EUCLID'S GEOMETRY IS JUST IN OUR MIND, RATHER THAN DESCRIBING THE REAL WORLD

Arturo Tozzi (corresponding Author),
Center for Nonlinear Science, University of North Texas
1155 Union Circle, #311427 Denton, TX 76203-5017 USA
tozziarturo@libero.it

James F. Peters
Department of Electrical and Computer Engineering, University of Manitoba
75A Chancellor's Circle Winnipeg, MB R3T 5V6 CANADA and
Department of Mathematics, Adıyaman University, 02040 Adıyaman, Turkey
James.Peters3@umanitoba.ca

The first definition (prior to the well-known five postulates) of Euclid describes the point as "that of which there is no part". Here we show how the Euclidean account of manifolds is untenable in our physical realm and that the concepts of points, lines, surfaces, volumes need to be revisited, in order to allow us to be able to describe the real world. Here we show that the basic object in a physical context is a traversal of spacetime via tiny subregions of spatial regions, rather than the Euclidean point. We also elucidate the psychological issues that lead our mind to think to points and lines as really existing in our surrounding environment.

Keywords: points; lines; brain; continuum; Aristotle

Our physical and biological world can be described in terms of mathematical definitions, postulates, relations, theorems, formulas, equations and algorithms. Indeed, despite the claims that math is a collection of abstractions invented by our minds, it is astonishingly successful in order to explain and quantify a lot of issues of our world (Wigner, 1960). Indeed, math elucidates many physical phenomena, such as the interaction of photons and matter, the interaction of light with prisms, the apparent influence of speed on moving masses, and so on. Novel branches of math - such as algebraic topology, group theory and functional analysis - leading to far-flung approaches that include nonlinear dynamics in physics and chemistry - start to unveil also the mysteries of otherwise previously unclassifiable issues, such as weather, thrust, interactions and structure of neuro-systems (albeit minds) and the collective behaviors of societies.

A huge portion of our mathematics of the physical world is based on the amazingly simple Euclidean geometry. Indeed, starting from very straightforward assumptions and theorems such as those found in Euclid's geometry, it is feasible to build also non-Euclidean geometries and complex manifolds able to explain issues such as those in quantum mechanics. One of the main components of Euclidean geometry is the point, that stands for the most fundamental object. The first definition of a point (prior to Euclid) is given by the Pythagoreans: a point is a monad having position. Euclid begins his geometry with the definition of a point [that of which there is no part] (Def. 1, Euclid, 300 BCE) and the extremities of a line are points (Def. 2). Euclid's Def. 1 is interpreted by T.L. Heath to mean that a point is that which is indivisible in parts. Therefore, we are confronted with a primitive notion defined only by axioms that it must satisfy, i.e., the point upon which the whole apparatus is built, meaning that geometry cannot be described in terms of previously defined real objects or structures. Here we ask whether the zero-substance point holds true in our physical world and extend our analysis also to other Euclidean objects, such as lines, surfaces, volumes and so on.

THE PHYSICAL UNFEASIBILITY OF EUCLIDEAN POINTS AND SURFACES

Despite the huge influence of the Euclidean geometry in current scientific approaches, its internal framework is fatally flawed from the very beginning, if we want to use it to describe physical (and also biological) phenomena. The geometric points described by Euclid, originally defined as "that which have no part", are not equipped with any diameter, area, volume, or other physical features such as height, density, weight. Many constructs of Euclidean geometry are represented by a set of points, that stand for infinite collections of objects that conform to certain axioms. In particular, the Euclidean point is posited as something featureless. However, it is widely recognized that our world is quantized, and that a minimum length does exist, *i.e*., the Planck size. The quantization of matter leads to one of the hallmarks of spacetime, namely, Planck's constant. This allowed scientists to formulate frameworks, such as string theories (Duff 1996; de Haro et al., 2013) and quantum gravity (Rovelli and Smolin, 1988; Muxin 2011), in which our reality is quantized, and therefore discontinuous. Thus, a real manifold must be made of separable units, and is not a continuum with no holes. However, such an approach *does not hold true* for the Euclidean manifolds. In the sequel, we will show how and why an Euclidean geometric structure (such as points, lines, surfaces, volumes) cannot display a real counterpart, and are therefore useless in the description of our physical world.

