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Neutrosophic Ideals of Γ-Semirings

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Abstract - Neutrosophic ideals of a Γ-semiring are introduced and studied in the sense of Smarandache[14], along with some operations such as intersection, composition, cartesian product etc. on them. Among the other results/characterizations, it is shown that all the operations are structure preserving.

Keywords - *Cartesian product, Homomorphism, Ideal, Intersection, Neutrosophic.*

1 Introduction

Uncertainties, which could be caused by information incompleteness, data randomness limitations of measuring instruments, etc., are pervasive in many complicated problems in biology, engineering, economics, environment, medical science and social science. We cannot successfully use the classical methods for these problems. To solve this problem, the concept of fuzzy sets was introduced by Zadeh [15] in 1965 where each element have a degree of membership and has been extensively applied to many scientific fields. As a generalization of fuzzy sets, the intuitionistic fuzzy set was introduced by Atanassov [1] in 1986, where besides the degree of membership of each element there was considered a degree of non-membership with (membership value + non-membership value) ≤ 1.

There are also several well-known theories, for instances, rough sets, vague sets, interval-valued sets etc. which can be considered as mathematical tools for dealing with uncertainties. In 1995, inspired from the sport games (winning/tie/defeating), votes, from (yes/NA/no), from decision making (making a decision/ hesitating/not making), from (accepted/pending/rejected) etc. and guided by the fact that the law of excluded middle did not work any longer in the modern logics, F. Smarandache [14] combined the non-standard analysis [4, 11] with a tri-component logic/set/probability theory and with philosophy and introduced *Neutrosophic set* which represents the main distinction between *fuzzy* and *intuitionistic fuzzy* logic/set. Here he included the middle component. i.e. the neutral/ indeterminate/unknown part (besides the truth/membership

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and falsehood/non-membership components that both appear in fuzzy logic/set) to distinguish between 'absolute membership and relative membership' or 'absolute non-membership and relative non-membership'(see, [6, 13]). There are also several authors [2, 3, 8] who have enriched the theory of neutrosophic sets.

Inspired from the above idea and motivated by the fact that 'semirings arise naturally in combinatorics, mathematical modelling, graph theory, automata theory, parallel computation system etc.', in the paper, I have used that to study the ideals, which play a central role in the structure theory and useful for many purposes, of Γ-semirings[10] - a generalization of semirings [5, 7] and obtain some of its characterizations.

2 Preliminaries

We recall the following results for subsequent use.

Definition 2.1. *Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ ($(a,\alpha,b) \mapsto a\alpha b$) satisfying the following conditions:*

(i) $(a+b)\alpha c = a\alpha c + b\alpha c$

(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$

(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$

(v) $0_S\alpha a = 0_S = a\alpha 0_S$

(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
For simplification we write 0 instead of 0_S and 0_Γ.

Definition 2.2. *A left ideal I of Γ-semiring S is a nonempty subset of S satisfying the following conditions:*

(i) *If $a, b \in I$ then $a+b \in I$*

(ii) *If $a \in I$, $s \in S$ and $\gamma \in \Gamma$ then $s\gamma a \in I$*

(iii) $I \neq S$.

A right ideal of S is defined in an analogous manner and an ideal of S is a nonempty subset which is both a left ideal and a right ideal of S.

Definition 2.3. *Let R, S be two Γ-semirings and $a, b \in R$, $\gamma \in \Gamma$. A function $f : R \to S$ is said to be a homomorphism if*

(i) $f(a+b) = f(a) + f(b)$

(ii) $f(a\gamma b) = f(a)\gamma f(b)$

(iii) $f(0_R) = 0_S$ *where* 0_R *and* 0_S *are the zeroes of* R *and* S *respectively.*

Definition 2.4. *A neutrosophic set* A *on the universe of discourse* X *is defined as* $A = \{< x, A^T(x), A^I(x), A^F(x) >, x \in X\}$, *where* $A^T, A^I, A^F : X \to]^-0, 1^+[$ *and* $^-0 \leq A^T(x) + A^I(x) + A^F(x) \leq 3^+$. *From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of* $]^-0, 1^+[$. *But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of* $]^-0, 1^+[$. *Hence we consider the neutrosophic set which takes the value from the subset of [0, 1].*

3 Main Results

Throughout this section unless otherwise mentioned S denotes a Γ-semiring.

