On the representation theorem for the stochastic differential equations with fractional Brownian motion and jump by Probability measures transform

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Abstract

In this paper we prove Girsanov theorem for fractional Brownian motion and jump measures and consider representation form for the stochastic differential equations in transfer Probability space.

Keywords: Girsanov theorem, probability measures transform, fractional Brownian motion, jump measure

1 Introduction

Girsanov theorem is foundation of probability measures transform. After the introduction of Girsanov theorem by Brownian motion, one with jump measures is considered ([1], [3]) and Girsanovs transform for Backward Stochastic differential equation is also proved([2]).

Girsanov theorem by fractional Brownian motion is considered in [4].

In this paper we prove Girsanov theorem for fractional Brownian motion and jump measures and consider representation form for the stochastic differential equations in transfer Probability space.

If $(\Omega_H, \mathscr{F}_H, \mathbf{P}_H)$ is the probability space driven by fractional Brownian motion and $(\Omega_\nu, \mathscr{F}_\nu, \mathbf{P}_\nu)$ is one derived by pure jump Levy processes, one to consider is $(\Omega, \mathscr{F}, \mathbf{P}) = (\Omega_H \times \Omega_\nu, \mathscr{F}_H \otimes \mathscr{F}_\nu, \mathbf{P}_H \otimes \mathbf{P}_\nu)$.[6]

On this space fractional Brownian motion, skorohod integral by Poisson random measures, definition and property of Malliavin derivative and Ito formula, etc are on the basis of [5].

2 Probability measures transform

The stochastic differential equations to consider are as follows.

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB_H(t) + \int_{|z|>0} C(t, X(t), z)\tilde{\mu}(dt, dz) \qquad (1)$$

$$X(0) = X_0$$

Here, $\{B_H(t)\}_{t\in[0,T]}$ is fractional Brownian motion with parameter H $(0 < H < 1)$, that is,

$$\mathbf{E}(B_H(t)) = 0$$

$$C_H(s,t) = \mathbf{E}(B_H(s)B_H(t)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

Also, $\{\mu(dt, dz)\}$ is Poisson integral random measure, its intensity is

$\nu(dt, dz)$ and $\widetilde{\mu}(dt, dz) = \mu(dt, dz) - \nu(dt, dz)$ denotes the compensated version of $\mu(dt, dz)$.

Coefficients $a(t, x), b(t, x), c(t, x, z)$ are measurable and bounded with respect to every variable.

$\{\mathscr{F}_t\}_{t \in [0, T]}$ is given by

$$\mathscr{F}_t := \sigma\{B_H(s), \mu([0, s], \Gamma), 0 \le s \le t, \Gamma \in \mathbf{B}^1\}$$

Then, function space $\mathbf{L}_H^2([0, T])$ and $\mathbf{F}_\nu$ are defined as follows with kernel

function $\phi_H(s, t)$ $(\phi_H(s, t) = H(2H-1)|t-s|^{2H-2})$ and Levy measure $\nu(dz)$

$$\mathbf{L}_H^2([0, T]) = \{f \; ; \; \|f\|_\phi^2 = \int_0^T \int_0^T f(s) f(t) \phi_H(s, t) ds dt < \infty\}$$

$$\langle f, g \rangle_\phi = \int_0^T \int_0^T f(s) g(t) \phi_H(s, t) ds dt \,, \qquad f, g \in \mathbf{L}_H^2([0, T])$$

$$\|f\|_{\phi, t}^2 = \int_0^t \int_0^t f(s_1) f(s_2) \phi_H(s_1, s_2) ds_1 ds_2 \,, \quad f(s) \in \mathbf{L}_H^2([0, T])$$

$$\mathbf{L}_\nu = \{g \; ; \; \int_0^T \int_{|z|>0} |g(s, z)|^2 \nu(dz) ds < \infty,$$

Consider the following linear stochastic differential equation (Dolyan equation) by above fractional Brownian motion and integral random measure.