Paraphrasing Aristotle's account of continuity and indivisibility (ARISTOTLE PHYSICS Book VI, Part 1), things are said to be "continuous" if their extremities are one, "in contact" if their extremities are joined together, and "in succession" if there is nothing of their own kind intermediate between them. Nothing that is termed "continuous" can be composed of "indivisibles": to make an example, an Euclidean line cannot be composed of points, the line being continuous and the point indivisible. Indeed, the extremities of two Euclidean points can neither be one (since of an indivisible there can be no extremity as distinct from some other part), nor together (since that which has no parts can have no extremity, the extremity and the thing of which the extremity is distinct). Moreover, if that which is continuous is composed of Euclidean points, these points must be either continuous, or in contact with one another. Therefore, indivisibles cannot be continuous: one thing can be in contact with another only if whole is in contact with whole or part with part or part with whole. But since indivisibles have no parts, they must be in contact with one another as whole with whole. And if they are in contact with one another as whole with whole, they will not be continuous. For Euclid, that which is continuous has distinct parts, and these parts into which it is indivisible need to be spatially separate. In particular, concerning the line, an Euclidean point cannot be in succession to another in such a way that length can be composed of points. Indeed, things are in succession, provided there is nothing of their own kind intermediate between them, whereas that which is intermediate between points is always a line, or a region of a surface. Again, if the Euclidean length (i.e., the line) could be composed of indivisibles, it could be divided into indivisibles, since each is divisible into the parts of which it is composed. But, as stated above, no continuous thing is divisible into things without parts. Nor can there be anything of any other kind intermediate between the parts: if there could be any such thing, it must be either indivisible or divisible, and if it is divisible, it must be divisible either into indivisibles or into divisibles that are infinitely divisible, in which case it is continuous. In sum, because the Euclidean line is either continuous or depicted as an infinite set of zero-mass points, it does not describe real lines, that are made of a set of contiguous multi-mass regions.

The problem of a physical continuum becomes more murky, if we consider what means mathematically for a mapping between spatial regions to be continuous. Let $f: X \to Y$ be a mapping on X into Y, so that $f(x) \in Y$ for $x \in X$. In the context of proximity spaces, a point *x* is near a region or set of points *A*, provided $f(x)$ is near $f(A)$ (Naimpally, 1970). In other words, a function proximally continuous, provided the function preserves pairs of sets so that whenever $\{x\}, A$ are near (in the same proximity), so too $f(x), f(A)$ are near (Peters, 2014). Put another way, a function f is continuous, provided the image $f(x)$ of every point of adherence $x \in X$ is a point of adherence in the set $f(A)$ (Smirnov, 1952). Again from a mathematics perspective, it is an axiom in proximity space theory that the closeness of a point to a set implies that the point is a part of the set. By contrast, a physically continuous mapping from a computational proximity perspective (Peters, 2016) imply that the closeness of a physical region such as an atom x to another physical region such as a collection of atoms A means that x is the in the neighbourhood of A, not that x is in A. In other words, as anticipated by old philosophers too (Autrecourt, 1340), closeness of points and sets in mathematics is quite different from the closeness of points and sets in the physical world. Instead of the conventional closeness of points and sets, the axiomatic foundations of the descriptive closeness of points and sets has been recently introduced (DiConcilio, et al., 2018). In the context of descriptive proximity, the closeness of points and sets is viewed in terms of overlapping descriptions of the points and the sets. In other words, a feature vector that describes a point and the feature vector that describes a set overlap without requiring that the descriptive closeness of a point x and set A implies the inclusion of the point *x* in the set *A*.