Definition 3.1. *Let* $\mu = (\mu^T, \mu^I, \mu^F)$ *be a non-empty neutrosophic subset of a* Γ-*semiring* S *(i.e. anyone of* $\mu^T(x)$, $\mu^I(x)$ *or* $\mu^F(x)$ *not equal to zero for some* $x \in S$*). Then* μ *is called a neutrosophic left ideal of* S *if*

(i) $\mu^T(x + y) \geq \min\{\mu^T(x), \mu^T(y)\}$, $\mu^T(x\gamma y) \geq \mu^T(y)$

(ii) $\mu^I(x + y) \geq \frac{\mu^I(x)+\mu^I(y)}{2}$, $\mu^I(x\gamma y) \geq \mu^I(y)$

(iii) $\mu^F(x + y) \leq \max\{\mu^F(x), \mu^F(y)\}$, $\mu^F(x\gamma y) \leq \mu^F(y)$.

for all $x, y \in S$ *and* $\gamma \in \Gamma$.
Similarly we can define neutrosophic right ideal of S.

Example 3.2. *Let* S *be the additive commutative semigroup of all non-positive integers and* Γ *be the additive commutative semigroup of all non-positive even integers. Then* S *is a* Γ-*semiring if* $a\gamma b$ *denotes the usual multiplication of integers* a, γ, b *where* $a, b \in S$ *and* $\gamma \in \Gamma$. *Define a neutrosophic subset* μ *of* S *as follows*

$$\mu(x) = \begin{cases} (1, 0, 0) & \text{if } x = 0 \\ (0.8, 0.3, 0.4) & \text{if } x \text{ is even} \\ (0.3, .02, 0.7) & \text{if } x \text{ is odd} \end{cases}$$

Then the neutrosophic set μ *of* S *is a neutrosophic ideal of* S.

Theorem 3.3. *A neutrosophic set* μ *of a* Γ-*semiring* S *is a neutrosophic left ideal of* S *if and only if any level subsets* $\mu_t^T := \{x \in S : \mu^T(x) \geq t,\ t \in [0, 1]\}$, $\mu_t^I := \{x \in S : \mu^I(x) \geq t,\ t \in [0, 1]\}$ *and* $\mu_t^F := \{x \in S : \mu^F(x) \leq t,\ t \in [0, 1]\}$ *are left ideals of* S.

Proof. Assume that the neutrosophic set μ of S is a neutrosophic left ideal of S. Then anyone of μ^T, μ^I or μ^F is not equal to zero for some $x \in S$ i.e., in other words anyone of μ_t^T, μ_t^I or μ_t^F is not equal to zero for all $t \in [0, 1]$. So it is sufficient to consider that all of them are not equal to zero.
Suppose $x, y \in \mu_t = (\mu_t^T,\ \mu_t^I,\ \mu_t^F)$, $s \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} &\mu^T(x + y) \geq \min\{\mu^T(x), \mu^T(y)\} \geq \min\{t, t\} = t \\ &\mu^I(x + y) \geq \tfrac{\mu^I(x)+\mu^I(y)}{2} \geq \tfrac{t+t}{2} = t \\ &\mu^F(x + y) \leq \max\{\mu^F(x), \mu^F(y)\} \leq \max\{t, t\} = t \end{aligned}$$

which implies $x + y \in \mu_t^T, \mu_t^I, \mu_t^F$ i.e., $x + y \in \mu_t$. Also

$$\begin{array}{l} \mu^T(s\gamma x) \geq \mu^T(x) \geq t \\ \mu^I(s\gamma x) \geq \mu^I(x) \geq t \\ \mu^F(s\gamma x) \leq \mu^F(x) \leq t \end{array}$$

Hence $s\gamma x \in \mu_t$.
Therefore μ_t is a left ideal of S.
Conversely, suppose $\mu_t(\neq \phi)$ is a left ideal of S. If possible μ is not a neutrosophic left ideal. Then for $x, y \in S$ anyone of the following inequality is true.

$$\begin{array}{l} \mu^T(x+y) < \min\{\mu^T(x), \mu^T(y)\} \\ \mu^I(x+y) < \frac{\mu^I(x)+\mu^I(y)}{2} \\ \mu^F(x+y) > \max\{\mu^F(x), \mu^F(y)\} \end{array}$$