$$M(t) = 1 + \int_0^t \theta(s) M(s) dB_H(s) + \int_0^t \int_{|z|>0} \lambda(s, z) M(s) \widetilde{\mu}(ds, dz) \tag{2}$$

Here, $\theta(t) \in \mathbf{L}_H^2([0, T]), \lambda(t, z) \in \mathbf{L}_\nu$

Theorem 1. The solution for equation (2) is

$$\begin{aligned} M(t) = \exp\{ & \int_0^t \theta(s) dB_H(s) - \frac{1}{2}\|\theta\|_{\phi, t}^2 + \\ & + \int_0^t \int_{|z|>0} \ln(1 + \lambda(s, z)) \widetilde{\mu}(ds, dz) - \int_0^t \int_{|z|>0} [\lambda(s, z) - \ln(1 + \lambda(s, z))] \nu(dz) ds \} \end{aligned} \tag{3}$$

Proof. Let $Y(t)$ be as follows.

$$Y(t) = \int_0^t \theta(s)dB_H(s) - \frac{1}{2}\|\theta\|_{\phi,t}^2 +$$

$$+\int_0^t \int_{|z|>0} \ln(1+\lambda(s,z))\tilde{\mu}(ds,dz) - \int_0^t \int_{|z|>0} [\lambda(s,z) - \ln(1+\lambda(s,z))]\nu(dz)ds$$

Then, apply stochastic integral transform formula on function $F(y) = e^y$.

$$M(t) = F(Y(t)) =$$

$$= F(Y(0)) + \int_0^t F(Y(s))\theta(s)dB_H(s) - \frac{1}{2}\int_0^t\int_0^t F(Y(s_1))\theta(s_1)\theta(s_2)\phi_H(s_1,s_2)ds_1ds_2$$

$$+\frac{1}{2}\int_0^t\int_0^t F(Y(s_1))\theta(s_1)\theta(s_2)\phi_H(s_1,s_2)ds_1ds_2$$

$$+\int_0^t \int_{|z|>0} [F(Y(s)+\ln(1+\lambda(s,z))) - F(Y(s)) - \ln(1+\lambda(s,z))F(Y(s))]\nu(dz)ds$$

$$+\int_0^t \int_{|z|>0} [F(Y(s)+\ln(1+\lambda(s,z))) - F(Y(s))]\tilde{\mu}(ds,dz)$$

$$-\int_0^t \int_{|z|>0} F(Y(s))[\lambda(s,z) - \ln(1+\lambda(s,z))]\nu(dz)ds$$

$$= 1 + \int_0^t \theta(s)M(s)dB_H(s) + \int_0^t \int_{|z|>0} \lambda(s,z)M(s)\tilde{\mu}(ds,dz)$$

Note. In case that the integrand is suitable process, Skorohod integral by fractional Brownian motion is equal to Ito integral ([5]) and if and if only random process $\{F(t,\omega)\}$ is $(\mathscr{F}_t)$ - process, its chaos expansion $F(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot\,;t))$ is presented by

$$F(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot\,;t)\chi_{([0,t])^{\otimes n}}(\cdot))$$

Here, $\chi_{[0,t]^{\otimes n}}(\cdot)$ is point function.

And for random variable

$$G(\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot)) \in \mathbf{L}^2(\mathbf{P})$$

,semi conditional expectation is defined by

$$\tilde{\mathbf{E}}[G(\omega)|\mathscr{F}_s] = \sum_{n=0}^{\infty} I_n(f_n(\cdot)\chi_{([0,s])^{\otimes n}}(\cdot))$$

Lemma 1. The solution $\{M(t)\}_{t\in[0,T]}$ for equation (2) is $\mathscr{F}_t$ -semi martingale.