As for the line, the same applies also for other Euclidean structures, such as two-dimensional surfaces and three-dimensional volumes. Indeed, every continuous Euclidean manifold (whatsoever its dimensions may be) is divisible into divisibles that are infinitely divisible: if it were divisible into indivisibles, we should have an indivisible in contact with an indivisible, since the extremities of closed manifolds that are continuous with one another are one and are in contact. This means that a collection of n-dimensional Euclidean structures cannot give rise, when joined together, to a n+1 dimensional manifold. Therefore, the Euclidean points, lines, surfaces, volumes cannot exist in the real world. In particular, taking into account that physical space is quantized and that the minimum possible length is the Planck one, a real "point" cannot exhibit zero mass. Furthermore, at Planck length, another problem arises for an hypothetical basic "point": many theories predict that the minimum separable structure, *i.e*., the string, lies in 10-11 dimensions, rather than being *a*dimensional. Therefore, in the real world, also the simplest structure, *i.e*., the real point, must be a spatial region that displays nonzero mass in spacetime, otherwise it cannot exist. Furthermore, as stated above, it is also difficult to talk about real lines, surfaces, volumes, given the indefiniteness and vagueness of their Euclidean definition and structure.

FURTHER POSSIBILITIES TO COPE WITH THE PHYSICAL REALITY

In the previous paragraph, we showed that the concept of Euclidean points and n-dimensional manifolds do not hold true for the description of our real world, because every physical structure cannot be built starting from nonzero mass ones. An Euclidean line does not exist in the real world, because a real line must be composed of spatial subregions, and this is not compatible with the Euclidean tenets. It might be objected that a line could be made of points made of bosons, that, satisfying the Bose-Einstein condensate theory, can be described as superimposed. However, a real line in our world must be composed of matter, i.e., fermions that are subject to the principle of locality: therefore, a point made of fermions cannot be superimposed to another one, because they must occupy different locations in space. It might also be objected that a line could be made of different points that are not superimposed, rather they are separated by a small void, but it is not the case of the Euclidean line's description.
Is it feasible to provide definitions of points and manifolds other than the Euclidean ones, in order to overtake its fatal limitations in the description of physical matters?
Despite there exist several regions-based accounts of space/time, due to Roeper, Menger, Tarski and others (Shapiro and Hellman, 2017), the most of them, contrary to our approach, follow the Aristotelian theme that continua are not composed of points. A more recent view of a point by M. Simon (1901) moves closer to physical points. According to such an approach, a point is the limit of localization. More in keeping with the contemporary view of a point as a property of duration, a theory of extensive connection and atomicity of actual entities has been built (Whitehead, 1929). The notion of extensive connection brings us full circle to an expansion of cellular topology and physical proximity space theory, in which a point stands for the minimal part of overlapping regions of physical space, such as overlapping perimeters of vortex atoms that live together in motion continua with limited duration. In other words, a point is a finite region of space with non-zero area, while a line is drawn from one region to another (Peters, 2016).
One region A includes another region B, whenever every region connected to B is also connected to A (Whitehead, 1929). In other words, region A is connected to region B, provided there is a path from any subregion of B to a subregion of A. The extremity of a connecting path between subregions is itself a subregion with measurable area, albeit a tiny subregion that has the appearance of a point in Euclides' surveyer-like geometry.
We may state that a region of space is the maximum distance between subregions (intervals) of its bounding edge. This definition allows the description of a point as a single abstract spatial vertex: in other words, our physical reality can be described in terms of multiple surface spacetime vortices, in which each vortex is a funnel reaching down to a physical limit, called a vertex. Examples are provided in **Figure**. This approach permits us to remove the unwanted Euclidean geometry from the quantitative assessment of our world.