For the first inequality, choose $t_1 = \frac{1}{2}[\mu^T(x+y)+\min\{\mu^T(x), \mu^T(y)\}]$. Then $\mu^T(x+y) < t_1 < \min\{\mu^T(x), \mu^T(y)\}$ which implies $x, y \in \mu_{t_1}^T$ but $x + y \notin \mu_{t_1}^T$ - a contradiction.
For the second inequality, choose $t_2 = \frac{1}{2}[\mu^I(x+y)+\min\{\mu^I(x), \mu^I(y)\}]$. Then $\mu^I(x+y) < t_2 < \frac{\mu^I(x)+\mu^I(y)}{2}$ which implies $x, y \in \mu_{t_2}^I$ but $x + y \notin \mu_{t_2}^I$ - a contradiction.
For the third inequality, choose $t_3 = \frac{1}{2}[\mu^F(x + y) + \max\{\mu^F(x), \mu^F(y)\}]$. Then $\mu^F(x + y) > t_3 > \max\{\mu^F(x), \mu^F(y)\}$ which implies $x, y \in \mu_{t_3}^F$ but $x+y \notin \mu_{t_3}^F$ - a contradiction.
So, in any case we have a contradiction to the fact that μ_t is a left ideal of S.
Hence the result follows. □

Definition 3.4. *[9] Let μ and ν be two neutrosophic subsets of S. The intersection of μ and ν is defined by*

$$(\mu^T \cap \nu^T)(x) = \min\{\mu^T(x), \nu^T(x)\}$$

$$(\mu^I \cap \nu^I)(x) = \min\{\mu^I(x), \nu^I(x)\}$$

$$(\mu^F \cap \nu^F)(x) = \max\{\mu^F(x), \nu^F(x)\}$$

for all $x \in S$.

Proposition 3.5. *Intersection of a non-empty collection of neutrosophic left ideals is also a neutrosophic left ideal of S.*

Proof. Let $\{\mu_i : i \in I\}$ be a non-empty family of neutrosophic left ideals of a Γ-semiring S and $x, y \in S$, $\gamma \in \Gamma$. Then

$$\begin{aligned} (\bigcap_{i\in I}\mu_i^T)(x+y) &= \inf_{i\in I} \mu_i^T(x+y) \geq \inf_{i\in I} \{\min\{\mu_i^T(x), \mu_i^T(y)\}\} \\ &= \min\{\inf_{i\in I} \mu_i^T(x), \inf_{i\in I} \mu_i^T(y)\} \\ &= \min\{(\bigcap_{i\in I}\mu_i^T)(x), (\bigcap_{i\in I}\mu_i^T)(y)\}. \end{aligned}$$

$$\begin{aligned} (\bigcap_{i\in I}\mu_i^I)(x+y) &= \inf_{i\in I} \mu_i^I(x+y) \geq \inf_{i\in I} \frac{\mu_i^I(x)+\mu_i^I(y)}{2} \\ &= \frac{\inf_{i\in I} \mu_i^I(x)+\inf_{i\in I} \mu_i^I(y)}{2} \\ &= \frac{\bigcap_{i\in I}\mu_i^I(x)+\bigcap_{i\in I}\mu_i^I(y)}{2}. \end{aligned}$$

$$\begin{aligned}(\bigcap_{i\in I}\mu_i^F)(x+y) &= \sup_{i\in I}\ \mu_i^F(x+y) \le \sup_{i\in I}\ \{\max\{\mu_i^F(x), \mu_i^F(y)\}\}\\ &= \max\{\sup_{i\in I}\ \mu_i^F(x), \sup_{i\in I}\ \mu_i^F(y)\}\\ &= \max\{(\bigcap_{i\in I}\mu_i^F)(x), (\bigcap_{i\in I}\mu_i^F)(y)\}.\end{aligned}$$

$$(\bigcap_{i\in I}\mu_i^T)(x\gamma y) = \inf_{i\in I}\ \mu_i^T(x\gamma y) \ge \inf_{i\in I}\ \mu_i^T(y) = (\bigcap_{i\in I}\mu_i^T)(y).$$

$$(\bigcap_{i\in I}\mu_i^I)(x\gamma y) = \inf_{i\in I}\ \mu_i^I(x\gamma y) \ge \inf_{i\in I}\ \mu_i^I(y) = (\bigcap_{i\in I}\mu_i^I)(y).$$

$$(\bigcap_{i\in I}\mu_i^F)(x\gamma y) = \sup_{i\in I}\ \mu_i^F(x\gamma y) \le \sup_{i\in I}\ \mu_i^F(y) = (\bigcap_{i\in I}\mu_i^F)(y).$$

Hence $\bigcap_{i\in I}\mu_i$ is a neutrosophic left ideal of S. □

Proposition 3.6. *Let $f : R \to S$ be a morphism of Γ-semirings. Then*

(i) If ϕ is a neutrosophic left ideal of S, then $f^{-1}(\phi)$ [12] is a neutrosophic left ideal of R.

(ii) If f is surjective morphism and μ is a neutrosophic left ideal of R, then $f(\mu)$ [12] is a neutrosophic left ideal of S.