Proof. For any s, t $(0 \le s < t \le T)$, by equation (2),

$$\widetilde{\mathbf{E}}[M(t)|\mathscr{F}_s] =$$

$$= 1 + \widetilde{\mathbf{E}}[\int_0^t \theta(s_1)M(s_1)dB_H(s_1)|\mathscr{F}_s] + \widetilde{\mathbf{E}}[\int_0^t \int_{|z|>0} \lambda(s_1, z)M(s_1)\widetilde{\mu}(ds_1, dz)|\mathscr{F}_s]$$

$$= 1 + \int_0^t \theta(s_1)M(s_1)\chi_{(0\le s_1\le s)}(s_1)dB_H(s_1) + \int_0^t \int_{|z|>0} \lambda(s_1, z)M(s_1)\chi_{(0\le s_1\le s)}\widetilde{\mu}(ds_1, dz)$$

$$= 1 + \int_0^s \theta(s_1)M(s_1)dB_H(s_1) + \int_0^s \int_{|z|>0} \lambda(s_1, z)M(s_1)\widetilde{\mu}(ds_1, dz)$$

$$= M(s)$$

□

For any $f(t) \in \mathbf{L}^2{}_H([0, T]),\ g(t, z) \in \mathbf{F}_\nu$,

$$\varepsilon_0(t; f, g) := \exp\{\int_0^t f(s)dB_H(s) - \frac{1}{2}\|f\|^2_{\phi, t} +$$
$$+ \int_0^t \int_{|z|>0} \ln(1 + g(s, z))\widetilde{\mu}(ds, dz) - \int_0^t \int_{|z|>0} [g(s, z) - \ln(1 + g(s, z))]\nu(dz)ds\}$$

$$\varepsilon_1(t; f) := \exp\{\int_0^t f(s)dB_H(s) - \frac{1}{2}\|f\|^2_{\phi, t}\}$$

$$\varepsilon_2(t; g) = \exp\{\int_0^t \int_{|z|>0} \ln(1 + g(s, z))\widetilde{\mu}(ds, dz) - \int_0^t \int_{|z|>0} [g(s, z) - \ln(1 + g(s, z))]\nu(dz)ds\}$$

We can easily show that

$$\varepsilon_0(t; \theta, \lambda) = \varepsilon_1(t; \theta)\ \varepsilon_2(t; \lambda) = M(t)$$

Lemma 2. $\{\varepsilon_i(t;\ \cdot)\}_{t\in[0, T],\ i=0,1,2}$ is $\mathscr{F}_t$-exponential martingale.

$$\mathbf{E}[\varepsilon_i(t;\ \cdot)] = 1,\quad i = 0, 1, 2$$

The proof is certain by Theorem 1 and Lemma 1.

Let the new probability measure $\mathbf{P}^*$ define that the Radon-Nikodym derivative satisfies

$$\frac{d\mathbf{P}^*}{d\mathbf{P}}\Big|_{\mathscr{F}_t} = M(t),\ t \in [0, T]$$

3 Representation theorem

We can obtain the following theorems.

Theorem 2. For any $f(t) \in \mathbf{L}^2{}_H([0, T])$,

$\{B_H^*(t)\}_{t\in[0,T]}$ such that

$$\int_0^T f(s)dB_H^*(s) = \int_0^T f(s)dB_H(s) - \frac{1}{2}\langle f, \theta\rangle_\phi$$

is fractional Brownian motion with parameter H by the new probability measure $\mathbf{P}^*$.