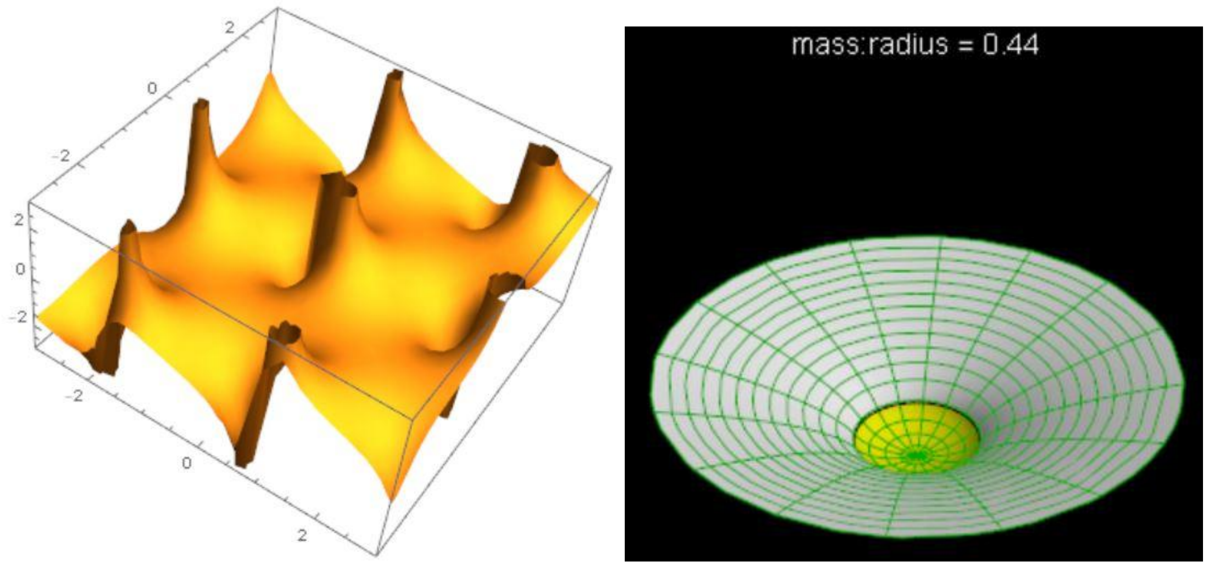

Figure. Examples of sample vortices. **Left**: a Weierstrass-Zeta mathematical model for multiple surface spacetime vortices, in which each vortex is a funnel reaching down to a physical limit called a vertex. **Right**: black hole vortex (initial mass radius 0.44 km) formation. It results from steadily increasing strong gravitational force that pulls neighboring chunks of matter into its funnel winding down to the squish level. Therefore, chunks of matter are squeezed to the diameter of a vertex.

CONCLUSIONS

Here we asked whether one of the tenets of Euclidean geometry, i.e., the point, holds true in our physical world. Given our negative answer, we concluded that the Euclidean manifolds cannot properly describe the real world. Our claim threats the very foundations of the Euclidean geometry, despite its current utmost importance in the description of the reality surrounding us. Indeed, the Euclidean geometry, after many centuries from its first formulation, still stands for the main tenet of the ensuing mathematical and physical developments. Starting from its postulates and theorems, it has been feasible to consider exception conditions, such as curved space, that are more in keeping with the relativistic spacetime. Furthermore, Euclidean geometry allowed to work our way toward non-Euclidean geometries (Johnstone 1983), R. Descartes's vortex geometry, sweeping comprehensive cellular complexes in Whiteheadean closure finite weak topology, V.G. Boltyanskii and V.A. Efremovich intuitive combinatorial topology, Edelsbrunner-Harer computational topology, Cech-Efremovich-Lodato-Naimpally-DiConcilio proximity spaces (Di Concilio A. 2013), K. Borsuk shape theory (Borsuk 1958-1959, Borsuk 1969), P.G. Tait's knots (1877). Such improvements led to J.S. Birman's braids, links, knots and mapping class groups that allow amazingly simple solutions to problems in quantum mechanics and to W.T. Kelvin's knotted and knitted vortex atoms (Kelvin, 1867). This also led to a deep understanding of Helmholz's discovery (Helmholz, 1867) of simple vortex rings and the vexing problem of vortex motion in perfect liquids, as well as the elucidation of thermodynamic properties of gases.

Despite the above-mentioned usefulness of Euclidean geometry, our results suggest that it's time to change our approach to physical issues, because Euclid describes totally abstract issues. It is also feasible to deliver a psychological explanation for the Euclid's successful mathematical description of the world. Indeed, taking into account recent neuroscientific findings, we are allowed to hypothesize why our mind seems to recognize in the surrounding environment Euclidean points, lines, surfaces, volumes, despite the fact that they cannot, according to our account, exist in reality. Over the last few years, studies of receptive field properties in several mammalian visual brain structures, such as the lateral geniculate nucleus and the primary sensitive cortex, have suggested the existence of cell classes with unique functional response properties (Van Hooser et al., 2005; Viswanathan and Nieder, 2017). To make an example, the cells of the cortical visual area V1 display basic receptive field properties, such as orientation selectivity, direction selectivity, bar length and end-stopping (Duffy and Hubel, 2007). Further, similar to the primary visual cortex, orientation sensitivity was closely reflected by the receptive fields properties also in extracortical structures, such as the lateral geniculate nucleus (Li et al., 2018). Therefore, many experimental suggestions dating back just a few years suggest that our brain is equipped with neurons, located in cortical and extra-cortical areas, that selectively fire when our sight perceives angles, surfaces, lines, contours. And such angles, surfaces, lines, contours do not exist in the real world, provided that the borders of the objects cannot be as well sharply delimited as suggested by Euclidean geometry.