Proof. Let $f : R \to S$ be a morphism of Γ-semirings.
(i) Let ϕ be a neutrosophic left ideal of S and $r, s \in R$, $\gamma \in \Gamma$.

$$\begin{aligned}f^{-1}(\phi^T)(r+s) &= \phi^T(f(r+s)) = \phi^T(f(r)+f(s))\\ &\ge \min\{\phi^T(f(r)), \phi^T(f(s))\} = \min\{(f^{-1}(\phi^T))(r), (f^{-1}(\phi^T))(s)\}.\end{aligned}$$

$$\begin{aligned}f^{-1}(\phi^I)(r+s) &= \phi^I(f(r+s)) = \phi^I(f(r)+f(s))\\ &\ge \tfrac{\phi^I(f(r))+\phi^I(f(s))}{2} = \tfrac{(f^{-1}(\phi^I))(r)+(f^{-1}(\phi^I))(s)}{2}.\end{aligned}$$

$$\begin{aligned}f^{-1}(\phi^F)(r+s) &= \phi^F(f(r+s)) = \phi^F(f(r)+f(s))\\ &\le \max\{\phi^F(f(r)), \phi^F(f(s))\} = \max\{(f^{-1}(\phi^F))(r), (f^{-1}(\phi^F))(s)\}.\end{aligned}$$

Again

$$\begin{aligned}(f^{-1}(\phi^T))(r\gamma s) &= \phi^T(f(r\gamma s)) = \phi^T(f(r)\gamma f(s))\\ &\ge \phi^T(f(s)) = (f^{-1}(\phi^T))(s).\end{aligned}$$

$$\begin{aligned}(f^{-1}(\phi^I))(r\gamma s) &= \phi^I(f(r\gamma s)) = \phi^I(f(r)\gamma f(s))\\ &\ge \phi^I(f(s)) = (f^{-1}(\phi^I))(s).\end{aligned}$$

$$\begin{aligned}(f^{-1}(\phi^F))(r\gamma s) &= \phi^F(f(r\gamma s)) = \phi^F(f(r)\gamma f(s))\\ &\le \phi^F(f(s)) = (f^{-1}(\phi^F))(s).\end{aligned}$$

Thus $f^{-1}(\phi)$ is a neutrosophic left ideal of R.

(ii) Suppose μ be a neutrosophic left ideal of R and $x^{'}, y^{'} \in S$, $\gamma \in \Gamma$. Then

$$\begin{aligned}(f(\mu^T))(x^{'}+y^{'}) &= \sup_{z\in f^{-1}(x^{'}+y^{'})} \mu^T(z) \geq \sup_{x\in f^{-1}(x^{'}), y\in f^{-1}(y^{'})} \mu^T(x+y) \geq \sup\{\min\{\mu^T(x), \mu^T(y)\}\} \\ &= \min\{\sup_{x\in f^{-1}(x^{'})} \mu^T(x), \sup_{y\in f^{-1}(y^{'})} \mu^T(y)\} = \min\{(f(\mu^T))(x^{'}), (f(\mu^T))(y^{'})\}.\end{aligned}$$

$$\begin{aligned}(f(\mu^I))(x^{'}+y^{'}) &= \sup_{z\in f^{-1}(x^{'}+y^{'})} \mu^I(z) \geq \sup_{x\in f^{-1}(x^{'}), y\in f^{-1}(y^{'})} \mu^I(x+y) \geq \sup \tfrac{\mu^I(x)+\mu^I(y)}{2} \\ &= \tfrac{1}{2}[\sup_{x\in f^{-1}(x^{'})} \mu^I(x) + \sup_{y\in f^{-1}(y^{'})} \mu^I(y)] = \tfrac{1}{2}[(f(\mu^I))(x^{'}) + (f(\mu^I))(y^{'})].\end{aligned}$$

$$\begin{aligned}(f(\mu^F))(x^{'}+y^{'}) &= \inf_{z\in f^{-1}(x^{'}+y^{'})} \mu^F(z) \leq \inf_{x\in f^{-1}(x^{'}), y\in f^{-1}(y^{'})} \mu^F(x+y) \leq \inf\{\max\{\mu^F(x), \mu^F(y)\}\} \\ &= \max\{\inf_{x\in f^{-1}(x^{'})} \mu^F(x), \inf_{y\in f^{-1}(y^{'})} \mu^F(y)\} = \max\{(f(\mu^F))(x^{'}), (f(\mu^F))(y^{'})\}.\end{aligned}$$