Proof. It is sufficient that the characteristic function of

$$\int_0^T f(s)dB_H^*(s)$$

is semi expectation by the new probability measure $\mathbf{P}^*$ and

$$\widetilde{\mathbf{E}}^* \exp\{iu\int_0^T f(s)dB_H^*(s)\} = \exp\{-\frac{u^2}{2}\|f\|_\phi^2\}.$$

$$\widetilde{\mathbf{E}}^* \exp\{iu\int_0^T f(s)dB_H^*(s)\} =$$

$$= \widetilde{\mathbf{E}}[\exp\{iu\int_0^T f(s)dB_H(s) - iu\frac{1}{2}\langle f, \theta\rangle_\phi\}\varepsilon_0(T; \theta, \lambda)]$$

$$= \widetilde{\mathbf{E}}[\exp\{\int_0^T (iuf(s) + \theta(s))dB_H(s) - iu\langle f, \theta\rangle_\phi - \frac{1}{2}\|\theta\|_\phi^2\}\varepsilon_2(T;\ \lambda)]$$

$$= \widetilde{\mathbf{E}}[\varepsilon_1(T; (iuf(s) + \theta(s)))\varepsilon_2(T;\ \lambda)]\exp\{-\frac{u^2}{2}\|f\|_\phi^2\}$$

$$= \exp\{-\frac{u^2}{2}\|f\|_\phi^2\}$$

Theorem 3. Random measure $\widetilde{\mu}^*(dt, dz)$ defined by

$$\widetilde{\mu}^*(dt, dz) = \mu(dt, dz) - \nu^*(dt, dz)$$

is $\mathcal{F}_t$-martingale by the new probability measure $\mathbf{P}^*$ and centralized Poisson integral random measure.

Here,

$$\nu^*(dt, dz) = (1 + \lambda(s, z))\nu(dt, dz)$$

Proof. We can similarly prove as in Theorem 2. That is ,

$$\widetilde{\mathbf{E}}^* \exp\{iu\int_0^T\int_{|z|>0} g(s,z)\widetilde{\mu}^*(ds,dz)\} =$$

$$= \widetilde{\mathbf{E}}[\exp\{iu\int_0^T\int_{|z|>0} g(s,z)\widetilde{\mu}(ds,dz) - iu\int_0^T\int_{|z|>0} g(s,z)\lambda(s,z)\nu(dz)ds\}\varepsilon_0(T;\theta,\lambda)]$$

$$= \widetilde{\mathbf{E}}[\exp\{\int_0^T\int_{|z|>0} (iug + \ln(1+\lambda))\widetilde{\mu}(ds,dz) -$$

$$-\int_0^T\int_{|z|>0} (g\lambda + \lambda - \ln(1+\lambda))\nu(dz)ds\}\varepsilon_1(T;\theta)]$$

$$= \widetilde{\mathbf{E}}[\varepsilon_1(T;\theta)\varepsilon_2(T;(iug+\ln(1+\lambda)))\exp\{\int_0^T\int_{|z|>0} (e^{iug+\ln(1+\lambda)} - 1 - iug(1+\lambda) - \lambda)\nu(dz)ds\}]$$

$$= \exp\{\int_0^T\int_{|z|>0} (e^{iug} - 1 - iug)\nu^*(dz)ds\}$$

Also, with respect to $\mathbf{P}^*$

$$\widetilde{\mathbf{E}}^* \varepsilon_2^*(t, iug) =$$

$$= \widetilde{\mathbf{E}}^*[\exp\{iu\int_0^t\int_{|z|>0} g(s,z)\widetilde{\mu}^*(ds,dz) - \int_0^t\int_{|z|>0} (e^{iug} - 1 - iug)\nu^*(dz)ds\}]$$

$$= 1$$

Consequently, the theorem is proved.

Theorem 4. The stochastic differential equation (1) in transfer Probability space is presented as follows.

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t\int_0^t b(s_1, X(s_1))\theta(s_2)\phi_H(s_1,s_2)ds_1 ds_2$$

$$+ \int_0^t\int_{|z|>0} c(s, X(s), z)\lambda(s,z)\nu(ds,dz)$$

$$+ \int_0^t b(s, X(s))dB_H^*(s) + \int_0^t\int_{|z|>0} c(s, X(s), z)\widetilde{\mu}^*(ds,dz)$$

The proof of above theorem follows from Theorem 2, 3.

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