes. Cantor concluded that the set of real numbers is a larger infinity than the set of natural numbers. That is to say the set of real numbers has a cardinal number greater than the cardinal number of the set of natural numbers; and he surmised that no set exists with a cardinal number between the two. This is referred to as the Continuum Hypothesis. In 1900 David Hilbert listed the Continuum Hypothesis as the first of his 23 unsolved problems in mathematics.

Here is the Continuum Hypothesis stated as a theorem in Hilbert's lecture:

"Every system of infinitely many real numbers, *i. e.,* every assemblage of numbers (or points), is either equivalent to the assemblage of natural integers, 1, 2, 3,... or to the assemblage of all real numbers and therefore to the continuum, that is, to the points of a line; as regards equivalence there are, therefore, only two assemblages of numbers, the countable assemblage and the continuum."[1] [2]

Kurt Gödel in 1940 showed that the Continuum Hypothesis (CH) cannot be disproved from Zermelo-Fraenkel set theory, even if the axiom of choice is added (making ZFC). Paul Cohen later showed that the CH cannot be proven from the ZFC axioms. The HC cannot be proved or disproved within ZFC, it is independent with regard to ZFC.

Since it has been shown that the CH is independent of the axioms of set theory is there another method available that can be employed to settle the question of its truth or falsity? That is the question that this paper will answer.

II. Abstract

This paper examines whether or not an analysis of the behavior of the continuous function $f(x) = C$, where C is any constant, on the interval (a, b) where a and b are real numbers and a < b, will provide

[1] Mathematical Problems
Lecture delivered before the International Congress of Mathematicians at Paris in 1900
By Professor David Hilbert
[2] Dr. Maby Winton Newson translated this address into English with the author's permission for Bulletin of the American Mathematical Society 8 (1902), 437-479. A reprint of appears in Mathematical Developments Arising from Hilbert Problems, edited by Felix Brouder, American Mathematical Society, 1976.

a method of proving the truth or falsity of the CH. The argument will be presented in three theorems and one corollary.

The first theorem proves, by construction, the countability of the domain d of $f(x) = C$ on the interval (a, b) where a and b are real numbers. The second theorem proves, by substitution, that the set of natural numbers $\boldsymbol{N}$ has the same cardinality as the subset of real numbers $\boldsymbol{S}$ on the given interval. The corollary extends the proof of theorem 2 to show that $\boldsymbol{N}$ and $\boldsymbol{R}$ are of the same cardinality. The third theorem proves, by logical inference, that the CH is true.

III. Given

1. The set of natural numbers $\boldsymbol{N}$. Let n stand for an element of $\boldsymbol{N}$ so that

$$\boldsymbol{N}, \{n \in \boldsymbol{N} \mid 1 \leq n\} \qquad (1)$$

2. The subset $\boldsymbol{S}$ of the set of real numbers $\boldsymbol{R}$. Let r stand for an element of $\boldsymbol{S}$ so that

$$\boldsymbol{S}, \{r \in \boldsymbol{S} \mid a < r < b\} \qquad (2)$$

where a and b are real numbers.

3. The continuous function on the interval (a, b)

$$f(x) = C \qquad (3)$$

where C is any constant and $a < x < b$.

4. Let the expression

$$\mathbf{E}_{i = (1, \infty)} f(x)\, d_i \qquad (4)$$

be taken to mean "evaluate the function $f(x)$ over the domain d where i is an index of the number of iterations of $\mathbf{E}$ and i = (1, 2 …)." Call the expression the evaluate function operator.

IV. Definitions

1. The domain d of $f(x)$ is the set

$$\boldsymbol{S}, \{r \in \boldsymbol{S} \mid a < r < b\} \qquad (2)$$

that is $d = \boldsymbol{S}$.
Since the function $f(x) = C$ is continuous over the interval (a, b) the domain d of $f(x) = C$ must, by definition, contain all real numbers between a and b.

2. The index i is the set

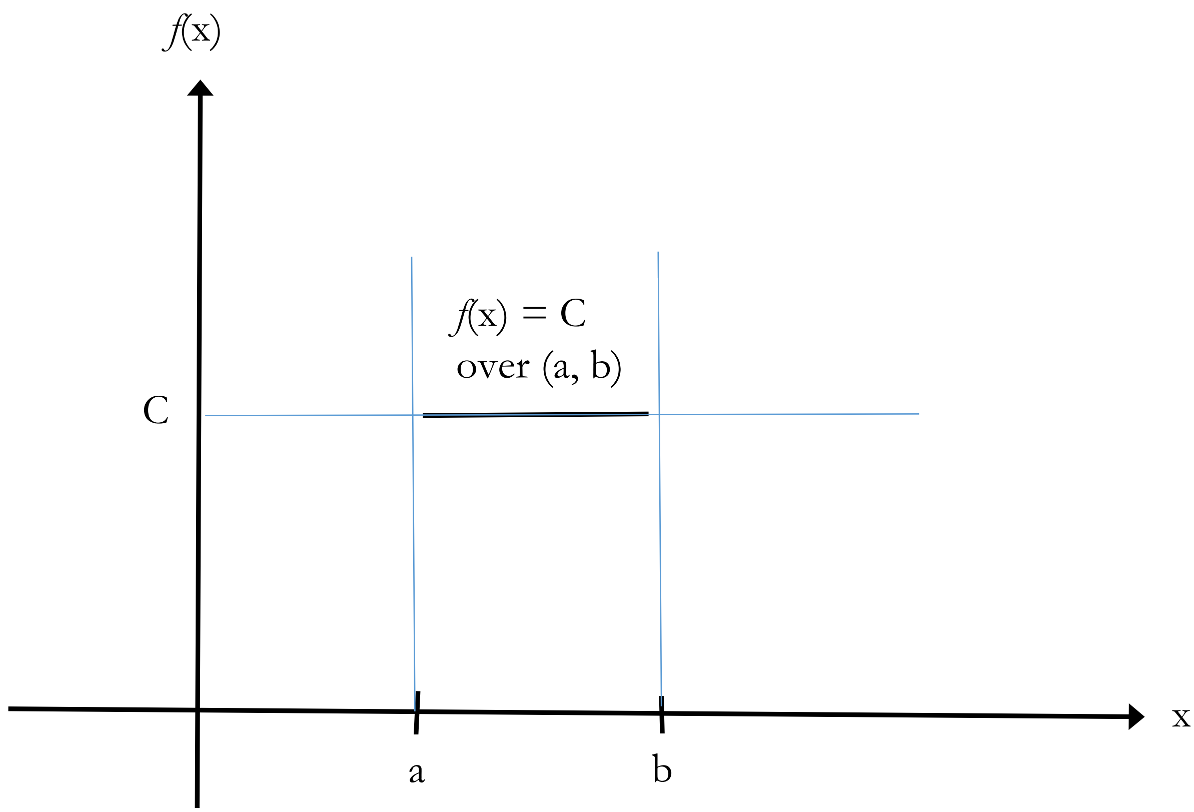
$$\boldsymbol{N}, \{n \in \boldsymbol{N} \mid 1 \leq n\} \qquad (1)$$