Again

$$\begin{aligned}f(\mu^T)(x^{'}\gamma y^{'}) &= \sup_{z\in f^{-1}(x^{'}\gamma y^{'})} \mu^T(z) \geq \sup_{x\in f^{-1}(x^{'}), y\in f^{-1}(y^{'})} \mu^T(x\gamma y) \\ &\geq \sup_{y\in f^{-1}(y^{'})} \mu^T(y) = f(\mu^T)(y^{'}). \\ f(\mu^I)(x^{'}\gamma y^{'}) &= \sup_{z\in f^{-1}(x^{'}\gamma y^{'})} \mu^I(z) \geq \sup_{x\in f^{-1}(x^{'}), y\in f^{-1}(y^{'})} \mu^I(x\gamma y) \\ &\geq \sup_{y\in f^{-1}(y^{'})} \mu^I(y) = f(\mu^I)(y^{'}). \\ f(\mu^F)(x^{'}\gamma y^{'}) &= \inf_{z\in f^{-1}(x^{'}\gamma y^{'})} \mu^F(z) \leq \inf_{x\in f^{-1}(x^{'}), y\in f^{-1}(y^{'})} \mu^F(x\gamma y) \\ &\leq \inf_{y\in f^{-1}(y^{'})} \mu^F(y) = f(\mu^F)(y^{'}).\end{aligned}$$

Thus $f(\mu)$is a neutrosophic left ideal of S.

□

Definition 3.7. *[9] Let μ and ν be two neutrosophic subsets of S. The cartesian product of μ and ν is defined by*

$$(\mu^T \times \nu^T)(x, y) = \min\{\mu^T(x), \nu^T(y)\}$$

$$(\mu^I \times \nu^I)(x, y) = \frac{\mu^I(x) + \nu^I(y)}{2}$$

$$(\mu^F \times \nu^F)(x, y) = \max\{\mu^F(x), \nu^F(y)\}$$

for all $x, y \in S$.

Theorem 3.8. *Let μ and ν be two neutrosophic left ideals of S. Then $\mu \times \nu$ is a neutrosophic left ideal of $S \times S$.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}(\mu^T \times \nu^T)((x_1, x_2) + (y_1, y_2)) &= (\mu^T \times \nu^T)(x_1 + y_1, x_2 + y_2)\\ &= \min\{\mu^T(x_1 + y_1), \nu^T(x_2 + y_2)\}\\ &\geq \min\{\min\{\mu^T(x_1), \mu^T(y_1)\}, \min\{\nu^T(x_2), \nu^T(y_2)\}\}\\ &= \min\{\min\{\mu^T(x_1), \nu^T(x_2)\}, \min\{\mu^T(y_1), \nu^T(y_2)\}\}\\ &= \min\{(\mu^T \times \nu^T)(x_1, x_2), (\mu^T \times \nu^T)(y_1, y_2)\}.\end{aligned}$$

$$\begin{aligned}(\mu^I \times \nu^I)((x_1, x_2) + (y_1, y_2)) &= (\mu^I \times \nu^I)(x_1 + y_1, x_2 + y_2)\\ &= \tfrac{\mu^I(x_1+y_1)+\nu^I(x_2+y_2)}{2}\\ &\geq \tfrac{1}{2}\{\tfrac{\mu^I(x_1)+\mu^I(y_1)}{2} + \tfrac{\nu^I(x_2)+\nu^I(y_2)}{2}\}\\ &= \tfrac{1}{2}\{\tfrac{\mu^I(x_1)+\nu^I(x_2)}{2} + \tfrac{\mu^I(y_1)+\nu^I(y_2)}{2}\}\\ &= \tfrac{1}{2}\{(\mu^I \times \nu^I)(x_1, x_2) + (\mu^I \times \nu^I)(y_1, y_2)\}.\end{aligned}$$

$$\begin{aligned}(\mu^F \times \nu^F)((x_1, x_2) + (y_1, y_2)) &= (\mu^F \times \nu^F)(x_1 + y_1, x_2 + y_2)\\ &= \max\{\mu^F(x_1 + y_1), \nu^F(x_2 + y_2)\}\\ &\leq \max\{\max\{\mu^F(x_1), \mu^F(y_1)\}, \max\{\nu^F(x_2), \nu^F(y_2)\}\}\\ &= \max\{\max\{\mu^F(x_1), \nu^F(x_2)\}, \max\{\mu^F(y_1), \nu^F(y_2)\}\}\\ &= \max\{(\mu^F \times \nu^F)(x_1, x_2), (\mu^F \times \nu^F)(y_1, y_2)\}.\end{aligned}$$

$$\begin{aligned}(\mu^T \times \nu^T)((x_1, x_2)\gamma(y_1, y_2)) &= (\mu^T \times \nu^T)(x_1\gamma y_1, x_2\gamma y_2) = \min\{\mu^T(x_1\gamma y_1), \nu^T(x_2\gamma y_2)\}\\ &\geq \min\{\mu^T(y_1), \nu^T(y_2)\} = (\mu^T \times \nu^T)(y_1, y_2).\end{aligned}$$