Our perception is perhaps an evolutive mechanism that allows us to cope with our external, rather visual, world, in order to avoid dangers and harms. The detection by sight of geometrical issues in the environment (such as points, lines, contours and so on) does not necessarily describe the real world, rather stands for a sort of internal map based on species-specific, innate interpretations of the physical reality. In Karl Friston's terms (2010), the brain is an inferential machine that compares the real world with an internal map, pre-existent in our mind. Therefore, it might be suggested that what we see is just partially real: indeed, we might ask whether other animals, such as, e.g., an ant equipped with olfactive-tactile perceptions rather than our visual ones, or a bat (Nagel, 1974), are able to recognize points and lines. Indeed, some response properties, such as linearity of spatial summation, contrast gain, dependence of receptive field size on eccentricity and interlaced arrangement of cortical neurons, vary from species to species (Van Hooser et al., 2005; Mazade and Alonso, 2017). To make an example, despite the fact that a squirrel has greater visual acuity and a physically larger V1 than some mammals that have orientation maps in V1, it is not provided of such maps (Van Hooser et al., 2005). In turn, electrophysiological recordings in awake and anesthetized turtles reveal that their dorsal cortex is sensitive to the spatial structure of natural images, and their receptive fields are not entirely uniform across space (Fournier et al., 2018). Therefore, the mapping and representation of visual space in turtle's three-layered cortex differ from those found in mammalian primary visual cortex. This means that, contrary to the mammalian primary visual cortex, the turtle dorsal cortex performs a global, rather than local, analysis of the visual scene (Fournier et al., 2018). This means that, in animals different from us, spatial information can be found both at the single-neuron and population scales and a clear retinotopic mapping of thalamocortical projections is not detectable.

REFERENCES

1) Aristotle. 350 B.C. Physics. The Internet Classic Archive, http://classics.mit.edu/Aristotle/physics.6.vi.html.
2) Borsuk K. 1958-1959. Concerning the classification of topological spaces from the standpoint of the theory of retracts. Fundamenta Mathematicae XLVI:177–190.
3) Borsuk K. 1969. Fundamental retracts and extensions of fundamental sequences, Fundamenta Mathematicae 1:55–85.
4) Autrecourt, Nicholas of. About 1340. The Universal Treatise. Marquette University Press, Milwaukee, Wisconsin, 1971.
5) de Haro S, Dieks D, 't Hooft G, Verlinde E. 2013. Forty Years of String Theory Reflecting on the Foundations. Foundations of Physics. 43 (1): 1–7.
6) Di Concilio A. 2013. Point-Free Geometries: Proximities and Quasi-Metrics. Mathematics in Computer Science 7(1). DOI:10.1007/s11786-013-0140-2.
7) Di Concilio A., Guadagni C., Peters JF, Ramanna S. 2018. Descriptive proximities. Properties and interplay between classical proximities and overlap. Mathematics in Computer Science 12, 91-106, DOI 10.1007/s11786-017-0328-y.
8) Duff M. 1996. M-theory (the theory formerly known as strings). International Journal of Modern Physics A. 11 (32): 6523–41.
9) Duffy KR, Hubel DH. 2007. Receptive field properties of neurons in the primary visual cortex under photopic and scotopic lighting conditions. Vision Research. Volume 47, Issue 19, September 2007, Pages 2569-2574.
10) Euclid, 300 BCE. Elements of Geometry, trans. by R. Fizpatrick, farside.ph.utexas.edu/Books/Euclid/Elements.pdf.
11) Fournier J, Müller CM, Schneider I, Laurent G. 2018. Spatial Information in a Non-retinotopic Visual Cortex. Neuron, 3;97(1):164-180.e7. doi: 10.1016/j.neuron.2017.11.017.