that is i = ***N***.

The index it built up from repeated iterations of $\mathbf{E}$ and provides a unique identifier for each evaluation of the function on a member of the domain.

V. Theorems 1 – 3 and Corollary 1

Theorem 1: The domain d of the continuous function $f(x) = C$ is countable over the interval (a, b).



Graph of $f(x) = C$ on the interval (a, b)

Proof by Construction: Invoking the evaluate function operator on $f(x) = C$ we have

$$\mathbf{E}_{i = (1, \infty)} f(x) = C\ d_i \text{ over } (a, b) \text{ where } i = (1, 2 \ldots) \qquad (5)$$

The data points arrived at as a result of evaluating the function over its domain can be formatted as an array showing the values of i, x and $f(x)$ respectively, depicted below.

i	1	2	3	…	n	…
	$\updownarrow$	$\updownarrow$	$\updownarrow$	…	$\updownarrow$	…
x	r_1	r_2	r_3	…	r_n	…
$f(x)$	C	C	C	…	C	…

The process of building the array of data points can go on indefinitely. There are an infinite number of elements in the domain and since the domain, by virtue of the fact that $f(x) = C$ is continuous over the interval (a, b) contains all real numbers between a and b, there will not be any missing real numbers in the array. Conversely no matter how long the process goes on there will be an infinite number of index of values of i to match with elements of the domain. There is exactly one i for every r and exactly one r for every i in the array. This shows that there exists a one to one correspondence between i and d and this completes the proof. The correspondence can be expressed as the bijective function,

$$f: \mathrm{i} \rightarrow \mathrm{d} \qquad (6)$$

Having shown that $f: \mathrm{i} \rightarrow \mathrm{d}$ exists we can assert that the cardinal numbers of the sets comprising i and d are equal and that the sets are the same size.

Theorem 2: The set of $\boldsymbol{N}$ of natural numbers

$$\boldsymbol{N}, \{n \in \boldsymbol{N} \mid 1 \leq n\} \qquad (1)$$

and the subset $\boldsymbol{S}$ of real numbers

$$\boldsymbol{S}, \{r \in \boldsymbol{S} \mid \mathrm{a} < r < \mathrm{b}\} \qquad (2)$$

have the same cardinality and, as a result. are the same size.

Proof by Substitution: Theorem 1 proves that $f: \mathrm{i} \rightarrow \mathrm{d}$ exits and definitions 1 and 2 establish that i = $\boldsymbol{N}$ and d = $\boldsymbol{S}$. Substituting $\boldsymbol{N}$ for i and $\boldsymbol{S}$ for d in the bijective function

$$f: \mathrm{i} \rightarrow \mathrm{d} \qquad (6)$$

we have

$$f: \boldsymbol{N} \rightarrow \boldsymbol{S} \qquad (7)$$

which completes the proof.

Corollary 1: Cantor showed that the open interval (a, b) is equinumerous with $\boldsymbol{R}$. Therefore, since d of f(x) = C has been defined as (a, b), and d = $\boldsymbol{S}$, $\boldsymbol{R}$ can be substituted for $\boldsymbol{S}$ in

$$f: \boldsymbol{N} \rightarrow \boldsymbol{S} \qquad (7)$$

which yields the bijective function

$$f: \boldsymbol{N} \rightarrow \boldsymbol{R} \qquad (8)$$

and demonstrates that the cardinal numbers of $\boldsymbol{N}$ and $\boldsymbol{R}$ are the same.

Theorem 3: The Continuum Hypothesis: there can be no infinite set with a cardinality strictly between that of the set of natural numbers $\boldsymbol{N}$ and the set of real numbers $\boldsymbol{R}$.

Proof: Corollary 1 demonstrates the bijective function,

$$f: \boldsymbol{N} \rightarrow \boldsymbol{R} \qquad (8)$$

exists and that a one to one correspondence between $\boldsymbol{N}$ and $\boldsymbol{R}$ exists. Therefore $\boldsymbol{N}$ and $\boldsymbol{R}$ must have the same cardinal number. It is self-evident that it is impossible for any infinite set to have a cardinal number in between two infinite sets with the same cardinal number. Since $\boldsymbol{N}$ and $\boldsymbol{R}$ have the same cardinal number it is not possible for any infinite set to have a carinal number between them. This completes the proof and confirms the truth of the hypothesis.

References

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