$$\begin{aligned}(\mu^I \times \nu^I)((x_1, x_2)\gamma(y_1, y_2)) &= (\mu^I \times \nu^I)(x_1\gamma y_1, x_2\gamma y_2) = \tfrac{\mu^I(x_1\gamma y_1)+\nu^I(x_2\gamma y_2)}{2}\\ &\geq \tfrac{\mu^I(y_1)+\nu^I(y_2)}{2} = (\mu^I \times \nu^I)(y_1, y_2).\end{aligned}$$

$$\begin{aligned}(\mu^F \times \nu^F)((x_1, x_2)\gamma(y_1, y_2)) &= (\mu^F \times \nu^F)(x_1\gamma y_1, x_2\gamma y_2) = \max\{\mu^F(x_1\gamma y_1), \nu^F(x_2\gamma y_2)\}\\ &\leq \max\{\mu^F(y_1), \nu^F(y_2)\} = (\mu^F \times \nu^F)(y_1, y_2).\end{aligned}$$

Hence $\mu \times \nu$ is a neutrosophic left ideal of $S \times S$.

□

Theorem 3.9. *Let μ be a neutrosophic subset of S. Then μ is a neutrosophic left ideal of S if and only if $\mu \times \mu$ is a neutrosophic left ideal of $S \times S$.*

Proof. Suppose μ be a neutrosophic subset of S. If μ is a neutrosophic left ideal of S then by Theorem 3.8, $\mu \times \mu$ is a neutrosophic left ideal of $S \times S$.
Conversely, suppose $\mu \times \mu$ is a neutrosophic left ideal of $S \times S$ and $x_1, x_2, y_1, y_2 \in S$, $\gamma \in \Gamma$. Then

$$\begin{aligned}\min\{\mu^T(x_1 + y_1), \mu^T(x_2 + y_2)\} &= (\mu^T \times \mu^T)(x_1 + y_1, x_2 + y_2)\\ &= (\mu^T \times \mu^T)((x_1, x_2) + (y_1, y_2))\\ &\geq \min\{(\mu^T \times \mu^T)(x_1, x_2), (\mu^T \times \mu^T)(y_1, y_2)\}\\ &= \min\{\min\{\mu^T(x_1), \mu^T(x_2)\}, \min\{\mu^T(y_1), \mu^T(y_2)\}\}.\end{aligned}$$

$$\begin{aligned}\tfrac{\mu^I(x_1+y_1)+\mu^I(x_2+y_2)}{2} &= (\mu^I \times \mu^I)(x_1+y_1, x_2+y_2)\\ &= (\mu^I \times \mu^I)((x_1,x_2)+(y_1,y_2))\\ &\geq \tfrac{(\mu^I\times\mu^I)(x_1,x_2)+(\mu^I\times\mu^I)(y_1,y_2)}{2}\\ &= \tfrac{1}{2}[\tfrac{\mu^I(x_1)+\mu^I(x_2)}{2} + \tfrac{\mu^I(y_1)+\mu^I(y_2)}{2}].\end{aligned}$$

$$\begin{aligned}\max\{\mu^F(x_1+y_1), \mu^F(x_2+y_2)\} &= (\mu^F \times \mu^F)(x_1+y_1, x_2+y_2)\\ &= (\mu^F \times \mu^F)((x_1,x_2)+(y_1,y_2))\\ &\leq \max\{(\mu^F \times \mu^F)(x_1,x_2), (\mu^F \times \mu^F)(y_1,y_2)\}\\ &= \max\{\max\{\mu^F(x_1),\mu^F(x_2)\}, \max\{\mu^F(y_1),\mu^F(y_2)\}\}.\end{aligned}$$

Now, putting $x_1 = x, x_2 = 0, y_1 = y$ and $y_2 = 0$, in the above inequalities and noting that $\mu^T(0) \geq \mu^T(x)$, $\mu^I(0) = 0$ and $\mu^F(0) \leq \mu^F(x)$ for all $x \in S$ we obtain

$$\begin{aligned}&\mu^T(x+y) \geq \min\{\mu^T(x), \mu^T(y)\}\\ &\mu^I(x+y) \geq \tfrac{\mu^I(x)+\mu^I(y)}{2}\\ &\mu^F(x+y) \leq \max\{\mu^F(x), \mu^F(y)\}.\end{aligned}$$