12) Helmholz, H., 1858. Uber integrale der hydrodynamics Gleichungenwelche der Wirbelbewegungentsprechen, J. fur die reine und angewandteMathematik, 55, 25-55, [On the integrals of the hydronamical equations, which express vortex motion], trans. by P.G. Tait, Phil. Mag. 33 (1867), 485-512.
13) Friston K. 2010. The free-energy principle: a unified brain theory? Nat Rev Neurosci 11(2):127-138.
14) Johnstone PT. 1983. The point of pointless topology. Bulletin (New Series) of the American Mathematical Society. 8 (1): 41–53.
15) Li H, Fang Q, Ge Y, Li Z, Meng J, Zhu J, Yu H. 2018. Relationship between the Dynamics of Orientation Tuning and Spatiotemporal Receptive Field Structures of Cat LGN Neurons. Neuroscience. 2018 May 1;377:26-39. doi: 10.1016/j.neuroscience.2018.02.024.
16) Mazade R, Alonso JM. 2017. Thalamocortical processing in vision. Vis Neurosci. 2017 Jan;34:E007. doi: 10.1017/S0952523817000049.
17) Muxin H. 2011. Cosmological constant in loop quantum gravity vertex amplitude. Physical Review D. 84 (6): 064010.
18) Nagel T. 1974. What Is It Like to Be a Bat?. The Philosophical Review. 83 (4): 435–450. doi:10.2307/2183914
19) Naimpally, S.A., Warrack, B.D. 1970. Proximity Spaces, Cambridge University Press, Cambridge, UK.
20) Peters JF. 2016. Two forms of proximal, physical geometry. Axioms, sewing regions together, classes of regions, duality and parallel fibre bundles, Advan. In Math.: Sci. J. 5(2), 241-268.
21) Peters JF. 2014. Topology of Digital Images. Visual Pattern Discovery in Proximity Spaces, Springer-Verlag, Berlin, DOI 10.1007/978-3-642-57845-2.
22) Peters JF. 2016. Computational Proximity. Excursions in the Topology of Digital Images, Springer International Publishing, Switzerland, DOI 10.1007/978-3-319-30262-1.
23) Rovelli C, Smolin L. 1988. Knot Theory and Quantum Gravity. Physical Review Letters. 61 (10): 1155–1958.
24) Shapiro S, Hellman G. 2017. Frege Meets Aristotle: Points as Abstracts. Philosophia Mathematica, Volume 25, Issue 1, 1 February 2017, Pages 73–90, https://doi.org/10.1093/philmat/nkv021.
25) Simon M. 1901. Eucid und die sechsplanimetrischen Bucher. Leipzig, Teubner, 8vo. Vi+141pp.
26) Smirnov JM. 1952. On proximity spaces [in Russian], Math. Sb. (N.S.) 31, no. 73, 543-574. Engish translation, Amer. Math. Soc. Trans. Ser. 2, no. 38 (1964), 5-35.
27) Tait PG. 1877. On knots, Trans. Roy. Soc. Edin. 28, 273-317.
28) Thomson W. (Lord Kelvin). 1867. On vortex atoms, Proc. Roy. Soc. Edin. 6, 94-105.
29) Thomson W. (Lord Kelvin). 1869. On vortex motion, Proc. Roy. Soc. Edin. 25, 217-260.
30) Van Hooser SD, Heimel JA, Nelson SB. 2005. Functional cell classes and functional architecture in the early visual system of a highly visual rodent. Prog Brain Res. 2005;149:127-45.
31) Viswanathan P, Nieder A. 2017. Visual Receptive Field Heterogeneity and Functional Connectivity of Adjacent Neurons in Primate Frontoparietal Association Cortices. J Neurosci. 2017 Sep 13;37(37):8919-8928. doi: 10.1523/JNEUROSCI.0829-17.2017. Epub 2017 Aug 11.
32) Whitehead AN. 1929. Process and Reality. An Essay in Cosmology, The Free Press, N.Y.
33) Wigner E. 1960. The Unreasonable Effectiveness of Mathematics in the Natural Sciences. In: Communications in Pure and Applied Mathematics, vol. 13, No. I. New York: John Wiley & Sons, Inc.