Next, we have

$$\begin{aligned}\min\{\mu^T(x_1\gamma y_1), \mu^T(x_2\gamma y_2)\} &= (\mu^T \times \mu^T)(x_1\gamma y_1, x_2\gamma y_2) = (\mu^T \times \mu^T)((x_1,x_2)\gamma(y_1,y_2))\\ &\geq (\mu^T \times \mu^T)(y_1,y_2) = \min\{\mu^T(y_1), \mu^T(y_2)\}.\end{aligned}$$

$$\begin{aligned}\tfrac{\mu^I(x_1\gamma y_1)+\nu^I(x_2\gamma y_2)}{2} &= (\mu^I \times \mu^I)((x_1,x_2)\gamma(y_1,y_2))\\ &\geq (\mu^I \times \mu^I)(y_1,y_2)\\ &= \tfrac{\mu^I(y_1)+\mu^I(y_2)}{2}.\end{aligned}$$

$$\begin{aligned}\max\{\mu^F(x_1\gamma y_1), \mu^F(x_2\gamma y_2)\} &= (\mu^F \times \mu^F)(x_1\gamma y_1, x_2\gamma y_2) = (\mu^F \times \mu^F)((x_1,x_2)\gamma(y_1,y_2))\\ &\leq (\mu^F \times \mu^F)(y_1,y_2) = \max\{\mu^F(y_1), \mu^F(y_2)\}.\end{aligned}$$

Taking $x_1 = x, x_2 = 0, y_1 = y$ and $y_2 = 0$, we obtain

$$\begin{aligned}&\mu^T(x\gamma y) \geq \mu^T(y)\\ &\mu^I(x\gamma y) \geq \mu^I(y)\\ &\mu^F(x\gamma y) \leq \mu^F(y).\end{aligned}$$

Hence μ is a neutrosophic left ideal of S. □

Definition 3.10. *Let μ and ν be two neutrosophic sets of a Γ-semiring S. Define composition of μ and ν by*

$$\begin{aligned}\mu^T o \nu^T(x) &= \sup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i} \{\min_i\{\mu^T(a_i), \nu^T(b_i)\}\}\\ &= 0, \ \textit{if } x \textit{ cannot be expressed as above}\end{aligned}$$

$$\mu^I o\nu^I(x) = \sup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i} \sum_{i=1}^{n} \frac{\mu^I(a_i)+\nu^I(b_i)}{2}$$
$$= 0, \text{ if } x \text{ cannot be expressed as above}$$
$$\mu^F o\nu^F(x) = \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i} \{\max_i\{\mu^F(a_i), \nu^F(b_i)\}\}$$
$$= 0, \text{ if } x \text{ cannot be expressed as above}$$

where $x, a_i, b_i \in S$ and $\gamma_i \in \Gamma$, for $i = 1, ..., n$.

Theorem 3.11. *If μ and ν be two neutrosophic left ideals of S then $\mu o\nu$ is also a neutrosophic left ideal of S.*

Proof. Suppose μ, ν be two neutrosophic ideals of S and $x, y \in S$, $\gamma \in \Gamma$. If $x + y$ cannot be expressed as $\sum_{i=1}^{n} a_i\gamma_i b_i$, for $a_i, b_i \in S$ and $\gamma_i \in \Gamma$, then there is nothing to prove. So, assume that $x + y$ have such an expression. Then

$$\begin{aligned}
&(\mu^T o\nu^T)(x+y)\\
&= \sup_{x+y=\sum_{i=1}^{n} a_i\gamma_i b_i} \{\min_i\{\mu^T(a_i), \nu^T(b_i)\}\}\\
&\geq \sup_{x=\sum_{i=1}^{n} c_i\delta_i d_i,\, y=\sum_{i=1}^{n} e_i\eta_i f_i} \{\min_i\{\mu^T(c_i), \nu^T(d_i), \mu^T(e_i), \nu^T(f_i)\}\}\\
&= \min\{\sup_{x=\sum_{i=1}^{n} c_i\delta_i d_i} \{\min_i\{\mu^T(c_i), \nu^T(d_i)\}\}, \sup_{y=\sum_{i=1}^{n} e_i\eta_i f_i} \{\min_i\{\mu^T(e_i), \nu^T(f_i)\}\}\}\\
&= \min\{(\mu^T o\nu^T)(x), (\mu^T o\nu^T)(y)\}.
\end{aligned}$$

$$\begin{aligned}
&(\mu^I o\nu^I)(x+y)\\
&= \sup_{x+y=\sum_{i=1}^{n} a_i\gamma_i b_i} \sum_{i=1}^{n} \frac{\mu^I(a_i)+\nu^I(b_i)}{2}\\
&\geq \sup_{x=\sum_{i=1}^{n} c_i\delta_i d_i,\, y=\sum_{i=1}^{n} e_i\eta_i f_i} \sum_{i=1}^{n} \frac{\mu^I(c_i)+\nu^I(d_i)+\mu^I(e_i)+\nu^I(f_i)}{2}\\
&\geq \frac{1}{2}[\sup_{x=\sum_{i=1}^{n} c_i\delta_i d_i} \sum_{i=1}^{n} \frac{\mu^I(c_i)+\nu^I(d_i)}{2}, \sup_{y=\sum_{i=1}^{n} e_i\eta_i f_i} \sum_{i=1}^{n} \frac{\mu^I(e_i)+\nu^I(f_i)}{2}]\\
&= \frac{(\mu^I o\nu^I)(x)+(\mu^I o\nu^I)(y)}{2}.
\end{aligned}$$

$$
\begin{aligned}
&(\mu^F o\nu^F)(x+y)\\
&= \inf_{x+y=\sum_{i=1}^{n} a_i\gamma_i b_i} \{\max_i\{\mu^F(a_i),\nu^F(b_i)\}\}\\
&\leq \inf_{x=\sum_{i=1}^{n} c_i\delta_i d_i,\, y=\sum_{i=1}^{n} e_i\eta_i f_i} \{\max_i\{\mu^F(c_i),\nu^F(d_i),\mu^F(e_i),\nu^F(f_i)\}\}\\
&= \max\{\inf_{x=\sum_{i=1}^{n} c_i\delta_i d_i} \{\max_i\{\mu^F(c_i),\nu^F(d_i)\}\}, \inf_{y=\sum_{i=1}^{n} e_i\eta_i f_i} \{\max_i\{\mu^F(e_i),\nu^F(f_i)\}\}\}\\
&= \max\{(\mu^F o\nu^F)(x),(\mu^F o\nu^F)(y)\}.
\end{aligned}
$$

$$
\begin{aligned}
(\mu^T o\nu^T)(x\gamma y) &= \sup_{x\gamma y=\sum_{i=1}^{n} a_i\alpha_i b_i} \{\min_i\{\mu^T(a_i),\nu^T(b_i)\}\}\\
&\geq \sup_{x\gamma y=\sum_{i=1}^{n} x\gamma e_i\eta_i f_i} \{\min_i\{\mu^T(x\gamma e_i),\nu^T(f_i)\}\}\\
&\geq \sup_{y=\sum_{i=1}^{n} e_i\eta_i f_i} \{\min_i\{\mu^T(e_i),\nu^T(f_i)\}\} = (\mu^T o\nu^T)(y).
\end{aligned}
$$

$$
\begin{aligned}
(\mu^I o\nu^I)(x\gamma y) &= \sup_{x\gamma y=\sum_{i=1}^{n} a_i\alpha_i b_i} \textstyle\sum_{i=1}^{n} \frac{\mu^I(a_i)+\nu^I(b_i)}{2}\\
&\geq \sup_{x\gamma y=\sum_{i=1}^{n} x\gamma e_i\eta_i f_i} \textstyle\sum_{i=1}^{n} \frac{\mu^I(x\gamma e_i)+\nu^I(f_i)}{2}\\
&\geq \sup_{y=\sum_{i=1}^{n} e_i\eta_i f_i} \textstyle\sum_{i=1}^{n} \frac{\mu^I(e_i)+\nu^I(f_i)}{2} = (\mu^I o\nu^I)(y).
\end{aligned}
$$

$$
\begin{aligned}
(\mu^F o\nu^F)(x\gamma y) &= \inf_{x\gamma y=\sum_{i=1}^{n} a_i\alpha_i b_i} \{\max_i\{\mu^F(a_i),\nu^F(b_i)\}\}\\
&\leq \inf_{x\gamma y=\sum_{i=1}^{n} x\gamma e_i\eta_i f_i} \{\max_i\{\mu^F(x\gamma e_i),\nu^F(f_i)\}\}\\
&\leq \inf_{y=\sum_{i=1}^{n} e_i\eta_i f_i} \{\max_i\{\mu^F(e_i),\nu^F(f_i)\}\} = (\mu^F o\nu^F)(y).
\end{aligned}
$$

Hence $\mu o\nu$ is a neutrosophic left ideal of S. □

4 Conclusion

In this paper, we have studied neutrosophic ideals of Γ-semirings in the sense of Smarandache[14] with some operations on them and obtain some of its characterizations. Our next aim is to use these results to study some other properties such *prime neutrosophic ideal*, *semiprime neutrosophic ideal*, *radicals* etc..

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